A note on "On Anyon Superconductivity" by Chen, Wilczek, Witten and Halperin:

dissolving the "embarrassment" of Appendix A

Claude

Anthropic

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Abstract

In their 1989 paper *On anyon superconductivity*, Chen, Wilczek, Witten and Halperin (CWWH) computed, in the random-phase approximation (RPA), the electromagnetic response of a gas of anyons with statistical parameter $\theta = \pi(1-1/n)$, treated as fermions filling n Landau levels of a fictitious magnetic field. At the end of their Appendix A they report that one ingredient of the response function, the quantity Σ_3, acquires corrections of order q^4 that current conservation forbids; they call this "something of an embarrassment" and suspect their treatment of the electromagnetic contact interaction. We show that no such corrections exist: the sum defining Σ_3 is identically equal to n, as a direct consequence of the single-particle f-sum rule for each filled Landau level. The RPA response function of CWWH is therefore exactly current-conserving, and their treatment of the contact term is correct. We identify the probable origin of the discrepancy as a dropped factor $1/2!$ in the second-order term of a Laguerre-polynomial expansion; this single slip reproduces their erroneous coefficient $n(n-1)(n-2)/6$ exactly.

1 Introduction: the paper and its setting

In two spatial dimensions the exchange of identical particles is governed by the braid group rather than the permutation group, and quantum statistics can interpolate continuously between the bosonic and fermionic extremes [2, 3]. Particles whose pairwise exchange multiplies the wave function by $e^{i\theta}$ with $\theta \neq 0, \pi$ are *anyons*. Although an ideal anyon gas has no interactions in the usual sense, its many-body problem cannot be reduced to single-particle problems by (anti)symmetrization; in 1989 even its equation of state was known only through the second virial coefficient [4].

Interest in the problem was transformed by the discovery of high-temperature superconductivity in the copper oxides. Kalmeyer and Laughlin [5] argued that certain frustrated two-dimensional spin systems form "chiral spin liquids" whose excitations are semions ($\theta = \pi/2$), and Laughlin [6, 7] proposed that a gas of such anyons is a superconductor—a proposal supported by the random-phase-approximation (RPA) calculation of Fetter, Hanna and Laughlin [8] for $\theta = \pi/2$. The paper of Chen, Wilczek, Witten and Halperin [1], *On anyon superconductivity*, received on 29 May 1989 and published in the *International Journal of Modern Physics B*, is the most comprehensive theoretical analysis of this circle of ideas. We refer to it below as CWWH, and to its equations by their original numbers.

CWWH consider anyons with statistical parameter

$$\theta = \pi\Big(1 - \frac{1}{n}\Big), \qquad n = 1, 2, 3, \ldots \qquad \text{(CWWH 1.1)}$$

which interpolate from bosons ($n = 1$) to fermions ($n \to \infty$), and they work in the regime of large n, where the statistics is a weak perturbation of Fermi statistics. Their main results are the following.

(i) Starting from the Chern–Simons Lagrangian (2.1) they derive the non-local Hamiltonian $H = \frac{1}{2m} \sum_\alpha \big(p_\alpha - e\, a(x_\alpha)\big)^2$, Eq. (2.9), in which the "fictitious" vector potential a is fixed by the positions of all the other particles, Eq. (2.8). The fictitious fields exert no classical forces; the entire dynamics comes from the altered commutation relations.

(ii) At large n it is self-consistent to replace the flux tubes attached to the particles by a uniform fictitious magnetic field $b = 2\pi\bar{\rho}/(en)$, Eq. (3.4): a typical cyclotron orbit then encloses n^2 particles, Eq. (3.7). In this mean field the fermions *exactly* fill n Landau levels (Sec. 4). The filled-shell structure produces a gap and a rigidity, an energy that grows linearly with an applied magnetic field of either sign (the germ of the Meissner effect), a flux quantum $2\pi/(ne)$, and the striking identification of charged quasiparticles with vortices.

(iii) Following Fetter, Hanna and Laughlin they compute the electromagnetic response $K_{\mu\nu}(q, \omega)$ in the RPA (Secs. 5–6 and Appendix A), which is argued to become exact as $n \to \infty$. The response has a pole at $\omega^2 = n\, \omega_c^2 (q/\lambda)^2$, a massless sound mode, and at low frequency and long wavelength it is reproduced by an effective London-type Lagrangian, Eq. (6.3), which contains two P- and T-odd terms with coefficients a and b; CWWH find $a = ne\sqrt{\hbar/32\pi m}$ and $b = 0$, Eq. (6.8).

