2.1. TIME-DEPENDENT HAMILTONIAN

Mixing of eigenstates by a time-dependent potential

For many time-dependent problems, most notably in spectroscopy, we often can partition the time-dependent Hamiltonian into a time-independent part that we can describe exactly and a time-dependent part

\[ H = H_0 + V(t) \]  \hspace{1cm} (2.1)

Here \( H_0 \) is time-independent and \( V(t) \) is a time-dependent potential, often an external field. Nitzan, Sec. 2.3., offers a nice explanation of the circumstances that allow us to use this approach. It arises from partitioning the system into internal degrees of freedom in \( H_0 \) and external degrees of freedom acting on \( H_0 \). If you have reason to believe that the external Hamiltonian can be treated classically, then eq. (2.1) follows in a straightforward manner. Then there is a straightforward approach to describing the time-evolving wavefunction for the system in terms of the eigenstates and energy eigenvalues of \( H_0 \). We know

\[ H_0 \hat{n} = E_n \hat{n}. \]  \hspace{1cm} (2.2)

The state of the system can be expressed as a superposition of these eigenstates:

\[ |\psi(t)\rangle = \sum_n c_n(t) |n\rangle \]  \hspace{1cm} (2.3)

The TDSE can be used to find an equation of motion for the expansion coefficients

\[ c_k(t) = \langle k | \psi(t) \rangle \]  \hspace{1cm} (2.4)

Starting with

\[ \frac{\partial |\psi\rangle}{\partial t} = -\frac{i}{\hbar} H |\psi\rangle \]  \hspace{1cm} (2.5)

\[ \frac{\partial c_k(t)}{\partial t} = -\frac{i}{\hbar} \langle k | H | \psi(t) \rangle \]  \hspace{1cm} (2.6)
inserting $\sum_n |n\rangle \langle n| = 1$

$$=-\frac{i}{\hbar} \sum_n \langle k | H | n \rangle c_n(t)$$

(2.7)

substituting eq. (2.1) we have:

$$\frac{\partial c_k(t)}{\partial t} = -\frac{i}{\hbar} \sum_n \langle k | \left[H_0 + V(t)\right] | n \rangle c_n(t)$$

$$=-\frac{i}{\hbar} \sum_n \left[E_n \delta_{kn} + V_{kn}(t)\right] c_n(t)$$

(2.8)

or,

$$\frac{\partial c_k(t)}{\partial t} + i E_k c_k(t) = -\frac{i}{\hbar} \sum_n V_{kn}(t) c_n(t).$$

(2.9)

If we make a substitution

$$c_m(t) = e^{-iE_m t/\hbar} b_m(t),$$

(2.10)

which defines a slightly different expansion coefficient, we can simplify considerably. Notice that $|b_k(t)|^2 = |c_k(t)|^2$. Also, $b_k(0) = c_k(0)$. In practice what we are doing is pulling out the “trivial” part of the time-evolution, the time-evolving phase factor for state $m$. The reasons will become clear later when we discuss the interaction picture. It is easy to calculate $b_k(t)$ and then add in the extra oscillatory term at the end. Now eq. (2.9) becomes

$$e^{-iE_k t/\hbar} \frac{\partial b_k}{\partial t} = -\frac{i}{\hbar} \sum_n V_{kn}(t) e^{-iE_k t/\hbar} b_n(t)$$

(2.11)

or

$$i \hbar \frac{\partial b_k}{\partial t} = \sum_n V_{kn}(t) e^{-i\omega n t} b_n(t)$$

(2.12)

This equation is an exact solution. It is a set of coupled differential equations that describe how probability amplitude moves through eigenstates due to a time-dependent potential. Except in simple cases, these equations can’t be solved analytically, but it’s often straightforward to integrate numerically.
Resonant Driving of Two-level System

As an example of the use of these equations, let’s describe what happens when you drive a two-level system with an oscillating potential.

\[
V(t) = V \cos \omega t = V_f(t) \tag{2.13}
\]

Note: This is what you expect for an electromagnetic field interacting with charged particles, i.e. dipole transitions. In a simple sense, the electric field is

\[
\overrightarrow{E}(t) = \overrightarrow{E}_0 \cos \omega t \tag{2.14}
\]

For a particle with charge \( q \) in a field \( \overrightarrow{E} \), the force on the particle is

\[
\overrightarrow{F} = q \overrightarrow{E} \tag{2.15}
\]

which is the gradient of the potential

\[
F_x = -\frac{\partial V}{\partial x} = qE_x \quad \Rightarrow \quad V = -qE_x x \tag{2.16}
\]

