A finite **senders-receivers game** is any \( \Gamma = (I, (T_i)_{i \in I}, p, J, (C_j)_{j \in J}, (u_k)_{k \in I \cup J}) \) with nonempty finite sets: \( I = \{\text{senders}\} \), \( T_i = \{i's\ types\} \), \( J = \{\text{receivers}\} \), \( C_j = \{j's\ actions\} \), \( I \cap J = \emptyset \), \( T = \times_{i \in I} T_i \), \( C = \times_{j \in J} C_j \), \( p \in \Delta(T) \) probabilities, \( u_k : C \times T \rightarrow \mathbb{R} \) utility payoffs. We may write \( c = (c_j)_{j \in J} = (c_{-j}, c_j) \in C \), \( t = (t_i)_{i \in I} = (t_{-i}, t_i) \in T \).

A direct **coordination mechanism** is any \( \mu : T \rightarrow \Delta(C) \).

Let \( U_j(\mu, c_j) = \sum_t p(t) \sum_{c_{-j}} \mu(c|t)u_j(c,t), \ \Hat{U}_j(\mu, d_j, c_j) = \sum_t p(t) \sum_{c_{-j}} \mu(c|t)u_j((c_{-j}, d_j),t), \)

\( U_i(\mu, t_i) = \sum_{t_{-i}} p(t) \sum_c \mu(c|t)u_i(c,t), \ \Hat{U}_i(\mu, s_i, t_i) = \sum_{t_{-i}} p(t) \sum_c \mu(c|t_{-i}, s_i)u_i(c,t). \) [proby-discounted]

An **incentive-compatible** (IC) mechanism is any \( \mu \) satisfying:

\( \mu(c|t) \geq 0 \ \forall c \in C, \forall t \in T; \ \sum_{c \in C} \mu(c|t) = 1 \ \forall t \in T; \)  \ [probability constraints]

\( U_j(\mu, c_j) \geq \Hat{U}_j(\mu, d_j, c_j) \ \forall c_j \in C_j, \ \forall d_j \in C_j, \ \forall j \in J; \)  \ [moral-hazard constraints]

\( U_i(\mu, t_i) \geq \Hat{U}_i(\mu, s_i, t_i) \ \forall t_i \in T_i, \ \forall s_i \in T_i, \ \forall i \in I. \)  \ [adverse-selection constraints]

\( \Gamma \) is an **elementary** game iff there exists some \( \mu^* : T \rightarrow \Delta(C) \) such that

\( U_j(\mu^*, c_j) > \Hat{U}_j(\mu^*, d_j, c_j) \ \forall c_j, \ \forall d_j \neq c_j, \ \forall j; \) and \( U_i(\mu^*, t_i) > \Hat{U}_i(\mu^*, s_i, t_i) \ \forall t_i, \ \forall s_i \neq t_i, \ \forall i. \)

**Fact:** If \( \Gamma \) is elementary, then almost all IC mechanisms satisfy all nontrivial incentive constraints strictly. (If \( \mu \) does not then \( (1-\varepsilon)\mu + \varepsilon \mu^* \) does.)
Consider the following \textit{primal linear programming problem}:

minimize $\sum_i \sum_{t} \pi_i(t_i) + \sum_j \sum_{c_j} \pi_j(c_j)$ over $\mu \geq 0$ & $\pi$ such that

$$\pi_j(c_j) + \sum_t \sum_{c^{-j}} p(t)\mu(c|t)(u_j(c,t) - u_j((c^{-j},d_j),t)) \geq 0 \ \forall c_j \in C_j, \ \forall d_j \in C_j, \ \forall j \in J; \ \lbrack \alpha_j(d_j|c_j) \rbrack$$

$$\pi_i(t_i) + \sum_{t^{-i}} \sum_{c} p(t)u_i(c,t)(\mu(c|t) - \mu(c|t^{-i},s_i)) \geq 0 \ \forall t_i \in T_i, \ \forall s_i \in T_i, \ \forall i \in I; \ \lbrack \alpha_i(s_i|t_i) \rbrack$$

$$\sum_c \mu(c|t) = 1 \ \forall t \in T. \ \lbrack \beta(t) \rbrack$$

The solutions to this LP are the incentive-compatible mechanisms, which exist (Nash) and yield optimal value 0 with $\pi = 0$. (Trivial constraints with $d_j = c_j$ and $s_i = t_i$ imply $\pi \geq 0$.)

The \textit{dual LP problem} is: maximize $\sum_t \beta(t)$ over $\alpha \geq 0$ & $\beta$ such that

$$\beta(t) + \sum_j \sum_{d_j} \alpha_j(d_j|c_j)p(t)(u_j(c,t) - u_j((c^{-j},d_j),t)) +$$

$$+ \sum_i \sum_{t_i} (\alpha_i(s_i|t_i)p(t)u_i(c,t) - \alpha_i(t_i|s_i)p(t^{-i},s_i)u_i(c,(t^{-i},s_i))) \leq 0 \ \forall t \in T, \ \forall c \in C; \ \lbrack \mu(c|t) \rbrack$$

$$\sum_{d_j} \alpha_j(d_j|c_j) = 1 \ \forall c_j \in C_j, \ \forall j \in J; \ \lbrack \pi_j(c_j) \rbrack$$

$$\sum_{s_i} \alpha_i(s_i|t_i) = 1 \ \forall t_i \in T_i, \ \forall i \in I. \ \lbrack \pi_i(t_i) \rbrack$$

\textit{Nontrivial dual solutions exist}, with some $\alpha_k(e_k|f_k) > 0$ & $e_k \neq f_k$, \textit{iff} $\Gamma$ is not elementary.

