A Note on Nash Bargaining with On-the-job Search

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August 25, 2003

1 Motivation

There appears to be some confusion as to the nature of the Nash bargaining solution in models with on-the-job search.[1] Nash posited four axioms that a bargaining solution should have. Applying those axioms to a bilateral bargaining problem between a worker and a firm, this implies the pair should choose a wage to maximize the Nash product,

$$(E(w) - U)(J(w) - V),$$

where $E(w)$ is the value of the worker if matched at a wage w, $J(w)$ is the value of the firm if matched at that wage, U is the worker's threat point (unemployment) and V is the firm's threat point (vacancy). If the derivative of E and J with respect to w is the same, as is the case in many search models without on-the-job search and with risk-neutral workers, then maximization of the Nash product leads to the first order condition

$$E(w) - U = J(w) - V.$$

The purpose of this note is to argue that introducing on-the-job search breaks that equivalence. Imposing the (incorrect) first order condition may lead to the possibility of a Pareto improvement, i.e. an increase in both $E(w)$ and $J(w)$. This is because a firm may prefer to pay a higher wage, raising $E(w)$, that reduces worker turnover, raising $J(w)$ (but by less than the increase in $E(w)$). Even if the Nash bargaining solution is not Pareto superior to the linear sharing rule, the two are not always equivalent.

2 Model

I establish this in a particular economy, but it should be clear that the example extends easily to many other scenarios.[2] The economy is composed of two types of agents, risk-

[1] A noteworthy example is Pissarides's (2000) *Equilibrium Unemployment Theory.* See equation (4.23).

[2] The assumption in Pissarides (2000) that all new jobs start with productivity 1 and then productivity follows a stochastic process in which future values of $p < 1$ would simplify the analysis considerably since all searchers would quit at any opportunity.

neutral workers and risk-neutral firms, both infinitely lived with discount rate r. Workers get z while unemployed and receive an endogenous wage w while employed. In addition, an employed worker may choose to pay a search cost σ which allows her to search with the same efficiency as an unemployed worker. A firm produces a stochastic amount of output p when employing a worker and pays the bargained wage w. This is a model with ex post heterogeneity: when a worker and firm meet, they realize their productivity p, drawn from a known cumulative distribution F with support $[0, \bar{p}]$. This realization is independent across workers and firms.

A searching worker, employed or unemployed, finds a job at rate λ. When an employed worker searches and meets another firm, she realizes the productivity p and then must reject one job and bargain with the other employer. Because of the preceding assumption, the worker's threat point is always unemployment, with value U. The firms' threat point is to end the job, with value 0. Jobs end exogenously at rate δ.

I assume that search effort is unobservable, and hence a worker and a firm only bargain over the wage. However, the pair recognizes the dependence of search effort on the wage, and this has a significant effect on the bargaining solution.

I look for an equilibrium in which a worker in a match with productivity p receives a wage $W(p)$, nondecreasing in p, and strictly increasing over the range for which workers search on the job. As a result, searching workers always move to more productive jobs. I conjecture that there is no other equilibrium.

3 Bellman Equations

Let $E(p, w)$ denote the expected present value of income for a worker employed in a job with productivity p and paid a wage w:

$$\begin{aligned} rE(p,w) &= w + \delta(U - E(p,w)) \\ &\qquad + \max\left\langle -\sigma + \lambda \int \max\left\langle E(p', W(p')) - E(p,w), 0 \right\rangle dF(p')\ ,\ 0 \right\rangle. \end{aligned}$$

The outer maximization reflects the binary search decision. The inner maximization reflects the decision to accept the alternative job. It should be clear that $E(p, w)$ is independent of productivity but is an increasing function of the bargained wage w. This implies there is an endogenous threshold w^* such that the worker searches if $w < w^*$ and does not search if $w \geq w^*$. This allows me to rewrite the worker's Bellman equation more compactly as

$$rE(p,w) = \begin{cases} w + \delta(U - E(p,w)) & \text{if } w \geq w^* \\ w + \delta(U - E(p,w)) - \sigma & \\ \qquad + \lambda \int_{W^{-1}(w)}^{\bar{p}} (E(p', W(p')) - E(p,w)) dF(p') & \text{if } w < w^* \end{cases}, \quad (1)$$

where I use the assumption that $w = W(p)$ is strictly increasing and hence invertible when $w < w^*$. The lower bound in the integral reflects the fact that by varying the wage, the firm can affect the turnover rate even if the worker searches. This implies that, even though $E(p, w)$ is linear in w when $w \geq w^*$, it is non-linear (actually convex) at smaller values of w.

The threshold w^* is defined by the condition that the worker prefers to search at any lower wage and weakly prefers not to search at a higher wage:

$$\sigma < \lambda \int_{W^{-1}(w)}^{\bar{p}} (E(p', W(p')) - E(p, w))dF(p') \tag{2}$$

if and only if $w < w^*$.