\( qx \) is just the \( x \) component of the dipole moment \( \mu \). So matrix elements in \( V \) look like:

\[
\langle k | V(t) | \ell \rangle = -qE_x \langle k | x | \ell \rangle \cos \omega t \tag{2.17}
\]

More generally,

\[
V = -\overrightarrow{E} \cdot \overrightarrow{\mu} \tag{2.18}
\]

We’ll look at this a bit more carefully later.

\[
V(t) = V \cos \omega t = -\overrightarrow{E}_0 \cdot \overrightarrow{\mu} \cos \omega t
\]

\[
V_{k\ell}(t) = V_{k\ell} \cos \omega t = -\overrightarrow{E}_0 \cdot \overrightarrow{\mu}_{k\ell} \cos \omega t \tag{2.19}
\]

We will now couple our two states \( |k\rangle \) and \( |\ell\rangle \) with the oscillating field. Let’s ask if the system starts in \( |\ell\rangle \) what is the probability of finding it in \( |k\rangle \) at time \( t \)?
The system of differential equations that describe this situation are:

\[
\frac{i\hbar}{\partial t} b_k(t) = \sum_n b_n(t) V_{kn}(t) e^{-i\omega_n t} = \sum_n b_n(t) V_{kn} e^{-i\omega_n t} \times \frac{1}{2} \left( e^{-i\omega t} + e^{+i\omega t} \right) \tag{2.20}
\]

Or more explicitly

\[
\begin{align*}
\frac{i\hbar}{2} b_k & = \frac{1}{2} b_{kl} V_{kl} \left[ e^{i(\omega_{kl} - \omega)t} + e^{i(\omega_{kl} + \omega)t} \right] + \frac{1}{2} b_k V_{kl} e^{i\omega t} + e^{-i\omega t} \\
\frac{i\hbar}{2} b_{\ell} & = \frac{1}{2} b_{\ell k} V_{\ell k} \left[ e^{i(\omega_{\ell k} - \omega)t} + e^{i(\omega_{\ell k} + \omega)t} \right] + \frac{1}{2} b_{\ell} V_{\ell k} e^{i\omega t} + e^{-i\omega t}
\end{align*}
\tag{2.21}
\]

Two of these terms can be dropped since (for our case) the diagonal matrix elements \( V_{ii} = 0 \). We also make the \textbf{secular approximation} (rotating wave approximation) in which the nonresonant terms are dropped. When \( \omega_{kl} = \omega \), terms like \( e^{\pm i\omega t} \) or \( e^{i(\omega_{kl} + \omega)t} \) oscillate very rapidly (relative to \( |V_{kl}|^{-1} \)) and so don’t contribute much to change of \( c_n \). (Remember that \( \omega_{kl} \) is positive). So we have:

\[
\begin{align*}
\dot{b}_k & = -\frac{i}{2\hbar} b_{kl} V_{kl} e^{i(\omega_{kl} - \omega)t} \\
\dot{b}_{\ell} & = -\frac{i}{2\hbar} b_{\ell k} V_{\ell k} e^{-i(\omega_{\ell k} - \omega)t}
\end{align*}
\tag{2.22}
\]

Note that the coefficients are oscillating out of phase with one another.

Now if we differentiate eq. (2.22):

\[
\dot{b}_k = -\frac{i}{2\hbar} \left[ b_{kl} V_{kl} e^{i(\omega_{kl} - \omega)t} + i(\omega_{kl} - \omega) b_{\ell} V_{\ell k} e^{i(\omega_{\ell k} - \omega)t} \right] \tag{2.24}
\]

Rewrite eq. (2.22):
\[ b_k = \frac{2i\hbar}{V_{kl}} \hat{b}_k e^{-i(\omega_{kl} - \omega)t} \]  

(2.25)

and substitute (2.25) and (2.23) into (2.24), we get linear second order equation for \( b_k \).

\[ \ddot{b}_k - i(\omega_{kl} - \omega)\dot{b}_k + \frac{|V_{kl}|^2}{4\hbar^2} b_k = 0 \]  

(2.26)

This is just the second order differential equation for a damped harmonic oscillator:

\[ a\dddot{x} + b\ddot{x} + cx = 0 \]  

(2.27)

\[ x = e^{-i\sqrt{\frac{b}{2a}}t} \left( A\cos\mu t + B\sin\mu t \right) \]  

\[ \mu = \frac{1}{2a} \left[ 4ac - b^2 \right]^{1/2} \]  