Let $D(c,t,\alpha) = \sum_j \sum_{d_j} \alpha_j(d_j|c_j)p(t)(u_j((c^{-j},d_j),t) - u_j(c,t)) +$

$$+ \sum_i \sum_{s_i} (\alpha_i(t_i|s_i)p(t^{-i},s_i)u_i(c,(t^{-i},s_i)) - \alpha_i(s_i|t_i)p(t)u_i(c,t)). \ \lbrack \alpha - \text{deviatn value @c,t} \rbrack$$

Then the dual optimum has $\beta(t) = \min_{c \in C} D(c,t,\alpha) \ \forall t \in T$, and $\sum_{t \in T} \beta(t) = 0$.

\textbf{Lemma:} Given any dual solution $\alpha$, for any mechanism $\mu: T \rightarrow \Delta(C)$:

$$\sum_{j \in J} \sum_{c_j} \sum_{d_j} \alpha_j(d_j|c_j)(\hat{U}_j(\mu,d_j,c_j) - U_j(\mu,c_j)) + \sum_{i \in I} \sum_{t_i} \sum_{s_i} \alpha_i(s_i|t_i)(\hat{U}_i(\mu,s_i,t_i) - U_i(\mu,t_i))$$

$$= \sum_{t \in T} \sum_{c \in C} \mu(c|t)D(c,t,\alpha) \geq \sum_{t \in T} \beta(t) = 0.$$

(\textit{Expected net gains of unilateral $\alpha$-deviations from $\mu$ must have a nonnegative sum.})
Given any Markov chain $\delta: X \rightarrow \Delta(X)$, we let $X/\delta$ denote the set of \textit{minimal nonempty $\delta$-absorbing subsets} of $X$. ($Y \subseteq X$ is $\delta$-absorbing iff $\delta(x|y) = 0 \ \forall y \in Y, \ \forall x \notin Y$.)

We define functions $\psi$ and $\phi$ so that, for any absorbing set $R \in X/\delta$ and any $x \in X$:

- $\phi(x|R,\delta)$ is the probability of $x$ in the unique $\delta$-stationary distribution on $R$, and
- $\psi(R|x,\delta)$ is the \textit{probability of a $\delta$-stochastic process reaching $R$ from $x$}. That is:

$$
\phi(y|R,\delta) = \sum_{z \in R} \phi(z|R,\delta) \delta(y|z) \ \forall y \in R;
\sum_{y \in R} \phi(y|R,\delta) = 1;
\phi(z|R,\delta) = 0 \text{ if } z \not\in R;
\psi(R|y,\delta) = 1 \text{ if } y \in R;
\psi(R|y,\delta) = 0 \text{ if } y \in \hat{R} \in X/\delta \text{ and } \hat{R} \neq R;
\psi(R|y,\delta) = \sum_{z \in X} \delta(z|y) \psi(R|z,\delta) \ \forall y \in X.
$$

\textbf{Dual solutions $\alpha$ define Markov chains} $\alpha_i: T_i \rightarrow \Delta(T_i)$ and $\alpha_j: C_j \rightarrow \Delta(C_j)$.

Given any dual solution $\alpha$ for the game $\Gamma$, let us define the \textit{$\alpha$-reduced game}:

- $\Gamma/\alpha = (I, (T_i/\alpha_i)_{i \in I}, q, (C_j/\alpha_j)_{j \in J}, (v_k)_{k \in I \cup J})$ where

$$
q(\tau) = \sum_{t \in T} p(t) \left( \prod_{i \in I} \psi(\tau_i|t_i,\alpha_i) \right),
\text{ and}
$$

$$
v_k(\gamma,\tau) = \sum_{t \in T} \sum_{c \in C} p(t) \left( \prod_{i \in I} \psi(\tau_i|t_i,\alpha_i) \right) \left( \prod_{j \in J} \phi(c_j|\gamma_j,\alpha_j) \right) u_k(c,t) / q(\tau).
$$

A mechanism $\lambda: \left( \times_{i \in I} T_i/\alpha_i \right) \rightarrow \Delta(\times_{j \in J} C_j/\alpha_j)$ for $\Gamma/\alpha$ induces a mechanism $\mu^\lambda$ on $\Gamma$:

$$
\mu^\lambda(c|t) = \sum_\tau \sum_\gamma \left( \prod_{i \in I} \psi(\tau_i|t_i,\alpha_i) \right) \lambda(\gamma|\tau) \left( \prod_{j \in J} \phi(c_j|\gamma_j,\alpha_j) \right).
$$

\textbf{Theorem:} If $\alpha$ is a dual solution for $\Gamma$, then any incentive-compatible mechanism $\lambda$ for the $\alpha$-reduced game $\Gamma/\alpha$ induces an incentive-compatible mechanism $\mu^\lambda$ for $\Gamma$.

\textbf{Fact:} For any finite $\Gamma$, iterative dual reduction yields an elementary reduced game.
In the reduced game $\Gamma / \alpha$, sender i's information is reduced to the absorbing set in $T_i / \alpha_i$ that was reached by an $\alpha_i$ stochastic process from i's true type in $T_i$, and receiver j's choices are reduced to the $\alpha_j$-stationary distributions on absorbing sets in $C_j / \alpha_j$.

To show that incentive-compatible mechanisms in $\Gamma / \alpha$ are still IC in $\Gamma$, we must show that, when all players are expected to act within this reduced strategic domain, one player could not gain by a deviation that uses his full information or choice set in $\Gamma$.

By the Lemma, for any mechanism in $\Gamma$, if each player considered deviating by $\alpha_i$ or $\alpha_j$, the sum of their expected gains from unilaterally deviating must be nonnegative.

But for players who are acting according to their reduced strategic options in $\Gamma / \alpha$, deviating by $\alpha_i$ or $\alpha_j$ would not change their probabilistic behavior (by construction).

Now suppose, contrary to the Theorem, that some j could expect to gain strictly by deviating to some $c_j$ in $C_j$ when $\gamma_j$ in $C_j / \alpha_j$ is the $\lambda$-recommended action for j.

Consider the scenario that differs from $\lambda$ only in that player j deviates to a uniform distribution over all of j's best-responses in $C_j$ when $\gamma_j$ is recommended.