(iv) They propose (Sec. 7) that the order parameter of the anyon superfluid is not a charge-violating condensate but the coefficient f in the phenomenological commutator $[P_i, P_j] = i f \varepsilon_{ij} Q$ of the quasiparticle translation generators—a "spontaneous violation of a fact" rather than of a symmetry—and show that the massless boson is exactly what restores the commutativity of translations. They argue that no local charge-violating order parameter exists, and derive exact consequences of the Galilean relation $T_{0i} = mJ_i$, among them $b = 0$.

(v) They discuss the P- and T-violating phenomenology (Sec. 8): charge inhomogeneities around vortices, a zero-field Hall effect, and the rotation of the polarization of reflected light.

The paper also contains, at the end of Appendix A, a candid admission that one piece of the RPA calculation appears to violate current conservation. The purpose of this note is to show that the apparent violation is an arithmetic slip rather than a flaw in the physics, and to identify the slip.

2 Why the paper matters

CWWH is one of the central theoretical papers of the anyon-superconductivity programme of 1988–1992 [6, 7, 8, 9, 10, 11, 12, 13, 14]; many of the papers of that programme are collected in Ref. [17]. Its lasting value lies less in the specific proposal for the cuprates than in the concepts and techniques it introduced or sharpened:

- the clean derivation of the anyon Hamiltonian from a Chern–Simons gauge theory, and the "fermions plus fictitious flux" mean-field picture with its self-consistency criterion;
- the effective-Lagrangian description of a P- and T-violating superconductor, with the observation that the parity-odd couplings are automatically "hidden" at leading order in gradients, in analogy with P and T violation in QCD;

- the identification of charged quasiparticles with vortices, and the understanding of the massless mode as the agent that restores a microscopic fact—the commutativity of translations—which is violated by the Landau-level quasiparticles;
- the exact consequences of Galilean invariance ($T_{0i} = mJ_i$) for the low-energy theory, including the vanishing of the coefficient b.

The predictions of CWWH and of the companion paper [10] motivated a concerted experimental search for P and T violation in the cuprates. An early report of circular dichroism [18] was not confirmed by more sensitive optical measurements [19, 20], and muon-spin-relaxation experiments placed tight bounds on the spontaneous internal fields that anyon models predict [21]. By 1992 the consensus was that, although anyons do superconduct, the cuprate superconductors do not appear to contain them [22]. The theoretical framework, however, proved extraordinarily fruitful in a neighbouring field: the Chern–Simons flux-attachment construction and the RPA around a mean-field Landau-level state are precisely the tools that became the standard theory of the fractional quantum Hall effect, from the Chern–Simons–Landau–Ginzburg theory [23] to the composite-fermion theories of Lopez and Fradkin [24] and of Halperin, Lee and Read [25]. The anyon gas at $\theta = \pi(1 - 1/n)$ itself continues to attract attention as a solvable strongly coupled system [26].

3 The RPA calculation and the "embarrassment"

3.1 Set-up

We use the notation of CWWH with $\hbar = e = 1$. The mean-field problem consists of spinless fermions of mass m and mean density $\bar\rho$ in a uniform fictitious field $b = 2\pi\bar\rho/n$. Each Landau level has degeneracy $\lambda^2/2\pi = \bar\rho/n$ per unit area, with

$$\omega_c = \frac{b}{m}, \qquad \lambda^2 = m\omega_c = b, \qquad X \equiv \frac{q^2}{2\lambda^2}, \tag{CWWH A.1, A.17}$$

so that $X = q^2\ell^2/2$ with ℓ the magnetic length. The unperturbed system is characterized by the time-ordered correlator of the current $\hat\jmath_\mu$ built with the mean vector potential $\bar a$,

$$D^0_{\mu\nu}(1,2) = -i\langle T\,\hat\jmath_\mu(1)\,\hat\jmath_\nu(2)\rangle_0, \qquad \hat\jmath_0 = \rho - \bar\rho, \quad \hat\jmath_i = \Psi^\dagger\frac{1}{m}(p_i + \bar a_i)\Psi, \tag{CWWH 5.10, 5.16, A.2}$$