(2.28)

With a little more work, and remembering the initial conditions \( b_k(0) = 0 \) and \( b_k(0) = 1 \), we find

\[ P_k = |b_k(t)|^2 = \frac{|V_{kl}|^2}{|V_{kl}|^2 + \hbar^2 (\omega_{kl} - \omega)^2} \sin^2 \Omega t \]  

(2.29)

Where the Rabi Frequency

\[ \Omega_R = \frac{1}{2\hbar} \left[ \left| V_{kl} \right|^2 + \hbar^2 (\omega_{kl} - \omega)^2 \right]^{1/2} \]  

(2.30)

Also,

\[ P_\ell = 1 - |b_k|^2 \]  

(2.31)

The amplitude oscillates back and forth between the two states at a frequency dictated by the coupling between them. [ Note a result we will return to later: Electric fields couple quantum states, creating coherences! ]

An important observation is the importance of resonance between the driving potential and the energy splitting between states. To get transfer of probability density you need the driving field to be at the same frequency as the energy splitting. On resonance, you always drive probability amplitude entirely from one state to another.
The efficiency of driving between $\ell$ and $k$ states drops off with detuning. Here plotting the maximum value of $P_{k\ell}$ as a function of frequency:

**Readings**

This lecture draws from


2.2. QUANTUM DYNAMICS

The motion of a particle is described by a complex wavefunction \( \psi(\vec{r},t) \) that gives the probability amplitude of finding a particle at point \( \vec{r} \) at time \( t \). If we know \( \psi(\vec{r},t_0) \), how does it change with time?

\[
\psi(\vec{r},t_0) \rightarrow \psi(\vec{r},t) \quad t > t_0 \quad (2.32)
\]

We will use our intuition here (largely based on correspondence to classical mechanics). We are seeking an equation of motion for quantum systems that is equivalent to Newton’s (or more accurately Hamilton’s) equations for classical systems.

We start by assuming causality: \( \psi(t_0) \) precedes and determines \( \psi(t) \). So will be deriving a deterministic equation of motion for \( \psi(\vec{r},t) \). Also, we assume time is a continuous parameter:

\[
\lim_{t \rightarrow t_0} \psi(t) = \psi(t_0) \quad (2.33)
\]

Define an operator that gives time-evolution of system.

\[
\psi(t) = U(t,t_0) \psi(t_0) \quad (2.34)
\]

This “time-displacement operator” or “propagator” is similar to the “space-displacement operator”

\[
\psi(r) = e^{ik(r-r_0)} \psi(r_0) \quad (2.35)
\]

which moves a wavefunction in space.
We also say that $U$ does not depend on the particular state of the system $|\psi\rangle$. This is necessary for conservation of probability, i.e. to retain normalization for the system. If

$$|\psi(t_0)\rangle = a_1|\varphi_1(t_0)\rangle + a_2|\varphi_2(t_0)\rangle$$

then

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$$

$$= U(t,t_0)a_1|\varphi_1(t_0)\rangle + U(t,t_0)a_2|\varphi_2(t_0)\rangle$$

$$= a_1|\varphi_1(t)\rangle + a_2|\varphi_2(t)\rangle$$

This is a reflection of the importance of linearity in quantum systems. While $|a_i(t)\rangle$ typically not equal to $|a_i(0)\rangle$,

$$\sum_n |a_n(t)|^2 = \sum_n |a_n(t_0)|^2$$

Properties of $U(t,t_0)$

1) **Unitary.** Note that for eq. (2.38) to hold and for probability density to be conserved, $U$ must be unitary

$$P = \langle \psi(t)|\psi(t)\rangle = \langle \psi(t_0)|U^\dagger U|\psi(t_0)\rangle$$

which holds only if $U^\dagger = U^{-1}$. In fact, this is the reason that equates unitary operators with probability conservation.

