## Dual Reduction and Elementary Games with Senders and Receivers

See GEB 21:183-202 (1997): https://doi.org/10.1006/game.1997.0573
https://home.uchicago.edu/~rmyerson/research/eldual2023notes.pdf
A finite senders-receivers game is any $\Gamma=\left(\mathrm{I},\left(\mathrm{T}_{\mathrm{i}}\right)_{\mathrm{i} \in \mathrm{I}}, \mathrm{p}, \mathrm{J},\left(\mathrm{C}_{\mathrm{j}}\right)_{\mathrm{j} \in \mathrm{J}},\left(\mathrm{u}_{\mathrm{k}}\right)_{\mathrm{k} \in \mathrm{IUJ}}\right)$
with nonempty finite sets: $\mathrm{I}=\{$ senders $\}, \mathrm{T}_{\mathrm{i}}=\{$ i's types $\}, \mathrm{J}=\{$ receivers $\}, \mathrm{C}_{\mathrm{j}}=\{\mathrm{j}$ 's actions $\}$,
$\mathrm{I} \cap \mathrm{J}=\varnothing, \mathrm{T}=x_{\mathrm{i} \in \mathrm{I}} \mathrm{T}_{\mathrm{i}}, \mathrm{C}=x_{\mathrm{j} \in \mathrm{J}} \mathrm{C}_{\mathrm{j}}, \mathrm{p} \in \Delta(\mathrm{T})$ probabilities, $\mathrm{u}_{\mathrm{k}}: \mathrm{C} \times \mathrm{T} \rightarrow \mathbb{R}$ utility payoffs.
We may write $\mathrm{c}=\left(\mathrm{c}_{\mathrm{j}}\right)_{\mathrm{j} \in \mathrm{J}}=\left(\mathrm{c}_{-\mathrm{j}}, \mathrm{c}_{\mathrm{j}}\right) \in \mathrm{C}, \mathrm{t}=\left(\mathrm{t}_{\mathrm{i}}\right)_{\mathrm{i} \in \mathrm{I}}=\left(\mathrm{t}_{-\mathrm{i}}, \mathrm{t}_{\mathrm{i}}\right) \in \mathrm{T}$.
A direct coordination mechanism is any $\mu: \mathrm{T} \rightarrow \Delta(\mathrm{C})$.
Let $\mathrm{U}_{\mathrm{j}}\left(\mu, \mathrm{c}_{\mathrm{j}}\right)=\sum_{\mathrm{t}} \mathrm{p}(\mathrm{t}) \sum_{\mathrm{c}-\mathrm{j}} \mu(\mathrm{c} \mid \mathrm{t}) \mathrm{u}_{\mathrm{j}}(\mathrm{c}, \mathrm{t}), \hat{\mathrm{U}}_{\mathrm{j}}\left(\mu, \mathrm{d}_{\mathrm{j}}, \mathrm{c}_{\mathrm{j}}\right)=\sum_{\mathrm{t}} \mathrm{p}(\mathrm{t}) \sum_{\mathrm{c}-\mathrm{j}} \mu(\mathrm{c} \mid \mathrm{t}) \mathrm{u}_{\mathrm{j}}\left(\left(\mathrm{c}_{-\mathrm{j}}, \mathrm{d}_{\mathrm{j}}\right), \mathrm{t}\right)$,
$\mathrm{U}_{\mathrm{i}}\left(\mu, \mathrm{t}_{\mathrm{i}}\right)=\sum_{\mathrm{t}-\mathrm{i}} \mathrm{p}(\mathrm{t}) \sum_{\mathrm{c}} \mu(\mathrm{c} \mid \mathrm{t}) \mathrm{u}_{\mathrm{i}}(\mathrm{c}, \mathrm{t}), \hat{\mathrm{U}}_{\mathrm{i}}\left(\mu, \mathrm{s}_{\mathrm{i}}, \mathrm{t}_{\mathrm{i}}\right)=\sum_{\mathrm{t}-\mathrm{i}} \mathrm{p}(\mathrm{t}) \sum_{\mathrm{c}} \mu\left(\mathrm{c} \mid \mathrm{t}_{\mathrm{i}}, \mathrm{s}_{\mathrm{i}}\right) \mathrm{u}_{\mathrm{i}}(\mathrm{c}, \mathrm{t})$. [proby-discounted]
An incentive-compatible (IC) mechanism is any $\mu$ satisfying:

$$
\begin{array}{ll}
\mu(\mathrm{c} \mid \mathrm{t}) \geq 0 \quad \forall \mathrm{c} \in \mathrm{C}, \forall \mathrm{t} \in \mathrm{~T} ; \quad \sum_{\mathrm{c} \in \mathrm{C}} \mu(\mathrm{c} \mid \mathrm{t})=1 \quad \forall \mathrm{t} \in \mathrm{~T} ; \quad \text { [probability constraints] } \\
\mathrm{U}_{\mathrm{j}}\left(\mu, \mathrm{c}_{\mathrm{j}}\right) \geq \hat{\mathrm{U}}_{\mathrm{j}}\left(\mu, \mathrm{~d}_{\mathrm{j}}, \mathrm{c}_{\mathrm{j}}\right) \forall \mathrm{c}_{\mathrm{j}} \in \mathrm{C}_{\mathrm{j}}, \forall \mathrm{~d}_{\mathrm{j}} \in \mathrm{C}_{\mathrm{j}}, \forall \mathrm{j} \in \mathrm{~J} ; \quad \text { [moral-hazard constraints] } \\
\mathrm{U}_{\mathrm{i}}\left(\mu, \mathrm{t}_{\mathrm{i}}\right) \geq \hat{\mathrm{U}}_{\mathrm{i}}\left(\mu, \mathrm{~s}_{\mathrm{i}}, \mathrm{t}_{\mathrm{i}}\right) \forall \mathrm{t}_{\mathrm{i}} \in \mathrm{~T}_{\mathrm{i}}, \forall \mathrm{~s}_{\mathrm{i}} \in \mathrm{~T}_{\mathrm{i}}, \forall \mathrm{i} \in \mathrm{I} . \quad \text { [adverse-selection constraints] }
\end{array}
$$

