Emile Borel and the foundations of game theory

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Borel's fundamental insights in the theory of games

Nineteenth-century economic theorists used mathematical optimization models to develop a deeper understanding of the rational determinants of supply and demand in markets, and game theory follows naturally from such rational choice theory. In 1838, Augustin Cournot studied competitive supply decisions by modeling firms as profit maximizers; and to analyze a market with a small number of profit-maximizing suppliers, he developed models which introduced game-theoretic analysis into the literature of modern economic theory. In the decades after Cournot, other economists developed models of utility-maximizing consumers to explain demand, but the application of these rational-maximizer models was generally limited to the analysis of supply and demand in markets. When economists studied Cournot's oligopoly game, its assumption of simultaneous independent decision-making was viewed as a seriously restrictive theoretical assumption, because real oligopolists might consider how their actions could influence their competitors' future behavior. So Cournot and his successors did not perceive, as we do now, that such game-theoretic models could provide a general theoretical framework for analyzing competitive behavior, not only in markets but also in any kind of economic, political, or social institutions.

The breakthrough came with a paper that Emile Borel presented to the French Académie des Sciences a century ago, on December 19, 1921. Borel approached the study of games with a mathematician's instinct for generalization and with a particular focus on questions of probability theory. In the first few paragraphs of this short paper, Borel provided three fundamental insights that have been essential to the development of modern game theory.

First, Borel introduced the basic concept of strategic normalization of extensive games, which enables us to interpret one-stage simultaneous-move strategic-form games as general models for any kind of competitive interaction. The key idea was the modern game-theoretic concept of *strategy*, which was introduced by Borel almost too quickly. In his first paragraph, Borel tells us that, in his game model, each of the given alternatives that a player can choose is to be interpreted as a "method of play" or "a code that determines for every possible circumstance what the person should do." Thus, when we assume that the players will make independent choices among their given alternatives, that does not mean that the game does not involve opportunities for players to influence each other, because a player's alternatives in our analysis could represent strategic plans that could include the player's plans for observable actions that might influence others' subsequent behavior and also could include the player's plans for his own subsequent responses to observed prior actions of others. John von Neumann began his 1928 paper by developing this idea more fully, with an explicit definition of extensive-form games which can then be reduced to strategic form, but the basic idea was right there in the first paragraph of Borel (1921).

In his second paragraph, Borel introduced the idea of a randomized strategy, because he understood that there are games where a rational player should not behave in a way that is perfectly predictable by others. Probability theory had developed from the analysis of games where uncertainty is generated by dice and cards. But Borel recognized that, in competitive games, a significant component of a player's uncertainty could also be generated by the strategic behavior of other players. Borel introduced randomized strategies as a basic model of how a player could create uncertainty about his behavior.

In his third paragraph, Borel applied the criterion of expected payoff maximization to analyze each player's optimal response to his opponent's randomized strategy. Borel was able to completely justify this criterion, decades before von Neumann and Morgenstern published their axiomatic derivation of expected utility, because he assumed that a player's payoff for any strategy pair was measured in terms of the player's probability of winning the game, so that the expected payoff from a randomized strategy would correspond to the player's ex-ante probability of winning. This idea is actually implicit in our standard justification of expected-payoff maximization for general von Neumann-Morgenstern utility scales, where an individual's utility for any given outcome can be quantified as the probability p such that the individual would be indifferent between this outcome and a simple lottery that gives probability p of the best possible outcome and probability 1-p of the worst possible outcome.

Using a linear transformation of such win-probability payoffs in a symmetric two-player game that will have one winner, Borel (1921) analyzed symmetric two-person zero-sum games. He suggested that our analysis should began by iteratively eliminating certain "bad" strategies, which were weakly dominated in the game. He then asserted that, for any symmetric two-person zero-sum game where each player has 3 undominated strategies, we can find a randomized strategy such that each player can guarantee himself the fair expected payoff of 0. In the same paragraph, however, Borel also claimed that this result could not be generalized to larger games where players have more than 3 strategies.[1] So in 1921, Borel correctly asserted that the minimax theorem, which von Neumann would prove in 1928, was true for 3x3 symmetric games, but Borel also wrongly claimed that this theorem was false for larger games.[2]

Further development of these insights by von Neumann and Nash

In von Neumann's 1928 paper, before proving the minimax theorem, he began by formally developing the idea of strategic normalization that Borel had sketched in 1921. Von Neumann explicitly developed a general mathematical notation for describing multi-stage games in extensive form, and then he showed how any such game can be represented within the simpler class of models that we now call strategic-form games, where the players' alternatives are strategies for how to play the multi-stage extensive game. Because these strategies could be chosen by the players before they begin their interactions, there is no loss of generality in assuming that the players make their strategic choices independently. Thus, a reader of von Neumann's paper could not miss the point that Borel made almost too quickly in 1921: that the analysis of one-stage games where players act independently can offer a general framework for analyzing rational behavior in any kind of competitive interactions. But this argument for the general assumption of independent strategic decisions was later obscured by von Neumann's subsequent emphasis on coalitional analysis for games with more than two players; and so it remained for John Nash (1950, 1951) to formulate the basic principles of equilibrium analysis for general games.

The fact that ex-ante strategic optimization imposes no conditions on a player's responses to zero-probability events means that some Nash equilibria of the strategic-form game might involve behavior that is recognizably irrational in the extensive form, and so game theorists

[1] In the 1953 English translation of Borel (1921) by Leonard J. Savage, the impossibility claim is misprinted as being for games with more than 7 strategies. Perhaps Savage was influenced by his reading of Borel's subsequent 1927 paper, but the reference to 7x7 games has no precedent in Borel's 1921 paper.

[2] Both claims were asserted without proof in 1921. In a subsequent 1924 paper, Borel gave a proof of his correct assertion about 3x3 games and was more careful to indicate the hypothetical nature of his claim about larger games. Weinstein (2022) insightfully analyzes what Borel might have been thinking when he claimed that it seemed easy to disprove the minimax theorem. Weinstein observes that, in a subsequent 1926 paper, Borel explicitly considered a conjecture that every symmetric two-person zero-sum game would have a randomized equilibrium in which all pure strategies yield the expected payoff 0, but counterexamples to this conjecture can indeed be found even among 4x4 games with no dominated strategies.

subsequently developed refined equilibrium concepts for analysis of dynamic games in extensive form.[3] But Borel's mathematical ability to see great potential generality in the simple structure of one-stage strategic-form games was essential to the development of game theory. We had to understand equilibrium analysis in strategic form before we could develop other refinements and generalizations in the much more complicated structures of extensive-form games.

From von Neumann's proof of the minimax theorem for two-person zero-sum games and Nash's subsequent proof of equilibrium existence for general games, we know that, for any finite game, it is possible to make probabilistic predictions about the players' behavior such that each player can know what is predicted for others and rationally behave according to prediction himself. As remarked by von Neumann in 1953, Borel's conjecture that many finite games would not have such probabilistic equilibria would have meant the impossibility of game theory as we know it. In his pioneering work to demonstrate the power of admitting probabilities into our descriptions of rational competitive behavior, Borel nonetheless believed that there would be some games where even the mathematics of probability would be inadequate to describe the degree of unpredictability that rational behavior would require.

This belief was based on Borel's perception that, in many real games, misleading others about one's strategy can be an essential component of skillful play. The best poker players may be particularly good at deceiving others when they bluff, but modern game-theoretic analysis has generally avoided any assumption that one player can systematically fool the others. If game theory taught that one player should bluff in some manner X when he has a weak hand, then the teachings of game theory would equally warn others of the possibility that this action X might indicate a weak hand. A general theory of games that is applicable to all players' decision problems cannot promise special analytical advantages to any one player.

In essence, the successful development of game theory for social science has been based on two fundamental assumptions: that players are *rational* agents who act to maximize their expected payoff given their information, and that players are *intelligent* enough to know everything about their situation that the game theorist knows.[4] To satisfy this intelligence assumption, game theorists must assume that their model of a game is common knowledge

[3] A key advance was the introduction of *sequential equilibrium* by Kreps and Wilson (1982).

[4] One can argue that, in biological applications, a human game theorist could have a more sophisticated understanding of a game than the animals or genes that play it. But in social science, game theory models are formulated by academic theorists to describe the behavior of practical men and women of the world. People tend to be appropriately skeptical of theoretical arguments about markets or politics that depend on the professional experts missing something about their business which is supposedly clear to academic theorists.

among its players. These assumptions of rationality and intelligence together lead us to Nash's concept of equilibrium. A complete probabilistic prediction about how the players should behave in a game could be rationally fulfilled by each player, when he intelligently understands the predicted behavior of others, only if the prediction is a Nash equilibrium.[5]

With this analytical methodology, even if it cannot suggest how a skillful player could fool others, game theory has substantial value for helping people to understand how a change in the rules of a game could cause the players' behavior to change when they have learned how to compete under the new rules. Much work in social science is aimed at helping us to better understand the institutions of our society, so that we can better evaluate proposals for institutional reform, but reforming an institution means changing the rules of the game that people in the institution will play. Before the rules have changed, when nobody has had any experience with the proposed new rules, it may be unclear how the players' strategies might change, and game theory gives us tools to probe these questions. We want to predict how individuals may behave in the institution when the reformed rules have become familiar to them and they have an intelligent understanding of their new competitive environment. When the goal of our analysis is to identify institutional reforms that could increase people's welfare, it is helpful to assume that each individual always acts rationally to maximize his or her own welfare, so that we can separate arguments for reforming the institution from arguments for better education of individuals. Thus, the fundamental assumptions of game theory are well suited to such questions, and the generality of game theory's mathematical framework has meant that it can be applied to help analyze proposals for reforming the rules of any kind of institution or organization in society.