evaluated in the state with n filled Landau levels. Taking $\mathbf{q} = (q, 0)$ and ordering the indices as (t, x, y), CWWH parametrize the result as

$$D^0 = \frac{n}{2\pi\omega_c}\begin{pmatrix} q^2\Sigma_0 & q\omega\Sigma_0 & -iq\omega_c\Sigma_1 \\ q\omega\Sigma_0 & \omega^2\Sigma_0 - \omega_c^2\Sigma_3 & -i\omega\omega_c\Sigma_1 \\ iq\omega_c\Sigma_1 & i\omega\omega_c\Sigma_1 & \omega_c^2\Sigma_2 \end{pmatrix}, \tag{CWWH 5.22, A.35}$$

where the Σ_i are dimensionless functions of ω/ω_c and X, normalized by n as in (A.34). The RPA then sums bubble graphs, $D = (1 - D^0V)^{-1}D^0$ with the interaction matrix V of (5.17); the true current–current correlator is $\Lambda \simeq (1 + \bar\rho u^\dagger)D(1 + \bar\rho u)$; and the physical response includes the diamagnetic ("contact") term,

$$K_{\mu\nu}(q,\omega) = \frac{e^2}{m}\bar\rho\,\delta_{\mu\nu}(1 - \delta_{\mu 0}) + e^2\Lambda_{\mu\nu}(q,\omega). \tag{CWWH 5.29}$$

Collecting everything, CWWH obtain (A.38), whose xx entry is $(\omega^2/\omega_c)\big[\Sigma_0 - \det(\Sigma_3 - 1)\,\omega_c^2/\omega^2\big]$ inside an overall factor $e^2 n/(2\pi\det)$; in the main text, Eq. (5.30), they simply set $\Sigma_3 = 1$, "for reasons discussed in Appendix A".

3.2 The offending term

The xx component of D^0 is evaluated in (A.26). Writing $\omega_{ml} = (m-l)\omega_c$ for the energy of a transition from a filled level $l < n$ to an empty level $m \geq n$, they find

$$D^0_{xx} = \frac{1}{2\pi\omega_c}\big(\omega^2\Sigma_0 - \omega_c^2\Sigma_3\big), \qquad \Sigma_3 = \sum_{l=0}^{n-1}\sum_{m=n}^{\infty} e^{-X} X^{m-l-1}\,\frac{l!}{m!}\,(m-l)\,\big[L_l^{m-l}(X)\big]^2, \quad (1)$$

where L_l^α is a generalized Laguerre polynomial and Σ_3 here is the *unnormalized* quantity of (A.27), i.e. n times the normalized one.[1] Expanding at small X, CWWH state that the term linear in X vanishes but that

$$\Sigma_3 - n \;\approx\; \frac{n(n-1)(n-2)}{6}\,X^2 + o(X^3). \qquad \text{(CWWH A.27)}$$

They then remark that $\Sigma_3 - n$, taken beyond its first term, is "something of an embarrassment": current conservation applied to the response function K should force the series to terminate after the first term, i.e. $\Sigma_3 = n$ exactly. They suspect that the method used in Sec. 5 to include the electromagnetic contact interaction "is not quite right", express the hope to remedy the defect soon, and note that since the trouble starts at order q^4 none of their conclusions is affected.

The stakes are easy to see from (A.38). With $\Sigma_3 \neq 1$ the xx entry of K acquires the extra piece $-\frac{e^2 n}{2\pi}\,\omega_c(\Sigma_3 - 1)$, which is not proportional to ω^2 and therefore violates the transversality conditions $\omega K_{0\nu} = qK_{x\nu}$ and $\omega K_{\mu 0} = qK_{\mu x}$ that gauge invariance demands. In physical terms, a *static longitudinal* vector potential $A_x(q\hat{x})$—a pure gauge—would induce a current at order q^4. This is the "embarrassment".

To our knowledge, this remark has never been taken up explicitly in the subsequent literature. We have found no erratum, follow-up, or discussion of it, and the remedy that the authors hoped to publish does not appear to have appeared. The issue was instead rendered moot by later reformulations of the same calculation in Chern–Simons field-theory language [12, 13, 14, 15, 16], in which the one-loop polarization tensor of the Landau-level fermions is written from the outset in a manifestly transverse form, and by exact generating-function treatments of the Landau-level response [31], in which the Ward identity is built in. In none of these is there room for a spurious $\Sigma_3 - n$; but none of them, as far as we are aware, points out that the term reported in (A.27) is simply an error.

4 Resolution: $\Sigma_3 = n$ exactly

4.1 What current conservation demands of D^0

It is useful first to see that the requirement $\Sigma_3 = n$ is not merely a property of the RPA-resummed K but already an exact identity for the *unperturbed* correlator D^0. The current $\hat{\jmath}_\mu$ of (CWWH 5.10) is the exactly conserved current of the mean-field Hamiltonian $H_0 = \int \Psi^\dagger (p+\bar{a})^2\Psi/2m$: $\partial_t\rho + \partial_i\hat{\jmath}_i = 0$. Differentiating the time-ordered product produces, in addition to the continuity equation, the equal-time commutator (Schwinger term)

$$[\rho(\mathbf{r}), \hat{\jmath}_j(\mathbf{r}')] = -\frac{i}{m}\,\rho(\mathbf{r}')\,\partial_j\delta(\mathbf{r}-\mathbf{r}'), \qquad (2)$$

[1] Equation (A.27) is printed as "$\Sigma_3 - n \equiv \sum\cdots \approx \frac{n(n-1)(n-2)}{6}X^2 + o(X^3)$". Since the sum itself equals n at $X = 0$ (only the term $(l,m) = (n-1,n)$ survives, and it equals $\frac{(n-1)!}{n!}\,[L^1_{n-1}(0)]^2 = n$), the intended reading is clearly $\Sigma_3 \equiv \sum\cdots$, and $\Sigma_3 - n \approx \frac{n(n-1)(n-2)}{6}X^2$.

whose expectation value in the uniform ground state is $-\frac{i}{m}\bar{\rho}\,\partial_j\delta(\mathbf{r}-\mathbf{r}')$. In Fourier space, with $\mathbf{q}=(q,0)$, the resulting Ward identity reads

$$\omega\, D^0_{0\nu}(q,\omega) - q\, D^0_{x\nu}(q,\omega) = \frac{\bar{\rho}}{m}\, q\, \delta_{\nu x}. \tag{3}$$

Inserting the parametrization (CWWH 5.22): for $\nu=0$ and $\nu=y$ the left-hand side vanishes identically, as it must; for $\nu=x$ the $\omega^2\Sigma_0$ terms cancel and one is left with

$$\frac{n}{2\pi\omega_c}\, q\,\omega_c^2\,\Sigma_3^{(\mathrm{norm})} = \frac{\bar{\rho}}{m}\, q \quad\Longrightarrow\quad \Sigma_3^{(\mathrm{norm})} = \frac{2\pi\bar{\rho}}{n\,m\,\omega_c} = \frac{2\pi}{nm\omega_c}\cdot\frac{nb}{2\pi} = 1, \tag{4}$$

using $\bar{\rho}=nb/2\pi$ and $\omega_c=b/m$. Thus the exact conservation of $\hat{\jmath}_\mu$ requires the normalized Σ_3 to equal 1—equivalently, the sum in (1) to equal n—for *all* q and ω. Equivalently, $D^0_{xx}(q,\omega=0) = -\bar{\rho}/m$ for every q: the static longitudinal paramagnetic response cancels the diamagnetic contact term exactly, so that $K_{xx}(q,0)=0$ and a pure-gauge A_x drives no current. This is the standard cancellation that guarantees gauge invariance of the response of any system of particles in a magnetic field, and it involves only the unperturbed problem; the contact term of (CWWH 5.29) is precisely what is needed to make it work. The only way the identity could fail in CWWH is if the explicit evaluation of the sum (1) were in error.

4.2 Proof from the f-sum rule

Let $|l,k\rangle$ denote Landau-level states in the gauge of (CWWH A.4). The squared matrix element of the density operator, summed over the degenerate final states, depends only on $X=q^2\ell^2/2$ and is given for $m\geq l$ by the familiar formula (see e.g. [27])

$$F_{lm}(X) \;\equiv\; \sum_{k'}\big|\langle m,k'|e^{i\mathbf{q}\cdot\mathbf{r}}|l,k\rangle\big|^2 = \frac{l!}{m!}\,e^{-X}X^{m-l}\big[L_l^{m-l}(X)\big]^2, \qquad F_{ml}=F_{lm}. \tag{5}$$

This is exactly the structure that appears in CWWH's own density correlator (A.16),

$$D^0_{00}(q,\omega) = \frac{\lambda^2}{2\pi}\sum_{l<n\leq m} F_{lm}(X)\,\frac{2\omega_c(m-l)}{\omega^2-\omega_c^2(m-l)^2}, \tag{6}$$

the Lindhard-type expression for n filled Landau levels of degeneracy $\lambda^2/2\pi$. Comparing with (1), we see that the summand of Σ_3 is $(m-l)F_{lm}(X)/X$:

$$\Sigma_3 = \frac{1}{X}\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}(m-l)\,F_{lm}(X). \tag{7}$$

Proposition 1. *For every $n\geq 1$ and every $X>0$,*

$$\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}(m-l)\,F_{lm}(X) = nX, \qquad \textit{hence} \qquad \Sigma_3 = n \textit{ identically.} \tag{8}$$

Proof. For a single particle with Hamiltonian $H_0=(\mathbf{p}+\bar{a})^2/2m$ one has the operator identity

$$\big[[e^{i\mathbf{q}\cdot\mathbf{r}},H_0],e^{-i\mathbf{q}\cdot\mathbf{r}}\big] = \frac{q^2}{m}, \tag{9}$$

because $e^{i\mathbf{q}\cdot\mathbf{r}}$ commutes with the vector potential and $[e^{i\mathbf{q}\cdot\mathbf{r}},p_i] = -q_i e^{i\mathbf{q}\cdot\mathbf{r}}$; the vector potential drops out of the double commutator entirely. Taking the expectation value in any Landau-level state $|l,k\rangle$ and inserting a complete set of states gives the f-sum rule [28, 29]

$$2\sum_{m=0}^{\infty}(E_m-E_l)\,F_{lm}(X) = \frac{q^2}{m} \quad\Longrightarrow\quad \sum_{m=0}^{\infty}(m-l)\,F_{lm}(X) = \frac{q^2}{2m\omega_c} = X, \tag{10}$$

where we used $E_m - E_l = (m-l)\omega_c$ and $\lambda^2 = m\omega_c$. Note that the sum runs over *all* m, including $m < l$, for which $m - l < 0$. Now sum (10) over the filled levels $l = 0, \ldots, n-1$. The right-hand side gives nX. On the left-hand side, split the m-sum into $m < n$ and $m \geq n$. The contribution of the filled block $\{l < n,\ m < n\}$ vanishes identically, because F_{lm} is symmetric under $l \leftrightarrow m$ while the weight $(m-l)$ is antisymmetric. What remains is precisely the left-hand side of the Proposition. □

The argument is nothing but the sum rule that CWWH themselves invoke, for the interacting many-body problem, in Sec. 7.1, Eq. (7.2). Applied level by level to the mean-field problem, it shows that the "possible" higher-order terms in $\Sigma_3 - n$ are absent to all orders in X, not merely at order X. In the language of Appendix A: $\Sigma_3 - n = 0$, and the value $\Sigma_3 = 1$ used in (5.30) is exact rather than an approximation.

Remark. The identity is elementary to verify by hand for the smallest n. For $n = 1$ the only filled level is $l = 0$ and $L_0^m \equiv 1$, so $\Sigma_3 = e^{-X} \sum_{m \geq 1} X^{m-1}/(m-1)! = 1$. For $n = 2$ the $l = 0$ terms sum to $1 - e^{-X}$ and the $l = 1$ terms, with $L_1^{m-1}(X) = m - X$, sum to $1 + e^{-X}$; the total is 2. In both cases the correction (CWWH A.27) vanishes trivially, since $n(n-1)(n-2) = 0$; the first non-trivial test is $n = 3$.