2) **Time continuity:**

$$U(t,t) = 1.$$  

(2.40)

3) **Composition property.** If we take the system to be deterministic, then it stands to reason that we should get the same wavefunction whether we evolve to a target time in one step $(t_0 \rightarrow t_2)$ or multiple steps $(t_0 \rightarrow t_1 \rightarrow t_2)$:
\[ U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0) \] (2.41)

Note, since \( U \) acts to the right, order matters:

\[
\begin{align*}
|\psi(t_2)\rangle &= U(t_2, t_1)U(t_1, t_0)|\psi(t_0)\rangle \\
&= U(t_2, t_1)|\psi(t_1)\rangle
\end{align*}
\] (2.42)

Equation (2.41) is already very suggestive of an exponential form. Furthermore, since time is continuous and the operator is linear it also suggests what we will see that the time propagator is only dependent on a time interval

\[ U(t_1, t_0) = U(t_1 - t_0) \] (2.43)

and

\[ U(t_2 - t_0) = U(t_2 - t_1)U(t_1 - t_0) \] (2.43)

4) **Time-reversal.** The inverse of the time-propagator is the time reversal operator. From eq. (2.41):

\[ U(t, t_0)U(t_0, t) = 1 \] (2.32)

\[ \therefore U^{-1}(t, t_0) = U(t_0, t). \] (2.33)

**Finding an equation of motion for \( U \)**

Let’s find an equation of motion that describes the time-evolution operator using the change of the system for an infinitesimal time-step, \( \delta t : U(t_0 + \delta t, t_0) \)

\[ \lim_{\delta t \to 0} U(t_0 + \delta t, t_0) = 1 \] (2.34)

We expect that for small \( \delta t \), the difference between \( U(t_0, t_0) \) and \( U(t_0 + \delta t, t_0) \) will be linear in \( \delta t \) (This is based on analogy to how we think of deterministic motion in classical systems)
We take $\hat{\Omega}$ to be a time-dependent Hermitian operator. We’ll see later why the second term must be imaginary. So, now we can write a differential equation for $U$. We know that

$$U(t + \delta t, t_0) = U(t, t_0) - i\hat{\Omega}(t_0)\delta t$$  \hspace{1cm} (2.35)$$

Knowing the change of $U$ during the period $\delta t$ allows us to write a differential equation for the time-development of $U(t, t_0)$. The equation of motion for $U$ is

$$\frac{d}{dt} U(t, t_0) = \lim_{\delta t \to 0} \frac{U(t + \delta t, t_0) - U(t, t_0)}{\delta t}$$

$$= \lim_{\delta t \to 0} \left[ U(t + \delta t, t) - 1 \right] U(t, t_0)$$

Where I have substituted eqn. (2.35) in the second step. So we have:

$$\frac{\partial U(t, t_0)}{\partial t} = -i\hat{\Omega} U(t, t_0)$$  \hspace{1cm} (2.38)$$

You can now see that the operator needed a complex argument, because otherwise probability density would not be conserved (it would rise or decay). Rather it oscillates through different states of the system.

We note that $\hat{\Omega}$ has units of frequency. Since (1) quantum mechanics says $E = \hbar \omega$ and (2) in classical mechanics the Hamiltonian generates time-evolution, we write

$$\hat{\Omega} = \frac{\hat{H}}{\hbar}$$  \hspace{1cm} (2.39)$$

Where $\hat{\Omega}$ can be a function of time. Then

$$i\hbar \frac{\partial}{\partial t} U(t, t_0) = \hat{H} U(t, t_0)$$  \hspace{1cm} (2.40)$$

Multiplying from right by $|\psi(t_0)\rangle$ gives the TDSE
\[
\frac{i\hbar}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \tag{2.41}
\]

We are also interested in the equation of motion for \( U^\dagger \) which describes the time-evolution of the conjugate wavefunctions. Following the same approach and recognizing that \( U^\dagger(t,t_0) \) acts to the left:

\[
\langle \psi(t) | = \langle \psi(t_0) | U^\dagger(t,t_0), \tag{2.42}
\]

we get

\[
-i\hbar \frac{\partial}{\partial t} U^\dagger(t,t_0) = U^\dagger(t,t_0) \hat{H} \tag{2.43}
\]

**Evaluating \( U(t,t_0) \): Time-independent Hamiltonian**

Direct integration of eqn. (2.40) suggests that \( U \) can be expressed as:

\[
U(t,t_0) = \exp \left[ -\frac{i}{\hbar} H(t-t_0) \right] \tag{2.44}
\]

Since \( H \) is an operator, we will define this operator through the expansion:

\[
\exp \left[ -\frac{iH}{\hbar} (t-t_0) \right] = 1 - \frac{iH}{\hbar} (t-t_0) + \left( \frac{iH}{\hbar} \right)^2 \frac{H(t-t_0)^2}{2} + \ldots \tag{2.45}
\]

Note \( H \) commutes at all \( t \). You can confirm the expansion satisfies the equation of motion for \( U \).