Since $\alpha$ deviations from this scenario would not affect any other behavior, the Lemma implies that j cannot expect to lose by further deviating from this scenario by $\alpha_j$ when $\gamma_j$ is recommended.

So if $c_j$ is a best response to $\gamma_j$ and $\alpha_i(d_j|c_j)>0$ then $d_j$ is also a best response to $\gamma_j$.

So j's best responses in $C_j$ are an $\alpha_j$-absorbing set, and so an $\alpha_j$-stationary option in $C_j / \alpha_j$ is also optimal for j, and (by $\lambda$ IC) it is $\gamma_j$, contradicting the supposition above.
Now suppose, *contrary to the Theorem*, that some type $t_i$ of some sender $i$ could expect to gain strictly by deviating from $\lambda$ by changing his report from $\tau_i$ to some other $\rho_i$. Consider the scenario that differs from $\lambda$ only in that all types $t_i$ with $\psi(\tau_i|t_i,\alpha_i)>0$ which strictly prefer misreporting $\rho_i$ do so when $\tau_i$ would be correct in $\lambda$, all those which would strictly lose by misreporting stay with $\tau_i$, and the indifferent randomize equally. Since $\alpha$ deviations from this scenario would not affect any other behavior, the Lemma implies that $i$ could not expect to lose by a further $\alpha_i$-deviation (each $t_i$ acting like another $s_i$ in this scenario with probability $\alpha_i(s_i|t_i)$).

But an $\alpha_i(s_i|t_i)>0$ probability of $t_i$ further imitating an $s_i$ with $\psi(\tau_i|s_i,\alpha_i)>0$ in this scenario would yield an expected strict loss when $t_i$ strictly prefers misreporting & $s_i$ does not, or when $t_i$ strictly prefers to not misreport & $s_i$ is willing to do so, and otherwise it would not make any difference for $i$.

So all of $i$'s types in the $\alpha_i$ chain that leads into $\tau_i$ must be willing to misreport $\rho_i$, while some supposed types strictly prefer such misreporting.

But the incentive-compatibility of $\lambda$ in $\Gamma/\alpha$ implies that the probability-weighted sum of these types' expected payoffs cannot be higher than what they get by reporting $\tau_i$, *contradicting the supposition above*.

This intrinsic alignment of preferences among $i$'s types in $T_i$ that are pooled in a reduced-game type $\tau_i$ shows that the incentive-compatible mechanisms of the reduced game will not depend on the relative weighting of utility payoffs for these types.
**Fact:** Consider a dual solution $\alpha$ and $i \in I$, $\tau_i \in T_i / \alpha_i$, and $r_i \in \tau_i$. Let $\lambda$ and $\eta$ be any two mechanisms for $\Gamma / \alpha$ that do not depend on $i$'s (reduced) type, and suppose that $U_i(\mu^\lambda, r_i) \geq U_i(\mu^n, r_i)$. Then $U_i(\mu^\lambda, t_i) \geq U_i(\mu^n, t_i)$ for every $t_i$ such that $\psi(t_i|t_i, \alpha_i) > 0$.

Suppose, *contrary to the Fact*, that some type $s_i$ has $\psi(t_i|s_i, \alpha_i) > 0$ & $U_i(\mu^n, s_i) > U_i(\mu^\lambda, s_i)$. Consider the scenario which coincides with $\mu^\lambda$ except that it switches to $\mu^n$ iff $i$'s type $t_i$ satisfies $U_i(\mu^n, t_i) > U_i(\mu^\lambda, t_i)$.

This scenario satisfies all the strategic restrictions of the reduced game for all players other than sender $i$. Thus, since $\alpha$ deviations from this scenario would not affect any other behavior, the Lemma implies that player $i$ could not expect to lose by an $\alpha_i$-deviation (each $t_i$ acting like another $\hat{t}_i$ in this scenario with probability $\alpha_i(\hat{t}_i|t_i)$).

An $\alpha_i(\hat{t}_i|t_i) > 0$ probability of $t_i$ imitating $\hat{t}_i$ can make a difference in this scenario only in two cases: (1) if $\hat{t}_i$ satisfies $U_i(\mu^n, \hat{t}_i) > U_i(\mu^\lambda, \hat{t}_i)$ but $U_i(\mu^\lambda, t_i) \geq U_i(\mu^n, t_i)$, or (2) if $t_i$ satisfies $U_i(\mu^n, t_i) > U_i(\mu^\lambda, t_i)$ but $U_i(\mu^\lambda, \hat{t}_i) \geq U_i(\mu^n, \hat{t}_i)$.

In case (1), the imitation cannot help player $i$ in type $t_i$, as it would just substitute $\eta$ for the weakly preferred $\lambda$. In case (2), the imitation strictly hurts player $i$ in type $t_i$, as it substitutes $\lambda$ for the strictly preferred $\eta$.

But $\psi(t_i|s_i, \alpha_i) > 0$ implies the existence of a positive $\alpha_i$-chain from $s_i$ to $r_i \in \tau_i$, and so the strict-loss case (2) must happen at least once, *contradicting the above supposition*.

**Corollary:** If $\{r_i, s_i\} \subseteq \tau_i$ then $U_i(\mu^\lambda, r_i) \geq U_i(\mu^n, r_i) \iff U_i(\mu^\lambda, s_i) \geq U_i(\mu^n, s_i)$. 
Examples with no senders (as in the 1997 paper):

<table>
<thead>
<tr>
<th></th>
<th>C1: \ C2:</th>
<th>c2</th>
<th>d2</th>
</tr>
</thead>
<tbody>
<tr>
<td>c1</td>
<td>3, 2</td>
<td>0, 0</td>
<td></td>
</tr>
<tr>
<td>d1</td>
<td>0, 0</td>
<td>2, 3</td>
<td></td>
</tr>
</tbody>
</table>

All incentive constraints can be satisfied strictly with $\mu(c_1,c_2)=0.5=\mu(d_1,d_2)$. So this game is elementary, and it has no nontrivial dual solutions.