$\Gamma$ is an elementary game iff there exists some $\mu^{*}: T \rightarrow \Delta(\mathrm{C})$ such that

$$
\mathrm{U}_{\mathrm{j}}\left(\mu^{*}, \mathrm{c}_{\mathrm{j}}\right)>\hat{\mathrm{U}}_{\mathrm{j}}\left(\mu^{*}, \mathrm{~d}_{\mathrm{j}}, \mathrm{c}_{\mathrm{j}}\right) \forall \mathrm{c}_{\mathrm{j}}, \forall \mathrm{~d}_{\mathrm{j}} \neq \mathrm{c}_{\mathrm{j}}, \forall \mathrm{j} ; \text { and } \mathrm{U}_{\mathrm{i}}\left(\mu^{*}, \mathrm{t}_{\mathrm{i}}\right)>\hat{\mathrm{U}}_{\mathrm{i}}\left(\mu^{*}, \mathrm{~s}_{\mathrm{i}}, \mathrm{t}_{\mathrm{i}}\right) \forall \mathrm{t}_{\mathrm{i}}, \forall \mathrm{~s}_{\mathrm{i}} \neq \mathrm{t}_{\mathrm{i}}, \forall \mathrm{i} .
$$

Fact: If $\Gamma$ is elementary, then almost all IC mechanisms satisfy all nontrivial incentive constraints strictly. (If $\mu$ does not then ( $1-\varepsilon$ ) $\mu^{+} \varepsilon \mu^{*}$ does.)

Consider the following primal linear programming problem:
minimize $\sum_{\mathrm{i}} \sum_{\mathrm{ti}} \pi_{\mathrm{i}}\left(\mathrm{t}_{\mathrm{i}}\right)+\sum_{\mathrm{j}} \sum_{\mathrm{cj}} \pi_{\mathrm{j}}\left(\mathrm{c}_{\mathrm{j}}\right)$ over $\mu \geq \mathbf{0} \& \pi$ such that

$$
\begin{aligned}
& \pi_{\mathrm{j}}\left(\mathrm{c}_{\mathrm{j}}\right)+\sum_{\mathrm{t}} \sum_{\mathrm{c}-\mathrm{j}} \mathrm{p}(\mathrm{t}) \mu(\mathrm{c} \mid \mathrm{t})\left(\mathrm{u}_{\mathrm{j}}(\mathrm{c}, \mathrm{t})-\mathrm{u}_{\mathrm{j}}\left(\left(\mathrm{c}_{-\mathrm{j}}, \mathrm{~d}_{\mathrm{j}}\right), \mathrm{t}\right)\right) \geq 0 \quad \forall \mathrm{c}_{\mathrm{j}} \in \mathrm{C}_{\mathrm{j}}, \forall \mathrm{~d}_{\mathrm{j}} \in \mathrm{C}_{\mathrm{j}}, \forall \mathrm{j} \in \mathrm{~J} ; \quad\left[\alpha_{\mathrm{j}}\left(\mathrm{~d}_{\mathrm{j}} \mathrm{c}_{\mathrm{j}}\right)\right] \\
& \left.\pi_{\mathrm{i}} \mathrm{t}_{\mathrm{i}}\right)+\sum_{\mathrm{t}-\mathrm{i}} \sum_{\mathrm{c}} \mathrm{p}(\mathrm{t}) \mathrm{u}_{\mathrm{i}}(\mathrm{c}, \mathrm{t})\left(\mu(\mathrm{c} \mid \mathrm{t})-\mu\left(\mathrm{c} \mid \mathrm{t}_{\mathrm{-}, \mathrm{~s}, \mathrm{~s} i}\right)\right) \geq 0 \quad \forall \mathrm{t}_{\mathrm{i}} \in \mathrm{~T}_{\mathrm{i}}, \forall \mathrm{~s}_{\mathrm{i}} \in \mathrm{~T}_{\mathrm{i}}, \quad \forall \mathrm{i} \in \mathrm{I} ; \quad\left[\alpha_{\mathrm{i}}\left(\mathrm{~s}_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}\right)\right] \\
& \sum_{\mathrm{c}} \mu(\mathrm{c} \mid \mathrm{t})=1 \quad \forall \mathrm{t} \in \mathrm{~T} . \quad[\beta(\mathrm{t})]
\end{aligned}
$$