To evaluate \( U \) for the time-independent Hamiltonian, we expand in a set of eigenkets:

\[
H|n\rangle = E_n|n\rangle \quad \sum_n |n\rangle \langle n| = 1 \tag{2.46}
\]

So we have
\[
U(t,t_0) = \sum_n \exp\left[-iH(t-t_0)/\hbar\right]|n\rangle\langle n|
\]
\[
= \sum_n |n\rangle \exp\left[-iE_n(t-t_0)/\hbar\right]\langle n|
\]  

(2.47)

and

\[
|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle
\]
\[
= \sum_n |n\rangle \langle n|\psi(t_0)\rangle \exp\left[\frac{-i}{\hbar}E_n(t-t_0)\right]
\]
\[
= \sum_n \langle n| c_n(t)
\]  

(2.48)

Expectation values of operators are given by

\[
\langle A(t) \rangle = \langle \psi(t)|A|\psi(t)\rangle
\]
\[
= \langle \psi(t_0)|U^\dagger(t,t_0)AU(t,t_0)|\psi(t_0)\rangle
\]  

(2.49)

For an initial state \(|\psi(t_0)\rangle = \sum_n c_n|n\rangle\), we showed in eq. 1.48 that

\[
\langle A \rangle = \sum_{n,m} c_m^* \langle m|m\rangle e^{i\omega_m(t-t_0)} \langle m|A|n\rangle e^{-i\omega_n(t-t_0)} \langle n|n\rangle c_n
\]
\[
= \sum_{n,m} c_m^* c_n A_m n e^{-i\omega_{nm}(t-t_0)}
\]
\[
= \sum_{n,m} c_m^* (t)c_n (t) A_{mn}
\]  

(2.50)

which is Tr(\(\rho(t)A\)). The correlation amplitude \(\langle \beta|\psi(t)\rangle\) was given in eq. 1.45.
**Evaluating the time-evolution operator: Time-dependent Hamiltonian**

At first glance it may seem straightforward to deal with. If $H$ is a function of time, then the formal integration of $i\hbar \partial U / \partial t = HU$ gives

$$U(t,t_0) = \exp \left[ -i \int_{t_0}^{t} H(t') dt' \right]$$  \hspace{1cm} (2.51)

We would define this exponential as an expansion in a series, and substitute into the equation of motion to confirm it:

$$U(t,t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t} H(t') dt' + \frac{1}{2!} \left( \frac{-i}{\hbar} \right)^2 \int_{t_0}^{t} \int_{t_0}^{t'} dt' \int_{t_0}^{t''} dt'' H(t') H(t'') + ...$$  \hspace{1cm} (2.52)

Then if we know the eigenstates of $H$, we could use eq. (2.46) to express $U$ as

$$U(t,t_0) = \sum_{n} \psi_{n}(t) \exp \left[ -i \int_{t_0}^{t} E_{n}(t') dt' \right]$$  \hspace{1cm} (2.53)

However, this is dangerous; we are not treating $H$ as an operator. We are assuming that the Hamiltonians at different times commute! $[H(t'), H(t'')] = 0$. It is only the case for special Hamiltonians with a high degree of symmetry, in which the eigenstates have the same symmetry at all times. This holds for instance in the case of a degenerate system, i.e., spin $\frac{1}{2}$ system, with a time-dependent coupling. Generally speaking this is not the case.

Now, let’s proceed a bit more carefully assuming that the Hamiltonian at different times does not commute. Integrate

$$\frac{\partial}{\partial t} U(t,t_0) = -i \hbar^{-1} H(t) U(t,t_0)$$  \hspace{1cm} (2.54)

To give:

$$U(t,t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t} d\tau \ H(\tau) U(\tau,t_0)$$  \hspace{1cm} (2.55)

This is the solution; however, it’s not very practical since $U(t,t_0)$ is a function of itself. But we can solve by iteratively substituting $U$ into itself. First Step:
\[ U(t,t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t} d\tau H(\tau) \left[ 1 - \frac{i}{\hbar} \int_{t_0}^{\tau} d\tau' H(\tau') U(\tau', t_0) \right] \]
\[ = 1 + \left( \frac{-i}{\hbar} \right) \int_{t_0}^{t} d\tau H(\tau) + \left( \frac{-i}{\hbar} \right)^2 \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\tau' H(\tau) H(\tau') U(\tau', t_0) \]

(2.56)

Note in the last term of this equation, the integration variable \( \tau' \) precedes \( \tau \). Pictorially, the area of integration is