Using this same logic, the value of a firm is

$$rJ(p, w) = \begin{cases} p - w - \delta J(p, w) & \text{if } w \geq w^* \\ p - w - \Big(\delta + \lambda\big(1 - F(W^{-1}(w))\big)\Big) J(p, w) & \text{if } w < w^* \end{cases} . \tag{3}$$

Note that as long as $F(W^{-1}(w^*)) < 1$, the firm's value drops discontinuously as the wage falls below w^*, whereas the worker's value is continuous. The firm's value function is also linear in $w \geq w^*$ but globally convex.

Finally, I follow much of the search literature and relax Nash's symmetry axiom. The asymmetric Nash bargaining solution requires that the wage maximize a weighted Nash product,

$$W(p) = \arg\max_w (E(p, w) - U)^\beta J(p, w)^{1-\beta}, \tag{4}$$

where $\beta \in (0, 1)$ is the worker's 'bargaining power'. It is important to note that the results presented below do not hold in the two extremes, $\beta = 0$ or $\beta = 1$. In the former case the worker always gets paid her reservation wage and so never incurs a positive search cost. In the latter case, the worker receives her full productivity and so correctly internalizes the search decision.

I do not endogenize the value of an unemployed worker, but it is clear how to do this. In particular, by manipulating the flow payoff to an unemployed worker z, it is possible to obtain any desired value for U.

4 Equilibrium

I claim that there are three intervals of p that characterize the solution to this problem:

- If $p \geq p_2$, the firm pays a wage of $W(p) = \beta p + (1 - \beta)rU$ and the worker chooses not to engage in on-the-job search. Moreover, in this parameter range the Nash bargaining solution is equivalent to the linear version of the first order condition, $(1 - \beta)\big(E(p, W(p)) - U\big) = \beta J(p, W(p))$.

- If $p_2 > p > p_1$, the firm pays a wage of $W(p_2) = \beta p_2 + (1 - \beta)rU \equiv w^*$. This is like an efficiency wage, in the sense that it induces the worker not to search, but at the cost of reducing the firm's surplus below the workers, $(1-\beta)\big(E(p,W(p))-U\big) > \beta J(p,W(p))$. Note that in this region, the worker is indifferent about searching on the job, but in equilibrium chooses not to search.

- If $p_1 > p$, the worker searches on-the-job. The surplus is again divided according to a linear version of the first order condition, $(1-\beta)\big(E(p,W(p))-U\big) = \beta J(p,W(p))$. While in the middle region the worker received a high wage as an inducement not to search, in this region the worker receives a lower wage than she would if she could somehow commit not to search on-the-job (conditional on the value of unemployment).

I prove this by starting with the highest productivity matches, $p \geq p_2$ and working down to less productive matches. Substituting the Bellman equations (1) and (3) into the Nash bargaining solution (4) gives

$$W(p) = \arg\max_{w} \left(\frac{w-rU}{r+\delta}\right)^{\beta}\left(\frac{p-w}{r+\delta}\right)^{1-\beta}.$$

Taking logs of the objective function and differentiating gives the necessary and sufficient first order condition, which can be manipulated to get

$$W(p) = \beta p + (1-\beta)rU.$$

Substituting this back into the Bellman equations (1) and (3) gives

$$\begin{aligned}(r+\delta)\big(E(p,W(p))-U\big) &= \beta(p-rU)\\ (r+\delta)J(p,W(p)) &= (1-\beta)(p-rU),\end{aligned} \tag{5}$$

consistent with the standard linear relationship between the worker's and firm's thresholds.

Next I solve for the threshold p_2. This is defined by the point where a worker who is paid $W(p_2)$ is indifferent about searching, and may be expressed using (2) as

$$\sigma = \lambda \int_{p_2}^{\bar{p}} \big(E(p',W(p')) - E(p_2,W(p_2))\big) dF(p') = \frac{\lambda\beta}{r+\delta}\int_{p_2}^{\bar{p}} (p'-p_2)dF(p'). \tag{6}$$

This uniquely defines p_2 as a decreasing function of σ. For the model to be interesting, I assume that $p_2 > rU$, so workers are willing to accept jobs at this threshold. For parameter values that failed this condition, perhaps because σ is too large, there would be no on-the-job search. In addition, the on-the-job search threshold satisfies $w^* = W(p_2)$. A slight drop in the wage below w^* does not hurt the worker by much — she simply begins to search for a job — but results in a discrete loss of surplus to the firm. This suggests that for values of p slightly below p_2, the Nash product is maximized by paying an 'efficiency' wage $W(p_2)$.