The solutions to this LP are the incentive-compatible mechanisms, which exist (Nash) and yield optimal value 0 with $\boldsymbol{\pi}=\mathbf{0}$. (Trivial constraints with $\mathrm{d}_{\mathrm{j}}=\mathrm{c}_{\mathrm{j}}$ and $\mathrm{s}_{\mathrm{i}}=\mathrm{t}_{\mathrm{i}}$ imply $\pi \geq \mathbf{0}$.)
The dual LP problem is: maximize $\sum_{\mathrm{t}} \beta(\mathrm{t})$ over $\alpha \geq \mathbf{0} \& \beta$ such that

$$
\begin{aligned}
& \beta(\mathrm{t})+\sum_{\mathrm{j}} \sum_{\mathrm{dj}} \alpha_{\mathrm{j}}\left(\mathrm{~d}_{\mathrm{j}} \mid \mathrm{c}_{\mathrm{j}}\right) \mathrm{p}(\mathrm{t})\left(\mathrm{u}_{\mathrm{j}}(\mathrm{c}, \mathrm{t})-\mathrm{u}_{\mathrm{j}}\left(\left(\mathrm{c}_{-\mathrm{j}}, \mathrm{~d}_{\mathrm{j}}\right), \mathrm{t}\right)\right)+ \\
& +\sum_{\mathrm{i}} \sum_{\mathrm{ti}}\left(\alpha_{\mathrm{i}}\left(\mathrm{~s}_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}\right) \mathrm{p}(\mathrm{t}) \mathrm{u}_{\mathrm{i}}(\mathrm{c}, \mathrm{t})-\alpha_{\mathrm{i}}\left(\mathrm{t}_{\mathrm{i}} \mid \mathrm{s}_{\mathrm{i}}\right) \mathrm{p}\left(\mathrm{t}_{-\mathrm{i}}, \mathrm{~s}_{\mathrm{i}}\right) \mathrm{u}_{\mathrm{i}}\left(\mathrm{c},\left(\mathrm{t}_{-\mathrm{i}}, \mathrm{~s}_{\mathrm{i}}\right)\right) \leq 0 \quad \forall \mathrm{t} \in \mathrm{~T}, \forall \mathrm{c} \in \mathrm{C} ; \quad[\mu(\mathrm{c} \mid \mathrm{t})]\right. \\
& \sum_{\mathrm{dj}} \alpha_{\mathrm{j}}\left(\mathrm{~d}_{\mathrm{j}} \mid \mathrm{c}_{\mathrm{j}}\right)=1 \quad \forall \mathrm{c}_{\mathrm{j}} \in \mathrm{C}_{\mathrm{j}}, \quad \forall \mathrm{j} \in \mathrm{~J} ; \quad\left[\tau_{\mathrm{j}}\left(\mathrm{c}_{\mathrm{j}}\right)\right] \\
& \sum_{\mathrm{si}} \alpha_{\mathrm{i}}\left(\mathrm{~s}_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}\right)=1 \quad \forall \mathrm{t}_{\mathrm{i}} \in \mathrm{~T}_{\mathrm{i}}, \quad \forall \mathrm{i} \in \mathrm{I} . \quad\left[\pi_{\mathrm{i}}\left(\mathrm{t}_{\mathrm{i}}\right)\right]
\end{aligned}
$$

Nontrivial dual solutions exist, with some $\alpha_{k}\left(\mathrm{e}_{\mathrm{k}} \mid \mathrm{f}_{\mathrm{k}}\right)>0$ \& $\mathrm{e}_{\mathrm{k}} \neq \mathrm{f}_{\mathrm{k}}$, iff $\Gamma$ is not elementary.
Let $\mathrm{D}(\mathrm{c}, \mathrm{t}, \alpha)=\sum_{\mathrm{j}} \sum_{\mathrm{dj}} \alpha_{\mathrm{j}}\left(\mathrm{d}_{\mathrm{j}} \mid \mathrm{c}_{\mathrm{j}}\right) \mathrm{p}(\mathrm{t})\left(\mathrm{u}_{\mathrm{j}}\left(\left(\mathrm{c}_{-\mathrm{j}}, \mathrm{d}_{\mathrm{j}}\right), \mathrm{t}\right)-\mathrm{u}_{\mathrm{j}}(\mathrm{c}, \mathrm{t})\right)+$

$$
+\sum_{\mathrm{i}} \sum_{\mathrm{si}}\left(\alpha_{\mathrm{i}}\left(\mathrm{t}_{\mathrm{i}} \mid \mathrm{s}_{\mathrm{i}}\right) \mathrm{p}\left(\mathrm{t}_{-\mathrm{i}}, \mathrm{~s}_{\mathrm{i}}\right) \mathrm{u}_{\mathrm{i}}\left(\mathrm{c},\left(\mathrm{t}_{-\mathrm{i}}, \mathrm{~s}_{\mathrm{i}}\right)\right)-\alpha_{\mathrm{i}}\left(\mathrm{~s}_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}\right) \mathrm{p}(\mathrm{t}) \mathrm{u}_{\mathrm{i}}(\mathrm{c}, \mathrm{t})\right) . \quad[\alpha \text {-deviatn value } @ \mathrm{c}, \mathrm{t}]
$$