Understanding the equilibrium coordination problem after Nash

The early work of Borel and von Neumann focused on two-person zero-sum games, where one player's gain would always equal the other player's loss. In a two-person zero-sum game, a player's equilibrium strategies are also *maximin* strategies which guarantee that this player's expected payoff could not be less than his expected payoff in equilibrium, even if the other player did not act according to the equilibrium. In a two-person zero-sum game, if (a_1,a_2)

[5] Aumann (1987) has cogently argued for the more general concept of correlated equilibrium as a fundamental implication of Bayesian rationality. The argument for Nash equilibrium depends on the assumption that all opportunities for the players to communicate or share information have been included in the extensive game model, so that their pre-play strategic decisions must be independent.

and (b_1,b_2) are two different Nash equilibria, then these equilibria must be *equivalent* (in the sense that each player i gets the same expected payoff $u_i(a_1,a_2)=u_i(b_1,b_2)$ in both equilibria), and these equilibria will also be *interchangeable* (in the sense that (a_1,b_2) and (b_1,a_2) are also Nash equilibria). However, these properties do not hold in more general games.

In general, a game could have many different equilibria that would yield different payoffs for the players. Such multiplicity of equilibria was initially seen as a disadvantage of Nash's solution concept, because it implies that rational players' behavior may not be uniquely determined by the economic structure of the game.[6] But Thomas Schelling (1960) suggested that the existence of games with multiple equilibria is a pervasive fact of life that needs to be studied and understood. In many games, there may be Nash equilibria that yield strictly lower expected payoffs for all players than other Pareto-superior Nash equilibria, and then all players would prefer to coordinate on one of these Pareto-superior equilibria. So games with multiple equilibria can be useful models for analyzing a society's need for coordination. This fundamental coordination problem could not be recognized as long as game theorists only studied two-person zero-sum games.

Thus, Schelling taught game theorists to see *commitment* and *coordination* as fundamental social problems. Here the commitment problem is that people cannot be expected to behave in ways that are not part of a Nash equilibrium, and the coordination problem is that people may need guidance in focusing on an equilibrium that is better for them. Schelling argued that such coordination can be provided by any publicly-observed aspect of the social or physical environment that tends to focus people's attention on one particular equilibrium. So Nash's theory of equilibria can be a framework for understanding the vital role of culture, tradition, and leadership in coordinating people's strategic expectations.[7]

In particular, the basis of power in society may be found in a socially recognized leader's ability to designate focal equilibria which then become self-fulfilling directives. The leader's basic choice set is among alternative equilibria in games that have multiple equilibria. Benefits of holding such focal power can be derived from the leader's ability to induce social coordination

[6] At one point, Luce and Raiffa (1957, p 104) even suggested that the multiplicity of equilibria in games like the Battle of Sexes game, along with the uniqueness of a noncooperative equilibrium for finitely repeated Prisoners' Dilemma games, might "sound a death knell" for Nash equilibrium as a principal solution concept for general games. Their concerns about how to interpret the multiplicity of equilibria in coordination games were answered by Schelling (1960), and their concerns about the equilibrium analysis of finitely repeated Prisoners' Dilemma games were answered by Kreps, Milgrom, Roberts, and Wilson (1982).

[7] An extensive review of these points is in Myerson (2009).

on equilibria that are more favorable to the interests of the leader or the leader's clients.

Game theorists generally assume that players have consistent strategic expectations, because intelligent players should understand any predictions that we make about behavior in their game. Although methodological concerns may have led us to this assumption, it is often reasonable to assume such consistency of beliefs in a well-ordered society that has coordinating leadership, where people can expect each other to comply with strategic norms that have been designated by recognized leaders in their domain of authority. But this argument suggests questions about whether perhaps we should admit a possibility of finding some inconsistency of beliefs in divided societies where rival groups are fighting for supreme power.

Borel's 1921 paper was written in the aftermath of the First World War, and he expressed some hope that game-theoretic analysis could contribute to our understanding of the problems of war. Today there is an extensive literature on game theory and armed conflict, which is reviewed in this volume by James Fearon. Here we may just observe that, when we understand war as a struggle for power over the designation of focal equilibria, then game-theoretic principles of equilibrium selection may differ depending on whether the players begin the game in a state of war or peace.[8] Peace is a relationship between nations that have a mutually accepted framework for resolving disputes and maintaining consistent strategic expectations in transactions between them; but war means that the adversaries cannot agree about how to coordinate on equilibria that might be better for them all. So Pareto-inferior equilibria may be more tenable predictions in game models about the potential duration of a war, which is assumed to exist at the start of the game, than in game models about the onset of war, where it is assumed that the players start with peaceful leadership.

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[8] See Fearon (1995) and Myerson (2021), where principles of equilibrium selection in war-onset models and war-duration models are discussed.

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