Formally, I claim that when $p \in (p_1, p_2)$, a non-empty interval, $W(p) = \beta p_2 + rU$, so the firm absorbs all the cost of lower productivity. Since this induces the worker not to search, the Bellman equations (1) and (3) satisfy

$$\begin{aligned}(r+\delta)\big(E(p,W(p))-U\big) &= \beta(p_2-rU)\\ (r+\delta)J(p,W(p)) &= p-\beta p_2-(1-\beta)rU.\end{aligned}$$

One can easily show that further wage increases reduce the Nash product, while a slight wage reduction leads to a jump down in the Nash product because of the discrete loss suffered by the firm. Note also that since $p < p_2$, $(1-\beta)\big(E(p, W(p)) - U\big) > \beta J(p, W(p))$.

To prove that the Nash bargaining solution leads to an efficiency wage, I compare the Nash product under the efficiency wage,

$$\frac{\big(\beta(p_2 - rU)\big)^{\beta}\big(p_1 - \beta p_2 - (1-\beta)rU\big)^{1-\beta}}{r+\delta}, \tag{7}$$

to the Nash product under a 'searching wage', a lower wage that induces the worker to search and that satisfies the first order condition from the maximization of the Nash product. More precisely, I characterize the productivity level p_1 at which the Nash product is bimodal with the two local maxima taking on a common value. At higher values, $p \in (p_1, p_2)$, the Nash product is maximized by paying the efficiency wage, while at lower values, $p < p_1$, it is maximized by paying the searching wage.

Since $p_1 \leq p_2$, the Bellman equation (1) for a searching worker with productivity p_1 reduces to

$$rE(p_1, w) = w + \delta(U - E(p_1, w)) - \sigma + \lambda \int_{p_1}^{\bar{p}} (E(p', W(p')) - E(p_1, w))dF(p'),$$

since the worker quits for any job with $p' > p_1$. Now replace the search cost σ using the indifference condition (6) and simplify:

$$\begin{aligned} rE(p_1, w) &= w + \delta(U - E(p_1, w)) - \lambda \int_{p_2}^{\bar{p}} \big(E(p', W(p')) - E(p_2, W(p_2))\big)dF(p') \\ &\quad + \lambda \int_{p_1}^{\bar{p}} (E(p', W(p')) - E(p_1, w))dF(p'), \\ &= w + \delta(U - E(p_1, w)) + \lambda \int_{p_2}^{\bar{p}} \big(E(p_2, W(p_2)) - E(p_1, w)\big)dF(p') \\ &\quad + \lambda \int_{p_1}^{p_2} (E(p', W(p')) - E(p_1, w))dF(p') \\ &= w + \delta(U - E(p_1, w)) + \lambda\big(1 - F(p_1)\big)\big(E(p_2, W(p_2)) - E(p_1, w)\big). \end{aligned}$$

The second equality breaks the second integral into two parts, while the third equality uses $W(p') = W(p_2)$ for $p' \in (p_1, p_2)$ and then solves the integral. Equivalently, using the Bellman equation (5) for $E(p_2, W(p_2))$ given above,

$$\begin{aligned} \big(r + \delta + \lambda(1 - F(p_1))\big)(E(p_1, w) - U) &= w - rU + \lambda\big(1 - F(p_1)\big)\big(E(p_2, W(p_2)) - U\big) \\ &= w - rU + \frac{\lambda\beta}{r+\delta}\big(1 - F(p_1)\big)\big(p_2 - rU\big) \end{aligned}$$

The Bellman equation for a filled job with productivity p_1 can be derived directly from equation (3):

$$\big(r + \delta + \lambda(1 - F(p_1))\big)J(p_1, w) = p_1 - w,$$

since the worker quits for any job with $p' > p_1$. Then the Nash bargaining solution with a searching wage at productivity p_1 is

$$W(p_1) = \arg\max_{w<w^*} \frac{\left(w - rU + \frac{\lambda\beta}{r+\delta}\big(1-F(p_1)\big)\big(p_2 - rU\big)\right)^{\beta}\left(p_1 - w\right)^{1-\beta}}{r+\delta+\lambda\big(1-F(p_1)\big)},$$

which gives the necessary and sufficient first order condition

$$W(p_1) = \beta p_1 + (1-\beta)\left(rU - \frac{\lambda\beta}{r+\delta}\big(1-F(p_1)\big)\big(p_2 - rU\big)\right).$$

Note that this is lower than the wage the worker would get if she could not search, $\beta p_1 + (1-\beta)rU$. The value of the Nash product at the highest searching wage is then given by

$$\beta^{\beta}(1-\beta)^{1-\beta}\left(\frac{p_1 - rU + \frac{\lambda\beta}{r+\delta}\big(1-F(p_1)\big)\big(p_2 - rU\big)}{r+\delta+\lambda\big(1-F(p_1)\big)}\right). \tag{8}$$