Then the dual optimum has $\beta(\mathrm{t})=\min _{\mathrm{c} \in \mathrm{C}} \mathrm{D}(\mathrm{c}, \mathrm{t}, \alpha) \forall \mathrm{t} \in \mathrm{T}$, and $\sum_{\mathrm{t} \in \mathrm{T}} \beta(\mathrm{t})=0$.
Lemma: Given any dual solution $\alpha$, for any mechanism $\mu: T \rightarrow \Delta(\mathrm{C})$ :

$$
\begin{aligned}
& \sum_{\mathrm{j} \in \mathrm{~J}} \sum_{\mathrm{cj}} \sum_{\mathrm{dj}} \alpha_{\mathrm{j}}\left(\mathrm{~d}_{\mathrm{j}} \mathrm{c}_{\mathrm{j}}\right)\left(\hat{\mathrm{U}}_{\mathrm{j}}\left(\mu, \mathrm{~d}_{\mathrm{j}}, \mathrm{c}_{\mathrm{j}}\right)-\mathrm{U}_{\mathrm{j}}\left(\mu, \mathrm{c}_{\mathrm{j}}\right)\right)+\sum_{\mathrm{i} \in \mathrm{I}} \sum_{\mathrm{ti}} \sum_{\mathrm{si}} \alpha_{\mathrm{i}}\left(\mathrm{~s}_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}\right)\left(\hat{\mathrm{U}}_{\mathrm{i}}\left(\mu, \mathrm{~s}_{\mathrm{i}} \mathrm{t}_{\mathrm{i}}\right)-\mathrm{U}_{\mathrm{i}}\left(\mu, \mathrm{t}_{\mathrm{i}}\right)\right) \\
& \quad=\sum_{\mathrm{t} \in \mathrm{~T}} \sum_{\mathrm{c} \in \mathrm{C}} \mu(\mathrm{c} \mid \mathrm{t}) \mathrm{D}(\mathrm{c}, \mathrm{t}, \alpha) \geq \sum_{\mathrm{t} \in \mathrm{~T}} \beta(\mathrm{t})=0 .
\end{aligned}
$$

(Expected net gains of unilateral $\alpha$-deviations from $\mu$ must have a nonnegative sum.)

Given any Markov chain $\delta: \mathrm{X} \rightarrow \Delta(\mathrm{X})$, we let $\mathrm{X} / \delta$ denote the set of minimal nonempty
$\delta$-absorbing subsets of $\mathrm{X} .(\mathrm{Y} \subseteq \mathrm{X}$ is $\delta$-absorbing iff $\delta(\mathrm{x} \mid \mathrm{y})=0 \forall \mathrm{y} \in \mathrm{Y}, \forall \mathrm{x} \notin \mathrm{Y}$.
We define functions $\psi$ and $\varphi$ so that, for any absorbing set $\mathrm{R} \in \mathrm{X} / \delta$ and any $\mathrm{x} \in \mathrm{X}$ :
$\varphi(\mathrm{x} \mid \mathrm{R}, \delta)$ is the probability of x in the unique $\delta$-stationary distribution on $\boldsymbol{R}$, and
$\psi(\mathrm{R} \mid \mathrm{x}, \delta)$ is the probability of a $\delta$-stochastic process reaching $\boldsymbol{R}$ from $\boldsymbol{x}$. That is:

$$
\begin{aligned}
& \varphi(\mathrm{y} \mid \mathrm{R}, \delta)=\sum_{z \in \mathrm{R}} \varphi(\mathrm{z} \mid \mathrm{R}, \delta) \delta(\mathrm{y} \mid \mathrm{z}) \forall \mathrm{y} \in \mathrm{R} ; \sum_{\mathrm{y} \in \mathrm{R}} \varphi(\mathrm{y} \mid \mathrm{R}, \delta)=1 ; \varphi(\mathrm{z} \mid \mathrm{R}, \delta)=0 \text { if } \mathrm{z} \notin \mathrm{R} ; \\
& \psi(\mathrm{R} \mid \mathrm{y}, \delta)=1 \text { if } \mathrm{y} \in \mathrm{R} ; \psi(\mathrm{R} \mid \mathrm{y}, \delta)=0 \text { if } \mathrm{y} \in \mathrm{R} \in \mathrm{X} / \delta \& \hat{\mathrm{R}} \neq \mathrm{R} ; \\
& \quad \psi(\mathrm{R} \mid \mathrm{y}, \delta)=\sum_{z \in \mathrm{X}} \delta(\mathrm{z} \mid \mathrm{y}) \psi(\mathrm{R} \mid \mathrm{z}, \delta) \forall \mathrm{y} \in \mathrm{X} .
\end{aligned}
$$

Dual solutions $\alpha$ define Markov chains $\alpha_{i}: T_{i} \rightarrow \Delta\left(T_{i}\right)$ and $\alpha_{j}: \mathrm{C}_{j} \rightarrow \Delta\left(\mathrm{C}_{\mathrm{j}}\right)$.
Given any dual solution $\alpha$ for the game $\Gamma$, let us define the $\alpha$-reduced game:

$$
\begin{aligned}
& \Gamma / \alpha=\left(I,\left(T_{i} / \alpha_{i}\right)_{i \in I}, q, J,\left(C_{j} / \alpha_{j}\right)_{j \in J},\left(v_{k}\right)_{k \in I U J}\right) \text { where } \\
& q(\tau)=\sum_{t \in T} p(t)\left(\prod_{i \in I} \psi\left(\tau_{i} \mid t_{i}, \alpha_{i}\right)\right), \text { and } \\
& v_{k}(\gamma, \tau)=\sum_{t \in T} \sum_{c \in C} p(t)\left(\prod_{i \in I} \psi\left(\tau_{i} \mid t_{i}, \alpha_{i}\right)\right)\left(\prod_{j \in J} \varphi\left(c_{j} \mid \gamma_{j}, \alpha_{j}\right)\right) u_{k}(\mathrm{c}, \mathrm{t}) / q(\tau) .
\end{aligned}
$$