Next Step:

\[ U(t,t_0) = 1 + \left( \frac{-i}{\hbar} \right) \int_{t_0}^{t} d\tau H(\tau) \]
\[ + \left( \frac{-i}{\hbar} \right)^2 \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\tau' H(\tau) H(\tau') \]
\[ + \left( \frac{-i}{\hbar} \right)^3 \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\tau' \int_{t_0}^{\tau'} d\tau'' H(\tau) H(\tau') H(\tau'') U(\tau'', t_0) \]

(2.57)

From this expansion, you should be aware that there is a time-ordering to the interactions. For the third term, \( \tau'' \) acts before \( \tau' \), which acts before \( \tau : t_0 \leq \tau'' \leq \tau' \leq \tau \).

Imagine you are starting in state \( |\psi_0\rangle = |\ell\rangle \) and you are working toward a target state \( |\psi\rangle = |k\rangle \). The possible paths and associated time variables are:
The expression for $U$ describes all possible paths between initial and final state. Each of these paths interfere in ways dictated by the acquired phase of our eigenstates under the time-dependent Hamiltonian. The solution for $U$ obtained from this iterative substitution is known as the (positive) time-ordered exponential

$$U(t,t_0) \equiv \exp\left[ \frac{-i}{\hbar} \int_{t_0}^{t} d\tau \, H(\tau) \right]$$

$$= \hat{T} \exp\left[ \frac{-i}{\hbar} \int_{t_0}^{t} d\tau \, H(\tau) \right]$$

$$= 1 + \sum_{n=1}^{\infty} \left( \frac{-i}{\hbar} \right)^n \int_{t_0}^{t'} d\tau_n \int_{t_0}^{\tau_n} d\tau_n \ldots \int_{t_0}^{\tau_1} d\tau_1 \, H(\tau_n) H(\tau_{n-1}) \ldots H(\tau_1)$$

(\hat{T} is known as the Tyson time-ordering operator.) In this expression the time-ordering is:

$$t_0 \rightarrow \tau_1 \rightarrow \tau_2 \rightarrow \tau_3 \ldots \tau_n \rightarrow t$$

$$t_0 \rightarrow \ldots \tau'' \rightarrow \tau' \rightarrow \tau$$

(2.59)

So, this expression tells you about how a quantum system evolves over a given time interval, and it allows for any possible trajectory from an initial state to a final state through any number of intermediate states. Each term in the expansion accounts for more possible transitions between different intermediate quantum states during this trajectory.

Compare the time-ordered exponential with the traditional expansion of an exponential:

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{-i}{\hbar} \right)^n \int_{t_0}^{t'} d\tau_n \ldots \int_{t_0}^{\tau_1} d\tau_1 \, H(\tau_n) H(\tau_{n-1}) \ldots H(\tau_1)$$

(2.60)
Here the time-variables assume all values, and therefore all orderings for \( H(\tau_i) \) are calculated. The areas are normalized by the \( n! \) factor. (There are \( n! \) time-orderings of the \( \tau_n \) times.)

We are also interested in the Hermitian conjugate of \( U(t,t_0) \), which has the equation of motion in eq. (2.43)

\[
\frac{\partial}{\partial t} U^\dagger(t,t_0) = \frac{i}{\hbar} U^\dagger(t,t_0) H(t) \tag{2.61}
\]

If we repeat the method above, remembering that \( U^\dagger(t,t_0) \) acts to the left:

\[
\langle \psi(t) \rangle = \langle \psi(t_0) | U^\dagger(t,t_0) \rangle \tag{2.62}
\]

then from

\[
U^\dagger(t,t_0) = U^\dagger(t_0,t_0) + \frac{i}{\hbar} \int_{t_0}^{t} d\tau U^\dagger(t,\tau) H(\tau) \tag{2.63}
\]

we obtain a negative-time-ordered exponential:

\[
U^\dagger(t,t_0) = \exp\left[ \frac{i}{\hbar} \int_{t_0}^{t} d\tau H(\tau) \right] = 1 + \sum_{n=1}^{\infty} \left( \frac{i}{\hbar} \right)^n \int_{t_0}^{t} d\tau_n \int_{t_0}^{\tau_n} d\tau_{n-1} \ldots \int_{t_0}^{\tau_2} d\tau_1 H(\tau_1) H(\tau_2) \ldots H(\tau_n) \tag{2.64}
\]

Here the \( H(\tau_i) \) act to the left.

**Readings**

This lecture draws from the following: