# 1 Asymptotically Normal Experiments

In many problems, we are concerned with calculating the limiting power of a sequence of tests under a sequence of distributions. More specifically, given $n$ i.i.d. observations from a distribution $P_n$ and a test $\phi_n$, we want to determine the behavior of $E_{P_n}[\phi_n]$ as $n$ tends to infinity. Comparing this limit for different tests provided a basis for comparing tests. Of course, ideally one would like to determine the "best" possible limit. In some instances, this can be done directly, but here we will explore an alternative approach that is based on a deeper connection between the asymptotic behavior of the likelihood ratio for the problem at hand and the exact behavior of a the likelihood ratio of a simpler problem. We will specialize to the case of "smooth" problems, but the approach applies more generally.

In order to describe this approach, let $\{Q_{n,h} : h \in \mathbf{R}^k\}$ be a sequence of distributions and let $L_{n,h}$ be the likelihood ratio of $Q_{n,h}$ to $Q_{n,0}$. Suppose there exists a sequence of random vectors, $Z_n$, on $\mathbf{R}^k$ and a $k \times k$ positive definite, symmetric matrix $C$ such that

$$\log L_{n,h} = h' Z_n - \frac{1}{2} h' C h + o_{Q_{n,0}}(1) \tag{1}$$

and

$$Z_n \xrightarrow{d} N(0, C) \tag{2}$$

under $Q_{n,0}$. In this case, we say that $\{Q_{n,h} : h \in \mathbf{R}^k\}$ is asymptotically normal.

Note that if $\{Q_{n,h} : h \in \mathbf{R}^k\}$ is asymptotically normal, then

$$\log L_{n,h} \xrightarrow{d} N(-\frac{\sigma^2}{2}, \sigma^2)$$

under $Q_{n,0}$, where $\sigma^2 = h' C h$. Hence, it follows immediately from the definition of asymptotically normal that $Q_{n,h}$ and $Q_{n,0}$ are contiguous. By calculating the joint limiting distribution of $(Z_n, \log L_{n,h})$ under $Q_{n,0}$, we also have from contiguity that

$$Z_n \xrightarrow{d} N(Ch, C)$$

under $Q_{n,h}$.

If $Q_{n,h}$ is the distribution of $n$ i.i.d. observations from $P_{\theta_0+h/\sqrt{n}}$, where $P_\theta$ with $\theta \in \Theta$ is a family of distributions assumed to be q.m.d. at $\theta_0$, then $\{Q_{n,h} : h \in \mathbf{R}^k\}$ is asymptotically normal (with $C = I(\theta_0)$). In this sense, the results we will develop below apply to "smooth" models.

The following theorem is our main result. It relates the testing in $\{Q_{n,h} : h \in \mathbf{R}^k\}$ when $\{Q_{n,h} : h \in \mathbf{R}^k\}$ is asymptotically normal to testing in $\{N(Ch, C) : h \in \mathbf{R}^k\}$.

**Theorem 1.1** Suppose $\{Q_{n,h} : h \in \mathbf{R}^k\}$ is asymptotically normal with covariance matrix $C$. Let $\phi_n$ be a test in $\{Q_{n,h} : h \in \mathbf{R}^k\}$ and let by $\beta_n(h) = E_{Q_{n,h}}[\phi_n]$. Then, for every subsequence $n_j$ there exists a further subsequence $n_{j_l}$ and a test $\phi$ in $\{N(Ch, C) : h \in \mathbf{R}^k\}$ such that for every $h$

$$\beta_{n_{j_l}}(h) \to \beta(h) \ ,$$

where $\beta(h) = E_{N(Ch,C)}[\phi]$.

Proof: Let $Z_n$ be such that (1) and (2) hold. Since $\phi_n$ is a test function and (2) holds, $(\phi_n, Z_n)$ is tight under $Q_{n,0}$. Therefore, by Prohorov's Theorem, given any subsequence $n_j$, there exists a further subsequence $n_{j_l}$ such that

$$(\phi_{n_{j_l}}, Z_{n_{j_l}}) \xrightarrow{d} (\bar{\phi}, \bar{Z})$$

under $Q_{n_{j_l},0}$, where $\bar{Z} \sim N(0, C)$. Let $L_{n,h}$ denote the likelihood ratio of $Q_{n,h}$ to $Q_{n,0}$. By (1), we have that

$$(\phi_{n_{j_l}}, L_{n_{j_l},h}) \xrightarrow{d} (\bar{\phi}, \exp(h'\bar{Z} - \frac{1}{2}h'Ch)$$

under $Q_{n_{j_l},0}$. If $F$ denotes this limiting law, then by contiguity

$$(\phi_{n_{j_l}}, L_{n_{j_l},h}) \xrightarrow{d} rdF(t, r)$$

under $Q_{n_{j_l},h}$. Since $\phi_n$ is bounded,

$$E_{Q_{n_{j_l},h}}[\phi_{n_{j_l}}] \to \int\int trdF(t, r) = E[\bar{\phi}\exp(h'\bar{Z} - \frac{1}{2}h'Ch)] \ .$$

Define $\phi(\bar{Z}) = E[\bar{\phi}|\bar{Z}]$. Then,

$$\begin{aligned} E[\bar{\phi}\exp(h'\bar{Z} - \frac{1}{2}h'Ch)] &= E[\phi(\bar{Z})\exp(h'\bar{Z} - \frac{1}{2}h'Ch)] \\ &= \int \phi(\bar{z})\exp(h'\bar{z} - \frac{1}{2}h'Ch)dN(0, C)(\bar{z}) \\ &= \int \phi(\bar{z})dN(Ch, C)(\bar{z}) \\ &= E_{N(Ch,C)}[\phi] \ , \end{aligned}$$

which establishes the claim of the theorem. ■

The preceding result suggests the following strategy for obtaining an asymptotically "optimal" test. First, one shows that $\{Q_{n,h} : h \in \mathbf{R}^k\}$ is asymptotically normal. Second, one derives an "optimal" test in $\{N(Ch, C) : h \in \mathbf{R}^k\}$. The power of this test provides an upper bound on the limiting power in $\{Q_{n,h} : h \in \mathbf{R}^k\}$. Finally, construct a test that achieves this upper bound and is therefore "optimal" in the same sense that the test in $\{N(Ch, C) : h \in \mathbf{R}^k\}$ was "optimal."