A mechanism $\lambda:\left(x_{i \in I} T_{i} / \alpha_{i}\right) \rightarrow \Delta\left(x_{j \in J} C_{j} / \alpha_{j}\right)$ for $\Gamma / \alpha$ induces a mechanism $\mu^{\lambda}$ on $\Gamma$ :

$$
\mu^{\lambda}(\mathrm{c} \mid \mathrm{t})=\sum_{\tau} \sum_{\gamma}\left(\prod_{i \in I} \psi\left(\tau_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}, \alpha_{\mathrm{i}}\right)\right) \lambda(\gamma \mid \tau)\left(\prod_{\mathrm{j} \in \mathrm{~J}} \varphi\left(\mathrm{c}_{\mathrm{j}} \mid \gamma_{\mathrm{j}}, \alpha_{\mathrm{j}}\right)\right) .
$$

Theorem: If $\alpha$ is a dual solution for $\Gamma$, then any incentive-compatible mechanism $\lambda$ for the $\alpha$-reduced game $\Gamma / \alpha$ induces an incentive-compatible mechanism $\mu^{\lambda}$ for $\Gamma$.

Fact: For any finite $\Gamma$, iterative dual reduction yields an elementary reduced game.

In the reduced game $\Gamma / \alpha$, sender i's information is reduced to the absorbing set in $T_{i} / \alpha_{i}$ that was reached by an $\alpha_{i}$ stochastic process from i's true type in $T_{i}$, and receiver $j$ 's choices are reduced to the $\alpha_{j}$-stationary distributions on absorbing sets in $C_{j} / \alpha_{j}$. To show that incentive-compatible mechanisms in $\Gamma / \alpha$ are still IC in $\Gamma$, we must show that, when all players are expected to act within this reduced strategic domain, one player could not gain by a deviation that uses his full information or choice set in $\Gamma$.

By the Lemma, for any mechanism in $\Gamma$, if each player considered deviating by $\alpha_{i}$ or $\alpha_{j}$, the sum of their expected gains from unilaterally deviating must be nonnegative.
But for players who are acting according to their reduced strategic options in $\Gamma / \alpha$, deviating by $\alpha_{i}$ or $\alpha_{j}$ would not change their probabilistic behavior (by construction).
Now suppose, contrary to the Theorem, that some j could expect to gain strictly by deviating to some $c_{j}$ in $C_{j}$ when $\gamma_{j}$ in $C_{j} / \alpha_{j}$ is the $\lambda$-recommended action for $j$.
Consider the scenario that differs from $\lambda$ only in that player j deviates to a uniform distribution over all of j 's best-responses in $\mathrm{C}_{\mathrm{j}}$ when $\gamma_{\mathrm{j}}$ is recommended.
Since $\alpha$ deviations from this scenario would not affect any other behavior, the Lemma implies that j cannot expect to lose by further deviating from this scenario by $\alpha_{\mathrm{j}}$ when $\gamma_{\mathrm{j}}$ is recommended.
So if $\mathrm{c}_{\mathrm{j}}$ is a best response to $\gamma_{\mathrm{j}}$ and $\alpha_{\mathrm{j}}\left(\mathrm{d}_{\mathrm{j}} \mid \mathrm{c}_{\mathrm{j}}\right)>0$ then $\mathrm{d}_{\mathrm{j}}$ is also a best response to $\gamma_{\mathrm{j}}$.
So ${ }^{j}$ 's best responses in $C_{j}$ are an $\alpha_{j}$-absorbing set, and so an $\alpha_{j}$-stationary option in $C_{j} / \alpha_{j}$ is also optimal for j , and (by $\lambda \mathrm{IC}$ ) it is $\gamma_{\mathrm{j}}$, contradicting the supposition above.

Now suppose, contrary to the Theorem, that some type $t_{i}$ of some sender i could expect to gain strictly by deviating from $\lambda$ by changing his report from $\tau_{i}$ to some other $\rho_{i}$.
Consider the scenario that differs from $\lambda$ only in that all types $t_{i}$ with $\psi\left(\tau_{i} \mid t_{i}, \alpha_{i}\right)>0$ which strictly prefer misreporting $\rho_{i}$ do so when $\tau_{i}$ would be correct in $\lambda$, all those which would strictly lose by misreporting stay with $\tau_{i}$, and the indifferent randomize equally.
Since $\alpha$ deviations from this scenario would not affect any other behavior, the Lemma implies that $i$ could not expect to lose by a further $\alpha_{i}$-deviation (each $t_{i}$ acting like another $s_{i}$ in this scenario with probability $\alpha_{i}\left(\mathrm{~s}_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}\right)$ ).
But an $\alpha_{i}\left(\mathrm{~s}_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}\right)>0$ probability of $\mathrm{t}_{\mathrm{i}}$ further imitating an $\mathrm{s}_{\mathrm{i}}$ with $\psi\left(\tau_{\mathrm{i}} \mid \mathrm{s}_{\mathrm{i}}, \alpha_{\mathrm{i}}\right)>0$ in this scenario would yield an expected strict loss when $t_{i}$ strictly prefers misreporting \& $s_{i}$ does not, or when $t_{i}$ strictly prefers to not misreport \& $\mathrm{s}_{\mathrm{i}}$ is willing to do so, and otherwise it would not make any difference for i .
So all of i's types in the $\alpha_{i}$ chain that leads into $\tau_{i}$ must be willing to misreport $\rho_{i}$, while some supposed types strictly prefer such misreporting.
But the incentive-compatibility of $\lambda$ in $\Gamma / \alpha$ implies that the probability-weighted sum of these types' expected payoffs cannot be higher than what they get by reporting $\tau_{\mathrm{i}}$, contradicting the supposition above.