<table>
<thead>
<tr>
<th></th>
<th>C1: \ C2:</th>
<th>c2</th>
<th>d2</th>
</tr>
</thead>
<tbody>
<tr>
<td>c1</td>
<td>5, 5</td>
<td>0, 5</td>
<td></td>
</tr>
<tr>
<td>d1</td>
<td>5, 0</td>
<td>1, 1</td>
<td></td>
</tr>
</tbody>
</table>

Dual solutions include $\alpha_1(d_1|c_1)=1$, $\alpha_1(c_1|d_1)=0$, $\alpha_2(d_2|c_2)=1$, $\alpha_2(c_2|d_2)=0$. $C_i/\alpha_i = \{\{d_i\}\}$. In the reduced game, the dominated actions $c_1$ and $c_2$ are eliminated.

<table>
<thead>
<tr>
<th></th>
<th>C1: \ C2:</th>
<th>c2</th>
<th>d2</th>
</tr>
</thead>
<tbody>
<tr>
<td>c1</td>
<td>7, 0</td>
<td>2, 5</td>
<td></td>
</tr>
<tr>
<td>d1</td>
<td>4, 3</td>
<td>6, 1</td>
<td></td>
</tr>
</tbody>
</table>

Dual solutions include $\alpha_1(d_1|c_1)=1$, $\alpha_1(c_1|d_1)=0.4$, $\alpha_2(d_2|c_2)=0.6$, $\alpha_2(c_2|d_2)=0.8$, and the reduced game has one absorbing set of actions $\{c_i,d_i\}$ for each player $i$. The $\alpha$-stationary strategies are the unique Nash equilibrium strategies: 

$$(2/7)[c_1]+(5/7)[d_1], (4/7)[c_2]+(3/7)[d_2].$$

The reduced game is $1\times1$ with the equilibrium payoffs (4.857, 2.143).
**Example of a 3×3 game with no senders:**

<table>
<thead>
<tr>
<th></th>
<th>C₁:</th>
<th>C₂:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>c₂</td>
<td>d₂</td>
</tr>
<tr>
<td>c₁</td>
<td>0, 0</td>
<td>1, 0</td>
</tr>
<tr>
<td>d₁</td>
<td>0, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>e₁</td>
<td>1, 0</td>
<td>0, 1</td>
</tr>
</tbody>
</table>

This game has a correlated equilibrium that randomizes uniformly over the six nondiagonal outcomes of the game, which satisfies strictly six incentive constraints and so implies that every dual solution must have:

\[ \alpha_1(e_1|c_1)=0, \alpha_1(c_1|d_1)=0, \alpha_1(d_1|e_1)=0, \alpha_2(e_2|c_2)=0, \alpha_2(c_2|d_2)=0, \alpha_2(d_2|e_2)=0. \]

But this game is not elementary. There is a dual solution with:

\[ \alpha_1(d_1|c_1)=1, \alpha_1(e_1|d_1)=1, \alpha_1(c_1|e_1)=1, \alpha_2(d_2|c_2)=1, \alpha_1(e_2|d_2)=1, \alpha_2(c_2|e_2)=1. \]

The dual reduction is the 1×1 game where each player j's only option is to randomize uniformly over \{c_j, d_j, e_j\}, yielding expected payoffs (1/3, 1/3).

Thus, dual reduction suggests that the correlated equilibrium that we described above may be imperfect in some sense.

**Sensitivity analysis:** Suppose that we change the diagonal payoffs from (0,0) to \((\varepsilon, \varepsilon)\) for some number \(\varepsilon\).

- If \(\varepsilon < 0\) then the game is elementary, as the above correlated equilibrium then satisfies all incentive constraints strictly.
- If \(\varepsilon > 1\) then the game is elementary, as \(\mu(c_1, c_2)=\mu(d_1, d_2)=\mu(e_1, e_2)=1/3\) then satisfies all incentive constraints strictly.
- If \(0 < \varepsilon < 1\) then no incentive constraint can be satisfied strictly, and a dual solution where all nontrivial dual variables are strictly positive is:

\[ \alpha_1(e_1|c_1)=\alpha_1(c_1|d_1)=\alpha_1(d_1|e_1)=\alpha_2(e_2|c_2)=\alpha_2(c_2|d_2)=\alpha_2(d_2|e_2)=\varepsilon, \]

\[ \alpha_1(d_1|c_1)=\alpha_1(e_1|d_1)=\alpha_1(c_1|e_1)=\alpha_2(d_2|c_2)=\alpha_1(e_2|d_2)=\alpha_2(c_2|e_2)=1-\varepsilon. \]

These games have nontransitive dual solutions [see below] only in two cases: when \(\varepsilon=0\), and when \(\varepsilon=1\).
A similar 3×3 example with one sender and one receiver:

<table>
<thead>
<tr>
<th></th>
<th>C₁:</th>
<th>C₂:</th>
<th>C₃:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>r₁</td>
<td>0, 0</td>
<td>1, 0</td>
</tr>
<tr>
<td>1/3</td>
<td>s₁</td>
<td>0, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>1/3</td>
<td>t₁</td>
<td>1, 0</td>
<td>0, 1</td>
</tr>
</tbody>
</table>

An IC μ has \(μ(b_2|r_1)=μ(c_2|r_1)=0.5, μ(a_2|s_1)=μ(c_2|s_1)=0.5, μ(a_2|t_1)=μ(b_2|t_1)=0.5\).

