Problem Set #2

1. Let P_n and Q_n be probability distributions with likelihood ratio $L_n = \frac{dQ_n}{dP_n}$. Denote by G_n the distribution of L_n under P_n. Suppose G_n converges weakly to a distribution G. Show that if G has mean one, then Q_n is contiguous w.r.t. P_n. (This celebrated result is sometimes referred to as Le Cam's First Lemma for historical reasons.)

2. Recall the following result from class:

   Consider sequences of probability distributions P_n and Q_n with likelihood ratio $L_n = \frac{dQ_n}{dP_n}$. Suppose $\log L_n \xrightarrow{d} N(\mu, \sigma^2)$ under P_n. Then, Q_n and P_n are mutually contiguous if and only if $\mu = -\frac{1}{2}\sigma^2$.

   We proved most of this result in class. Complete the proof.

3. Recall the symmetric location model studied in class. We were interested in comparing tests of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$ and derived the local asymptotic power functions of the t-test and sign test by considering a sequence of alternatives of the form $\theta_n = h/\sqrt{n}$. Suppose further that f is differentiable a.e. w.r.t. Lebesgue measure and that

$$0 < I_0 = \int \frac{f'(x)^2}{f(x)} dx < \infty \ ,$$

   so L_n, the likelihood ratio of P_{θ_n} to P_0, satisfies

$$\log L_n = \frac{1}{\sqrt{n}} \sum_{1 \leq i \leq n} -\frac{hf'(X_i)}{f(X_i)} - \frac{h^2}{2} I_0 + o_{P_0}(1) \ .$$

   Use this fact and contiguity to rederive the local asymptotic power functions of these tests.

4. Recall the symmetric location model studied in class. Prove that

$$-hE_0[G(|X_i|)\text{sign}(X_i)\frac{f'(X_i)}{f(X_i)}] = 2h \int f^2(x) dx \ .$$

   Evaluate this last expression when

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp(\frac{-x^2}{2}) \ .$$

5. Let $X_i, i = 1, \ldots, n$ be an i.i.d. sequence of random variables with distribtution $N(\theta, 1)$ where $\theta \in \mathbf{R}$. Let $0 < \alpha_n \to 0$, but such that $\sqrt{n}\alpha_n \to \infty$. Let S_n be Hodges' estimator; that is,

$$S_n = \begin{cases} \bar{X}_n & \text{if } |\bar{X}_n| > \alpha_n \\ 0 & \text{otherwise} \end{cases}.$$

Suppose θ_n is such that $\sqrt{n}\theta_n \to \infty$, but $\theta_n/\alpha_n \to 0$. Show that $\sqrt{n}(S_n - \theta_n)$ converges in probability to $-\infty$ under θ_n.

6. Consider the setup of the previous exercise. The Convolution Theorem says that superefficiency can happen for at most a set of θ values with Lebesgue measure 0. This suggests that superefficiency could possibly happen for values of θ in a countably infinite set. Can you find an estimator that is superefficient for all values of θ in a countably infinite set?