When $p_1 = p_2$, the two expressions for the Nash wage (7) and (8) are equal. It is easy to verify that $p_1 < p_2$ when $\beta < 1$, i.e. when the worker cannot unilaterally set the wage. This follows because (7) strictly exceeds (8) at $p_1 = p_2$. Conversely, when $p_1 = \beta p_2 + (1-\beta)rU$,[3] (7) evaluates to zero while (8) is strictly positive. By continuity, the crossing-point, where (7) is equal to (8), lies somewhere in between these two values, which determines the equilibrium value of $p_1 \in (\beta p_2 + (1-\beta)rU, p_2)$.[4]

Finally, for values of $p \le p_1$, the worker searches while employed. The Bellman equations (1) and (3) are

$$\big(r+\delta+\lambda(1-F(p))\big)(E(p,w)-U) = w - rU - \sigma + \lambda\int_p^{\bar{p}}(E(p',W(p'))-U)dF(p')$$
$$\big(r+\delta+\lambda(1-F(p))\big)J(p,w) = p - w.$$

The Nash bargaining solution (4) then gives

$$W(p) = \beta p + (1-\beta)\left(rU + \sigma - \lambda\int_p^{\bar{p}}(E(p',W(p'))-U)dF(p')\right),$$

which is again a lower wage than the worker would get in the absence of on-the-job search. Substituting this back into the Bellman equations, we get

$$(1-\beta)(E(p,W(p))-U) = \beta J(p,W(p)),$$

the usual linear sharing rule.

[3] Recall that $p_2 > rU$ or there would be no on-the-job search in equilibrium.

[4] I have not yet proven that there is a unique such p_1.

The Nash bargaining assumption has some surprising implications in this framework. For example, in the Burdett-Mortensen model, firms optimally post wages and so, by definition, can never gain from posting a different wage. A simple revealed preference argument then establishes that more productive firms earn more profits. That is no longer true in the bargaining model; a job producing slightly more than p_1 yields less profit than a job producing slightly less. The proof is straightforward. By construction, the value of the Nash product is equal at the efficiency and searching wages when productivity equals p_1. But the wage increases discontinuously when productivity rises from below p_1 to above p_1, and so must the workers' surplus. For the Nash product to be the same at the two modes, it follows that firms' surplus must fall discontinuously. If a firm could somehow unilaterally reduce the productivity of the match in order to raise its value, this could make it better off (but of course would hurt workers).

The final, and perhaps most important, point is that a naïve imposition of the linear sharing rule, as in Pissarides (2000), precludes some possible Pareto improvements, a clear violation of Nash's efficiency axiom. Take $p \in (p_1, p_2)$, so the Nash bargaining solution requires a wage of $W(p_2)$. This gives the worker and firm surpluses respectively equal to

$$\begin{aligned} E(p, W(p)) - U &= \frac{\beta(p_2 - rU)}{r + \delta} \\ J(p, W(p)) &= \frac{p - \beta p_2 - (1 - \beta) rU}{r + \delta}. \end{aligned}$$

If instead a worker-firm pair unilaterally agreed to a wage that satisfies the linear sharing rule, the arguments used to derive p_1 ensure that surpluses would equal

$$\begin{aligned} E(p, w) - U &= \frac{\beta \left(p - rU + \frac{\lambda\beta}{r+\delta}\big(1 - F(p_1)\big)(p_2 - rU)\right)}{r + \delta + \lambda(1 - F(p_1))} \\ J(p, w) &= \frac{(1 - \beta) \left(p - rU + \frac{\lambda\beta}{r+\delta}\big(1 - F(p_1)\big)(p_2 - rU)\right)}{r + \delta + \lambda(1 - F(p_1))}, \end{aligned}$$

since the worker would search and accept any job producing at least p_1, i.e. paying at least w^*. Clearly the linear sharing rule makes the worker worse off since it results in a lower wage; this can be confirmed algebraically. The firm faces a tradeoff between lower wage payments and less turnover, but a little algebra shows that the linear sharing rule reduces firm profits if and only if

$$p_2 - p < \frac{\lambda(1 - \beta)^2 \big(1 - F(p_1)\big)(p_2 - rU)}{\beta(r + \delta) + \lambda(1 - F(p_1))}.$$

This defines another critical threshold $p^* \in (p_1, p_2)$,[5] where the Nash bargaining solution is Pareto superior to the linear sharing rule if $p \in [p^*, p_2)$ and not comparable using the Pareto criterion if $p \in (p_1, p^*)$.

[5]It is straightforward to show from this equation that $p^* < p_2$. I have already proved that $p^* > p_1$ in the previous paragraph: at p_1, the firm makes strictly less profit from the efficiency wage than from the searching wage.