Its six strictly satisfied constraints imply that any dual solution has

\[ α_1(s_1|r_1)=α_1(t_1|s_1)=α_1(r_1|t_1)=0 \quad \text{and} \quad α_2(c_2|a_2)=α_2(a_2|b_2)=α_2(b_2|c_2)=0. \]

But this game is not elementary. A dual solution has

\[ α_1(t_1|r_1)=α_1(r_1|s_1)=α_1(s_1|t_1)=1 \quad \text{and} \quad α_2(b_2|a_2)=α_2(c_2|b_2)=α_2(a_2|c_2)=1. \]

In the reduced game, the sender has one reduced type (equally likely to be \(r_1, s_1, \text{or} \ t_1\)), and the receiver has one reduced action (randomizing uniformly over \(\{a_2, b_2, c_2\}\)).

Thus, dual reduction suggests that the incentive-compatible mechanism \(μ\) that we described above may be imperfect in some sense.

(Sensitivity analysis: Suppose we change the diagonal elements from \((0,0)\) to \((ε,ε)\), for some \(ε<1\).

If \(ε<0\), then the game becomes elementary, and the above-described incentive-compatible mechanism \(μ\) satisfies all incentive constraints strictly.

If \(0<ε<1\) then no incentive constraint can be satisfied strictly, and there is a dual solution with all nontrivial incentive constraints having strictly positive dual variables:

\[ α_1(s_1|r_1)=α_1(t_1|s_1)=α_1(r_1|t_1)=α_2(c_2|a_2)=α_2(a_2|b_2)=α_2(b_2|c_2)=ε, \]
\[ α_1(t_1|t_1)=α_1(r_1|s_1)=α_1(s_1|t_1)=α_2(b_2|a_2)=α_1(c_2|b_2)=α_2(a_2|c_2)=1−ε. \]
A 3x4 example with one sender and one receiver:

<table>
<thead>
<tr>
<th>p:</th>
<th>T₁: \ C₂:</th>
<th>a₂</th>
<th>b₂</th>
<th>c₂</th>
<th>d₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>r₁</td>
<td>3, 0</td>
<td>0, 3</td>
<td>0, 3</td>
<td>3, 0</td>
</tr>
<tr>
<td>1/3</td>
<td>s₁</td>
<td>9, 9</td>
<td>8, 8</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>1/3</td>
<td>t₁</td>
<td>0, 0</td>
<td>0, 0</td>
<td>8, 8</td>
<td>9, 9</td>
</tr>
</tbody>
</table>

Dual solutions include $\alpha_1(s₁|r₁) = \eta$, $\alpha_1(t₁|r₁) = 1-\eta$ for $1/3 \leq \eta \leq 2/3$,
$\alpha_2(b₂|a₂) = 1$, $\alpha_2(c₂|d₂) = 1$, with all other components of $\alpha$ being 0.

For the symmetric solution $\eta=1/2$, the reduced game looks like:

<table>
<thead>
<tr>
<th>p:</th>
<th>1's reduced type</th>
<th>{b₂}</th>
<th>{c₂}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>{s₁}~0.75[s₁]+0.25[r₁]</td>
<td>6, 6.75</td>
<td>0, 0.75</td>
</tr>
<tr>
<td>0.5</td>
<td>{t₁}~0.6[t₁]+0.4[r₁]</td>
<td>0, 1.2</td>
<td>4.8, 6</td>
</tr>
</tbody>
</table>

With $\eta=1/3$, an asymmetric reduced game on one end would be

<table>
<thead>
<tr>
<th>p:</th>
<th>1's reduced type</th>
<th>{b₂}</th>
<th>{c₂}</th>
</tr>
</thead>
<tbody>
<tr>
<td>4/9</td>
<td>{s₁}~0.75[s₁]+0.25[r₁]</td>
<td>6, 6.75</td>
<td>0, 0.75</td>
</tr>
<tr>
<td>5/9</td>
<td>{t₁}~0.6[t₁]+0.4[r₁]</td>
<td>0, 1.2</td>
<td>4.8, 6</td>
</tr>
</tbody>
</table>

With $\eta=2/3$, an asymmetric reduced game on the other end would be

<table>
<thead>
<tr>
<th>p:</th>
<th>1's reduced type</th>
<th>{b₂}</th>
<th>{c₂}</th>
</tr>
</thead>
<tbody>
<tr>
<td>5/9</td>
<td>{s₁}~0.6[s₁]+0.4[r₁]</td>
<td>4.8, 6</td>
<td>0, 1.2</td>
</tr>
<tr>
<td>4/9</td>
<td>{t₁}~0.75[t₁]+0.25[r₁]</td>
<td>0, 0.75</td>
<td>6, 6.75</td>
</tr>
</tbody>
</table>

All these reduced games are elementary, with strict mechanism ($\{s₁\}→\{b₂\}, \{t₁\}→\{c₂\}$).
**Games with transitive dual solutions.**

For any game $\Gamma$, we say that a dual solution $\alpha$ is *maximal* iff its support (the set of $(k,x_k,y_k)$ such that $\alpha_k(x_k|y_k)>0$) is maximal among all dual solutions to the game. We may define the *dual magnitude* of a game to be the number of nonzero nontrivial components in a maximal dual solution for the game.

We say that the game $\Gamma$ has *transitive* dual solutions iff, in any maximal dual solution $\alpha$, for any player $k \in I \cup J$ and any three of $k$'s alternative types or actions $\{x_k, y_k, z_k\}$, if $\alpha_k(x_k|y_k)>0$ and $\alpha_k(y_k|z_k)>0$ then $\alpha_k(x_k|z_k)>0$.

Any $(I, (T_i)_{i \in I}, p, J, (C_j)_{j \in J})$ as above may be called a *framework* for senders-receivers games, which can then be defined by specifying a utility vector in $\mathbb{R}^{(I \cup J) \times C \times T}$.

Given this framework, the game defined by a utility vector $u$ may then be denoted $\Gamma(u)$. We say that the type distribution $p$ has *full support* iff $p(t) > 0$ for all $t \in T$.

We have seen 3x3 examples of games which do not have transitive dual solutions, but this failure of dual transitivity is very sensitive to small changes in the utilities.