This intrinsic alignment of preferences among i's types in $\mathrm{T}_{\mathrm{i}}$ that are pooled in a reduced-game type $\tau_{i}$ shows that the incentive-compatible mechanisms of the reduced game will not depend on the relative weighting of utility payoffs for these types.

Fact: Consider a dual solution $\alpha$ and $i \in I, \tau_{i} \in T_{i} / \alpha_{i}$, and $r_{i} \in \tau_{i}$. Let $\lambda$ and $\eta$ be any two mechanisms for $\Gamma / \alpha$ that do not depend on $i^{\prime}$ (reduced) type, and suppose that $\mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \mathrm{r}_{\mathrm{i}}\right) \geq \mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \mathrm{r}_{\mathrm{i}}\right)$. Then $\mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \mathrm{t}_{\mathrm{i}}\right) \geq \mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \mathrm{t}_{\mathrm{i}}\right)$ for every $\mathrm{t}_{\mathrm{i}}$ such that $\psi\left(\tau_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}, \alpha_{\mathrm{i}}\right)>0$.
Suppose, contrary to the Fact, that some type $\mathrm{s}_{\mathrm{i}}$ has $\psi\left(\tau_{\mathrm{i}} \mid \mathrm{s}_{\mathrm{i}}, \alpha_{\mathrm{i}}\right)>0 \& \mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \mathrm{s}_{\mathrm{i}}\right)>\mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \mathrm{s}_{\mathrm{i}}\right)$. Consider the scenario which coincides with $\mu^{\lambda}$ except that it switches to $\mu^{\eta}$ iff i's type $t_{i}$ satisfies $\mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \mathrm{t}_{\mathrm{i}}\right)>\mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \mathrm{t}_{\mathrm{i}}\right)$.
This scenario satisfies all the strategic restrictions of the reduced game for all players other than sender $i$. Thus, since $\alpha$ deviations from this scenario would not affect any other behavior, the Lemma implies that player i could not expect to lose by an $\alpha_{i}$-deviation (each $t_{i}$ acting like another $\hat{t}_{i}$ in this scenario with probability $\alpha_{i}\left(\hat{t}_{i} \mid t_{i}\right)$ ).
An $\alpha_{i}\left(\hat{\mathrm{t}}_{\mathrm{i}} \mid \mathrm{t}_{\mathrm{i}}\right)>0$ probability of $\mathrm{t}_{\mathrm{i}}$ imitating $\hat{\mathrm{t}}_{\mathrm{i}}$ can make a difference in this scenario only in two cases: (1) if $\hat{\mathrm{t}}_{\mathrm{i}}$ satisfies $\mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \hat{\mathrm{t}}_{\mathrm{i}}\right)>\mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \hat{\mathrm{t}}_{\mathrm{i}}\right)$ but $\mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \mathrm{t}_{\mathrm{i}}\right) \geq \mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \mathrm{t}_{\mathrm{i}}\right)$, or (2) if $\mathrm{t}_{\mathrm{i}}$ satisfies $\mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \mathrm{t}_{\mathrm{i}}\right)>\mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \mathrm{t}_{\mathrm{i}}\right)$ but $\mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \hat{\mathrm{t}}_{\mathrm{i}}\right) \geq \mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \hat{\mathrm{t}}_{\mathrm{i}}\right)$.
In case (1), the imitation cannot help player i in type $t_{i}$, as it would just substitute $\eta$ for the weakly preferred $\lambda$. In case (2), the imitation strictly hurts player $i$ in type $t_{i}$, as it substitutes $\lambda$ for the strictly preferred $\eta$.
But $\psi\left(\tau_{i} \mid \mathrm{s}_{\mathrm{i}}, \alpha_{\mathrm{i}}\right)>0$ implies the existence of a positive $\alpha_{\mathrm{i}}$-chain from $\mathrm{s}_{\mathrm{i}}$ to $\mathrm{r}_{\mathrm{i}} \in \tau_{\mathrm{i}}$, and so the strict-loss case (2) must happen at least once, contradicting the above supposition.
Corollary: If $\left\{\mathrm{r}_{\mathrm{i}}, \mathrm{s}_{\mathrm{i}}\right\} \subseteq \tau_{\mathrm{i}}$ then $\mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \mathrm{r}_{\mathrm{i}}\right) \geq \mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \mathrm{r}_{\mathrm{i}}\right)<\Rightarrow \mathrm{U}_{\mathrm{i}}\left(\mu^{\lambda}, \mathrm{s}_{\mathrm{i}}\right) \geq \mathrm{U}_{\mathrm{i}}\left(\mu^{\eta}, \mathrm{s}_{\mathrm{i}}\right)$.