**Claim.** Given any framework $(I, (T_i)_{i \in I}, p, J, (C_j)_{j \in J})$ for senders-receivers games, suppose that the type distribution $p$ has full support. Then the set of utility functions $u$ such that $\Gamma(u)$ has transitive dual solutions is dense in the utility space $\mathbb{R}^{(I \cup J) \times C \times T}$.

**Proof:** Given any nonempty open set in the utility space, we can select $u$ in this set so that $\Gamma(u)$ has a dual magnitude that is minimal in the set. Then we can show that this $\Gamma(u)$ with locally minimal dual magnitude will have transitive dual solutions.
**Proof:** Given any nonempty open set in the utility space, we can select \( u \) in this set so that \( \Gamma(u) \) has a dual magnitude [integer \( \geq 0 \)] that is minimal in the set. Then this \( \Gamma(u) \) with locally minimal dual magnitude will have transitive dual solutions.

Let \( \alpha \) be any maximal dual solution for \( \Gamma(u) \).

\( \Gamma(u) \) also has an IC mechanism \( \mu \) that is *maximally* incentive compatible, satisfying with strict inequality every incentive constraint that has a zero component in \( \alpha \).

If \( \Gamma(u) \) does not have transitive dual solutions then either

1. **(case 1)** some receiver \( j \) has an action \( d_j \) such that some other actions \( e_i \) and \( c_j \) satisfy \( \alpha_j(e_j|d_j)>0 \) and \( \alpha_j(d_j|c_j)>0 \) but \( \alpha_j(e_j|c_j)=0 \); or
2. **(case 2)** some sender \( i \) has a type \( s_i \) such that some other types \( r_i \) and \( t_i \) satisfy \( \alpha_i(t_i|s_i)>0 \) and \( \alpha_i(s_i|r_i)>0 \) but \( \alpha_i(t_i|r_i)=0 \).

In case 1, for any small \( \varepsilon>0 \), we can construct \( \tilde{u} \) to be the same as \( u \) except that:

\[
\tilde{u}_j((c_{-j}, d_j), t) = (1-\varepsilon)u_j((c_{-j}, d_j), t) + \varepsilon \sum_{e_j} \alpha_j(e_j|d_j)u_j((c_{-j}, e_j), t), \quad \forall c_{-j}, \forall t.
\]

In case 2, for any small \( \varepsilon>0 \), we can construct \( \tilde{u} \) to be the same as \( u \) except that

\[
p(t_{-i}, s_i)\tilde{u}_i(c, t_{-i}, s_i) = (1-\varepsilon)p(t_{-i}, s_i)u_i(c, t_{-i}, s_i) + \varepsilon \sum_{r_i} \alpha_i(s_i|r_i)p(t_{-i}, r_i)u_i(c, t_{-i}, r_i) \quad \forall t_{-i}, \forall c.
\]

Then any \( \mu \) that is maximally incentive compatible for \( \Gamma(u) \) will also be incentive compatible for \( \Gamma(\tilde{u}) \), but its set of strictly satisfied constraints will expand to include the incentive constraint for the dual variable \( \alpha_j(d_j|c_j) \) [in case 1] or \( \alpha_i(t_i|s_i) \) [in case 2].

So in each case, this construction would yield utility vectors \( \tilde{u} \) that are in the open set but give \( \Gamma(\tilde{u}) \) a strictly smaller dual magnitude than \( \Gamma(u) \), contradicting the selection of \( u \).
Concluding note:
Dual reduction identifies incentive constraints that are hard to satisfy with strict perfection, and it models them as inseparable alternatives in a reduced game. Iterative dual reduction of all such inseparable actions and inseparable types yields an elementary reduced game where all incentive constraints can be satisfied strictly. Thus, dual reduction allows us to analyze games without any knife-edge imperfection issues, because any such issues in the original game have been identified and embedded into the structure of the reduced game.

Abstract: Consider the incentive constraints that define the incentive-compatible mechanisms of a senders-receivers game. Duals of these linear constraints form Markov chains on the senders' type sets and the receivers' action sets. The minimal nonempty absorbing sets of these Markov chains can be interpreted as the types and actions in a dual-reduced game. Any incentive-compatible mechanism of a dual-reduced game induces an equivalent incentive-compatible mechanism for the original game. We say that a game is elementary if all nontrivial incentive constraints can be satisfied as strict inequalities in incentive-compatible mechanisms. Any senders-receivers game can be reduced to an elementary game by iterative dual reduction.

https://home.uchicago.edu/~rmyerson/research/eldual2023notes.pdf
**Dual Reduction and Elementary Games with Senders and Receivers**

Addendum On the Density of Games with Transitive Dual Solutions

For any game \( \Gamma \), we say that a dual solution \( \alpha \) is **maximal** iff its support (the set of \( (k,x_k,y_k) \) such that \( \alpha_k(x_k|y_k) > 0 \)) is maximal among all dual solutions to the game. A maximal dual solution must exist, because the union of the supports of any two dual solutions will be the support of the average of the two given dual solutions, which is also a dual solution.

We may define the **dual magnitude** of a game to be the number of nonzero nontrivial components in a maximal dual solution for the game. (So an elementary game would have dual magnitude equal to 0.)

An incentive-compatible mechanism \( \mu \) is **maximal** iff its set of incentive constraints satisfied as strict inequalities is maximal. A maximal IC mechanism must also exist. An incentive constraint has a strictly positive dual variable in a maximal dual solution if and only if the constraint is binding with 0 slack in any maximal incentive-compatible mechanism.

We say that the game \( \Gamma \) has **transitive** dual solutions iff, in any maximal dual solution \( \alpha \), for any player \( k \in I \cup J \) and any three of \( k \)'s alternative types or actions \( \{x_k, y_k, z_k\} \), if \( \alpha_k(x_k|y_k) > 0 \) and \( \alpha_k(y_k|z_k) > 0 \) then \( \alpha_k(x_k|z_k) > 0 \).

For any given senders-receivers game \( \Gamma = (I, (T_i)_{i \in I}, p, J, (C_j)_{j \in J}, (u_k)_{k \in I \cup J}) \), for any players \( i \in I \) and \( j \in J \), and any \( c_j \in C_j \) or \( t_i \in T_i \), let

\[
A_j(c_j) = \{d_j \in C_j | \text{a maximal dual solution } \alpha \text{ has } \alpha_j(d_j|c_j) > 0 \},
\]

\[
Q_i(t_i) = \{s_i \in T_i | \text{a maximal dual solution } \alpha \text{ has } \alpha_i(t_i|s_i) > 0 \}.
\]

So \( \Gamma \) has transitive dual solutions iff \( \forall i \in I, \forall j \in J \):

\[
d_j \in A_j(c_j) \Rightarrow A_j(d_j) \subseteq A_j(c_j), \text{ and } s_i \in Q_i(t_i) \Rightarrow Q_i(s_i) \subseteq Q_i(t_i).
\]

Any elementary game satisfies dual transitivity, because its \( A_j(c_j) \) & \( Q_i(t_i) \) are all empty sets.

Any \( (I, (T_i)_{i \in I}, p, J, (C_j)_{j \in J}) \) as above may be called a **framework** for senders receivers games, which can then be defined by specifying a utility vector in \( \mathbb{R}^{(I \cup J) \times C \times T} \). Given this framework, the game defined by a utility vector \( u \) may then be denoted \( \Gamma(u) \).

We say that the type distribution \( p \) has **full support** iff \( p(t) > 0 \) for all \( t \in T \).

We have seen 3x3 examples of games which do not have transitive dual solutions, but this failure of dual transitivity is very sensitive to small changes in the utilities.

We can now show that the games with transitive dual solutions are dense in the set of games on any given framework with a full-support type distribution.

**Claim.** Given any framework \( (I, (T_i)_{i \in I}, p, J, (C_j)_{j \in J}) \) for senders-receivers games, suppose that the type distribution \( p \) has full support. Then the set of utility functions \( u \) such that \( \Gamma(u) \) has transitive dual solutions is dense in \( \mathbb{R}^{(I \cup J) \times C \times T} \).
(That is, for any senders-receivers game $\Gamma(u)$ where the dual solutions are not transitive, for any $\varepsilon>0$, there exist other utility vectors $\tilde{u}$ such that $\Gamma(\tilde{u})$ has transitive dual solutions and no component of $\tilde{u}$ differs by more than $\varepsilon$ from the corresponding component of $u$.)

**Proof:** Contrary to the claim, suppose that we can find some nonempty open subset of $\mathbb{R}^{|I\times J|\times C\times T}$ such that all utility vectors in this set induce games where the dual solutions are not transitive.

Let us select $u$ to be any utility vector in this set such that $\Gamma(u)$ has a dual magnitude which is minimal among the dual magnitudes of all games with utilities in this open set. (Such a magnitude-minimizing $u$ can be found, even on an open set, because there are only finitely many possible values for the dual magnitudes of these games.)

Then let $\alpha$ denote a maximal dual solution for $\Gamma(u)$.

Since the dual solutions of $\Gamma(u)$ are not transitive, either *(case 1)* some receiver $j$ has some action $d_j$ such that we can find other actions $c_j$ and $e_j$ in $C_j$ satisfying $d_j \in A_j(c_j)$, $e_j \in A_j(d_j)$, but $c_j \not\in A_j(e_j)$; or *(case 2)* some sender $i$ has some type $s_i$ such that we can find other type $t_i$ and $r_i$ satisfying $s_i \in Q_i(t_i)$, $r_i \in Q_i(s_i)$, but $r_i \not\in Q_i(t_i)$.

First consider case 1, where $u$ has a violation of transitivity for some action $d_j$ of some receiver $j$.

For any small $\varepsilon>0$, we can construct $\tilde{u}$ to be the same as $u$ except that player $j$'s payoff from choosing action $d_j$ is changed to:

$$
\tilde{u}_j((c_j,d_j),t) = (1-\varepsilon)u_j((c_j,d_j),t) + \varepsilon \sum_{e_j \in A_j(d_j)} \alpha_j(e_j|d_j)u_j((c_j,e_j),t), \ \forall c_j, \ \forall t.
$$

That is, player $j$'s expected $\tilde{u}$-payoff from $d_j$ would as if $j$ imagined that choosing $d_j$ would involve an $\varepsilon$ chance of (accidentally) deviating to another action in $A_j(d_j)$ according to the $\alpha_j(\bullet|d_j)$ distribution; but $j$'s imagined deviation would not affect other players' payoffs.

We now show that any maximal incentive-compatible mechanism $\mu$ for the game $\Gamma(u)$ will also be incentive compatible for the new game $\Gamma(\tilde{u})$, and all the incentive constraints that $\mu$ satisfies with positive slack for $\Gamma(u)$ will also be satisfied with positive slack for $\Gamma(\tilde{u})$. We only need to consider incentive constraints involving $d_j$, because all other payoffs in $\tilde{u}$ remain the same as in $u$. In the game $\Gamma(u)$, by the complementary slackness of primal and dual solutions, $j$ is indifferent between choosing $d_j$ and any other $e_j$ in $A_j(d_j)$ when $d_j$ is recommended by any $\mu$ that is incentive-compatible in $\Gamma(u)$; and so $j$'s imagined chances of deviating from $d_j$ to another action $e_j$ in $A_j(d_j)$ would not affect $j$'s expected payoffs from choosing $d_j$ when it is recommended by $\mu$ in $\Gamma(\tilde{u})$. So for any mechanism $\mu$ that is incentive compatible in $\Gamma(u)$, the slack in primal incentive constraints corresponding to any $\alpha_j(e_j|d_j)$ would remain the same in $\Gamma(\tilde{u})$ as in $\Gamma(u)$. On the other hand, for primal incentive constraints corresponding to any $\alpha_j(d_j|c_j)$, the fact that $j$ would gain by deviating in $\Gamma(u)$ from $\mu$'s recommendation of $c_j$ to $d_j$ or any other $e_j$ implies that $j$ would also not gain by deviating from any recommended $c_j$ to the imagined mixture of $d_j$ and other $e_j$ that $j$ gets from the action $d_j$ in $\Gamma(\tilde{u})$. Thus, any $\mu$ that is incentive compatible in $\Gamma(u)$ is also incentive compatible in $\Gamma(\tilde{u})$, and any incentive constraint that $\mu$ satisfies with positive slack in $\Gamma(u)$ must also be satisfied with positive slack in $\Gamma(\tilde{u})$ (for all sufficiently small $\varepsilon$). So any dual variable which is 0 in the maximal dual solutions $\mu$ for $\Gamma(u)$ must remain 0 in all dual solutions for $\Gamma(\tilde{u})$. 

2
But as u does not have transitive dual solutions (by supposition), the case-1 conditions \( d_j \in A_j(c), \) \( e_j \in A_j(d_j), \) \& \( e_j \not\in A_j(c) \) for \( \Gamma(u) \) together imply that the incentive constraint for the dual variable \( \alpha_j(d_j|c) \) would now be satisfied strictly in \( \Gamma(\bar{u}) \) (with \( \varepsilon > 0 \)) for any \( \mu \) that is maximally incentive compatible in \( \Gamma(u) \); and so the dual variable \( \alpha_j(d_j|c) \) which is strictly positive in maximal dual solutions for \( \Gamma(u) \) must be 0 in dual solutions for \( \Gamma(\bar{u}) \). Thus, the dual magnitude of the game \( \Gamma(\bar{u}) \) must be strictly smaller than the dual magnitude of the game with utilities \( u \). But with \( \varepsilon \) sufficiently small, this \( \bar{u} \) will be in the given open set, contradicting the selection of \( u \) above.

Now consider case 2, where \( u \) has a violation of transitivity for some type \( s_i \) of some sender \( i \).

In case 2, for any small \( \varepsilon > 0 \), we can construct \( \bar{u} \) to be the same as \( u \) except that, if player \( i \)'s type is \( s_i \) then

\[
p(t-i,s_i)\tilde{u}(c,t-i,s_i) = (1-\varepsilon)p(t-i,s_i)u_i(c,t-i,s_i) + \varepsilon\sum_{r_i \in Q(s_i)} \alpha_i(s_i|r_i)p(t-i,r_i)u_i(c,t-i,r_i).
\]

That is, player \( i \)'s \( \tilde{u} \)-payoff with type \( s_i \) would be as if \( s_i \) imagined that there was a small chance of \( i \)'s having had another type \( r_i \) in \( Q(s_i) \) which, with probability \( \varepsilon \alpha_i(s_i|r_i) \), chose to identify itself as \( s_i \) and then forgot about this self-delusion. The payoffs of other players' or other types of \( i \) would not be affected by this imagined self-delusion of \( i \)'s type \( s_i \), however.

We now show that any maximal incentive-compatible mechanism \( \mu \) for the game \( \Gamma(u) \) will also be incentive compatible for this new game \( \Gamma(\bar{u}) \), and all the incentive constraints that \( \mu \) satisfies with positive slack for \( \Gamma(u) \) will also be satisfied with positive slack for \( \Gamma(\bar{u}) \). We only need to consider incentive constraints for type \( s_i \) of player \( i \), because all other payoffs in \( \tilde{u} \) remain the same as in \( u \). In the game \( \Gamma(u) \), by complementary slackness of primal and dual solutions, any type \( r_i \) in \( Q(s_i) \) would be indifferent between reporting type \( s_i \) and reporting its true type, and weakly prefers reporting its true type over any other type \( t_i \) in the mechanism \( \mu \); so in \( \Gamma(\bar{u}) \) where \( i \) is unsure whether he is getting \( u \)-payoffs for type \( s_i \) or for some other type \( r_i \) in \( Q(s_i) \), participating honestly in the mechanism \( \mu \) is still optimal for type \( s_i \). Thus, any \( \mu \) that is incentive compatible in \( \Gamma(u) \) is also incentive compatible in \( \Gamma(\bar{u}) \), and any incentive constraint that \( \mu \) satisfies with positive slack in \( \Gamma(u) \) must also be satisfied with positive slack in \( \Gamma(\bar{u}) \) for all sufficiently small \( \varepsilon \). So any dual variable which is 0 in the maximal dual solutions \( \mu \) for \( \Gamma(u) \) must remain 0 in all dual solutions for \( \Gamma(\bar{u}) \).

Now, however, the case-2 dual-intransitivity conditions \( s_i \in Q(t_i), r_i \in Q(s_i), \) \& \( r_i \not\in Q(t_i) \) for \( \Gamma(u) \) would together imply that the incentive constraint for the dual variable \( \alpha_i(t_i|s_i) \) would be satisfied strictly in \( \Gamma(\bar{u}) \) (with \( \varepsilon > 0 \)) for any \( \mu \) that is maximally incentive compatible in \( \Gamma(u) \); and so the dual variable \( \alpha_i(t_i|s_i) \) which is strictly positive in maximal dual solutions for \( \Gamma(u) \) must be 0 in dual solutions for \( \Gamma(\bar{u}) \). Thus, the dual magnitude of the game with payoffs \( \tilde{u} \) must be strictly smaller than the dual magnitude of the game with utilities \( u \). But with \( \varepsilon \) sufficiently small, this \( \tilde{u} \) will also be in the given open set, again contradicting the magnitude-minimizing selection of \( u \).

These contradictions for cases 1 and 2 together prove that, for utilities in any nonempty open subset of \( \mathbb{R}^{(I,J) \times C \times T} \), the minimal dual magnitude can be achieved only at games that have transitive duals. Thus, utilities that yield games with transitive duals are dense in \( \mathbb{R}^{(I,J) \times C \times T} \).