## Examples with no senders (as in the 1997 paper):

| $\mathrm{C}_{1}: \backslash \mathrm{C}_{2}:$ | $c_{2}$ | $d_{2}$ |
| :--- | ---: | ---: |
| $c_{1}$ | 3,2 | 0,0 |
| $d_{1}$ | 0,0 | 2,3 |

All incentive constraints can be satisfied strictly with $\mu\left(c_{1}, c_{2}\right)=0.5=\mu\left(d_{1}, d_{2}\right)$. So this game is elementary, and it has no nontrivial dual solutions.

| $\mathrm{C}_{1}: \backslash \mathrm{C}_{2}:$ | $c_{2}$ | $d_{2}$ |
| :--- | ---: | ---: |
| $c_{1}$ | 5,5 | 0,5 |
| $d_{1}$ | 5,0 | 1,1 |

Dual solutions include $\alpha_{1}\left(d_{1} \mid c_{1}\right)=1, \alpha_{1}\left(c_{1} \mid d_{1}\right)=0, \alpha_{2}\left(d_{2} \mid c_{2}\right)=1, \alpha_{2}\left(c_{2} \mid d_{2}\right)=0$.
$\mathrm{C}_{\mathrm{i}} / \alpha_{\mathrm{i}}=\left\{\left\{d_{i}\right\}\right\}$. In the reduced game, the dominated actions $c_{1}$ and $c_{2}$ are eliminated.

| $\mathrm{C}_{1}: \backslash \mathrm{C}_{2}:$ | $c_{2}$ | $d_{2}$ |
| :--- | ---: | ---: |
| $c_{1}$ | 7,0 | 2,5 |
| $d_{1}$ | 4,3 | 6,1 |

Dual solutions include $\alpha_{1}\left(d_{1} \mid c_{1}\right)=1, \alpha_{1}\left(c_{1} \mid d_{1}\right)=0.4, \alpha_{2}\left(d_{2} \mid c_{2}\right)=0.6, \alpha_{2}\left(c_{2} \mid d_{2}\right)=0.8$, and the reduced game has one absorbing set of actions $\left\{c_{i}, d_{i}\right\}$ for each player i.
The $\alpha$-stationary strategies are the unique Nash equilibrium strategies:
$(2 / 7)\left[c_{1}\right]+(5 / 7)\left[d_{1}\right],(4 / 7)\left[c_{2}\right]+(3 / 7)\left[d_{2}\right]$.
The reduced game is $1 \times 1$ with the equilibrium payoffs (4.857, 2.143).

Example of a $3 \times 3$ game with no senders:

| $\mathrm{C}_{1}: \backslash \mathrm{C}_{2}:$ | $c_{2}$ | $d_{2}$ | $e_{2}$ |
| :--- | :---: | ---: | ---: |
| $c_{1}$ | 0,0 | 1,0 | 0,1 |
| $d_{l}$ | 0,1 | 0,0 | 1,0 |
| $e_{1}$ | 1,0 | 0,1 | 0,0 |

This game has a correlated equilibrium that randomizes uniformly over the six nondiagonal outcomes of the game, which satisfies strictly six incentive constraints and so implies that every dual solution must have:

$$
\alpha_{1}\left(e_{1} \mid c_{1}\right)=0, \alpha_{1}\left(c_{1} \mid d_{1}\right)=0, \alpha_{1}\left(d_{1} \mid e_{1}\right)=0, \alpha_{2}\left(e_{2} \mid c_{2}\right)=0, \alpha_{2}\left(c_{2} \mid d_{2}\right)=0, \alpha_{2}\left(d_{2} \mid e_{2}\right)=0
$$

But this game is not elementary. There is a dual solution with:

$$
\alpha_{1}\left(d_{1} \mid c_{1}\right)=1, \alpha_{1}\left(e_{1} \mid d_{1}\right)=1, \alpha_{1}\left(c_{1} \mid e_{1}\right)=1, \alpha_{2}\left(d_{2} \mid c_{2}\right)=1, \alpha_{1}\left(e_{2} \mid d_{2}\right)=1, \alpha_{2}\left(c_{2} \mid e_{2}\right)=1
$$

The dual reduction is the $1 \times 1$ game where each player $j$ 's only option is to randomize uniformly over $\left\{c_{j}, d_{j}, e_{j}\right\}$, yielding expected payoffs $(1 / 3,1 / 3)$.
Thus, dual reduction suggests that the correlated equilibrium that we described above may be imperfect in some sense.

A similar $3 \times 3$ example with one sender and one receiver:

| $\mathrm{p}:$ | $\mathrm{T}_{1}: \backslash \mathrm{C}_{2}:$ | $a_{2}$ | $b_{2}$ | $c_{2}$ |
| :--- | :--- | ---: | ---: | ---: |
| $1 / 3$ | $r_{1}$ | 0,0 | 1,0 | 0,1 |
| $1 / 3$ | $s_{1}$ | 0,1 | 0,0 | 1,0 |
| $1 / 3$ | $t_{1}$ |  | 1,0 | 0,1 |

An IC $\mu$ has $\mu\left(b_{2} \mid r_{1}\right)=\mu\left(c_{2} \mid r_{1}\right)=0.5, \mu\left(a_{2} \mid s_{1}\right)=\mu\left(c_{2} \mid s_{1}\right)=0.5, \mu\left(a_{2} \mid t_{1}\right)=\mu\left(b_{2} \mid t_{1}\right)=0.5$. Its six strictly satisfied constraints imply that any dual solution has

$$
\alpha_{1}\left(s_{1} \mid r_{1}\right)=\alpha_{1}\left(t_{1} \mid s_{1}\right)=\alpha_{1}\left(r_{1} \mid t_{1}\right)=0 \text { and } \alpha_{2}\left(c_{2} \mid a_{2}\right)=\alpha_{2}\left(a_{2} \mid b_{2}\right)=\alpha_{2}\left(b_{2} \mid c_{2}\right)=0 .
$$

But this game is not elementary. A dual solution has

$$
\alpha_{1}\left(t_{1} \mid r_{1}\right)=\alpha_{1}\left(r_{1} \mid s_{1}\right)=\alpha_{1}\left(s_{1} \mid t_{1}\right)=1 \text { and } \alpha_{2}\left(b_{2} \mid a_{2}\right)=\alpha_{2}\left(c_{2} \mid b_{2}\right)=\alpha_{2}\left(a_{2} \mid c_{2}\right)=1 .
$$

In the reduced game, the sender has one reduced type (equally likely to be $r_{1}, s_{1}$, or $t_{1}$ ), and the receiver has one reduced action (randomizing uniformly over $\left\{a_{2}, b_{2}, c_{2}\right\}$ ).
Thus, dual reduction suggests that the incentive-compatible mechanism $\mu$ that we described above may be imperfect in some sense.

A $3 \times 4$ example with one sender and one receiver:

| $\mathrm{p}:$ | $\mathrm{T}_{1}: \backslash \mathrm{C}_{2}:$ | $a_{2}$ | $b_{2}$ | $c_{2}$ | $d_{2}$ |  |
| :---: | :--- | ---: | ---: | ---: | ---: | :--- |
| $1 / 3$ | $r_{1}$ | 3,0 | 0,3 | 0,3 | 3,0 | [bad type] |
| $1 / 3$ | $s_{1}$ | 9,9 | 8,8 | 0,0 | 0,0 | [left good type] |
| $1 / 3$ | $t_{1}$ | 0,0 | 0,0 | 8,8 | 9,9 | [right good type] |

Dual solutions include $\alpha_{1}\left(s_{1} \mid r_{1}\right)=\eta, \alpha_{1}\left(t_{1} \mid r_{l}\right)=1-\eta$ for $1 / 3 \leq \eta \leq 2 / 3$, $\alpha_{2}\left(b_{2} \mid a_{2}\right)=1, \alpha_{2}\left(\mathrm{c}_{2} \mid \mathrm{d}_{2}\right)=1$, with all other components of $\alpha$ being 0 .
For the symmetric solution $\eta=1 / 2$, the reduced game looks like:
p:
1's reduced type
$\left\{b_{2}\right\}$
$\left\{c_{2}\right\}$
$0.5 \quad\left\{s_{l}\right\} \sim(2 / 3)\left[s_{1}\right]+(1 / 3)\left[r_{1}\right]$
5.33, 6.33
0, 1
$0.5\left\{t_{1}\right\} \sim(2 / 3)\left[t_{1}\right]+(1 / 3)\left[r_{1}\right]$
0,1
5.33, 6.33

With $\eta=1 / 3$, an asymmetric reduced game on one end would be

| $\mathrm{p}:$ | 1's reduced type | $\left\{b_{2}\right\}$ | $\left\{c_{2}\right\}$ |
| :---: | :---: | :---: | :---: |
| $4 / 9$ | $\left\{s_{1}\right\} \sim 0.75\left[s_{1}\right]+0.25\left[r_{1}\right]$ | $6,6.75$ | $0,0.75$ |
| $5 / 9$ | $\left\{t_{1}\right\} \sim 0.6\left[t_{1}\right]+0.4\left[r_{1}\right]$ | $0,1.2$ | $4.8,6$ |

With $\eta=2 / 3$, an asymmetric reduced game on the other end would be

| p: | 1 's reduced type | $\left\{b_{2}\right\}$ | $\left\{c_{2}\right\}$ |
| :---: | :---: | :---: | :--- |
| $5 / 9$ | $\left\{s_{1}\right\} \sim 0.6\left[s_{1}\right]+0.4\left[r_{1}\right]$ | $4.8,6$ | $0,1.2$ |
| $4 / 9$ | $\left\{t_{1}\right\} \sim 0.75\left[t_{1}\right]+0.25\left[r_{1}\right]$ | $0,0.75$ | $6,6.75$ |

All these reduced games are elementary, with strict mechanism $\left(\left\{s_{1}\right\} \rightarrow\left\{b_{2}\right\},\left\{t_{1}\right\} \rightarrow\left\{c_{2}\right\}\right)$.

## Concluding note:

Dual reduction identifies incentive constraints that are hard to satisfy with strict perfection, and it models them as inseparable alternatives in a reduced game. Iterative dual reduction of all such inseparable actions and inseparable types yields an elementary reduced game where all incentive constraints can be satisfied strictly.
Thus, dual reduction allows us to analyze games without any knife-edge imperfection issues, because any such issues in the original game have been identified and embedded into the structure of the reduced game.

Abstract: Consider the incentive constraints that define the incentive-compatible mechanisms of a senders-receivers game. Duals of these linear constraints form Markov chains on the senders' type sets and the receivers' action sets. The minimal nonempty absorbing sets of these Markov chains can be interpreted as the types and actions in a dual-reduced game. Any incentive-compatible mechanism of a dualreduced game induces an equivalent incentive-compatible mechanism for the original game. We say that a game is elementary if all nontrivial incentive constraints can be satisfied as strict inequalities in incentive-compatible mechanisms. Any sendersreceivers game can be reduced to an elementary game by iterative dual reduction.
https://home.uchicago.edu/~rmyerson/research/eldual2023notes.pdf