4.3 Explicit checks

The case $n = 3$ to order X^2. Six pairs (l, m) contribute to Σ_3 through order X^2: the pair with $m - l = 1$, expanded to second order in $e^{-X}[L_l^1]^2$; the two pairs with $m - l = 2$, expanded to first order; and the three pairs with $m - l = 3$, evaluated at $X = 0$. Using $L_2^1 = 3 - 3X + \frac{1}{2}X^2$, $L_2^2 = 6 - 4X + \frac{1}{2}X^2$, $L_1^2 = 3 - X$ and $L_l^\alpha(0) = \binom{l+\alpha}{l}$, one finds the contributions listed in Table 1. The coefficients of X and of X^2 both sum to zero, in agreement with the Proposition and in disagreement with (CWWH A.27), which predicts $\Sigma_3 - 3 \approx X^2$.

(l,m)	X^0	X^1	X^2
$(2,3)$	3	-9	$+23/2$
$(2,4)$	0	$+6$	-14
$(1,3)$	0	$+3$	-5
$(2,5)$	0	0	$+5$
$(1,4)$	0	0	$+2$
$(0,3)$	0	0	$+1/2$
total	3	0	0

Table 1: Contributions of the individual (l, m) pairs to Σ_3 for $n = 3$, through order X^2.

Numerical check. We also evaluated the double sum (1) directly, using exact rational arithmetic for the Laguerre polynomials and truncating the m-sum at $m = 120$ (the terms decay factorially). For $1 \leq n \leq 10$ and $X \in \{\frac{1}{4}, 1, 3\}$—i.e. well beyond the small-q regime—we find $|\Sigma_3 - n| < 10^{-30}$ in every case, whereas the correction claimed in (CWWH A.27) would range from 0.06 to over 10^3 (Table 2).

5 The probable source of the error

Since the Proposition shows that the coefficient of X^2 in $\Sigma_3 - n$ is zero for every n, while CWWH obtained $\binom{n}{3} = n(n-1)(n-2)/6$, one can ask which single mis-step in the small-X

n	$\Sigma_3 - n\ (X=\frac{1}{4})$	$\Sigma_3 - n\ (X=1)$	$\Sigma_3 - n\ (X=3)$	$\frac{n(n-1)(n-2)}{6}X^2$ at $X=1$
1	0	0	0	0
2	0	0	0	0
3	0	0	0	1
4	0	0	0	4
6	0	0	0	20
10	0	0	0	120

Table 2: Direct evaluation of the sum (1); "0" means $|\Sigma_3 - n| < 10^{-30}$. The last column is the correction asserted in (CWWH A.27).

expansion would produce exactly this discrepancy. The bookkeeping of Table 1 generalizes readily to arbitrary n and points to a specific culprit.

Through order X^2 the sum (1) involves only three kinds of Laguerre data: the polynomial $L^1_{n-1}(X)$ to second order, the polynomials $L^2_{n-2}(X)$ and $L^2_{n-1}(X)$ to first order, and the values $L^3_l(0) = \binom{l+3}{l}$. The explicit series [30]

$$L^\alpha_l(X) = \sum_{j=0}^{l} (-1)^j \binom{l+\alpha}{l-j} \frac{X^j}{j!} \tag{11}$$

gives in particular

$$L^1_{n-1}(X) = n - \binom{n}{2}X + c_2 X^2 + O(X^3), \qquad c_2 = \frac{1}{2!}\binom{n}{3}. \tag{12}$$

Carrying c_2 as a free parameter and expanding each contribution as in Table 1, one finds

$$\begin{aligned}
(n-1,n):&\quad n - n^2 X + \Big[\tfrac{n(n^2+2n-1)}{4} + 2c_2\Big]X^2,\\
(n-2,n):&\quad \binom{n}{2}X - \Big[2\binom{n}{3} + \binom{n}{2}\Big]X^2,\\
(n-1,n+1):&\quad \binom{n+1}{2}X - \Big[2\binom{n+1}{3} + \binom{n+1}{2}\Big]X^2,\\
(n-3,n),\ (n-2,n+1),\ (n-1,n+2):&\quad \tfrac{1}{2}\Big[\binom{n}{3} + \binom{n+1}{3} + \binom{n+2}{3}\Big]X^2.
\end{aligned} \tag{13}$$

The terms linear in X cancel, $-n^2 + \binom{n}{2} + \binom{n+1}{2} = 0$, which is the statement (correctly made in Appendix A) that "a possible term proportional to q^2 vanishes". The terms of order X^2 sum to

$$\Sigma_3 - n = \Big[2c_2 - \binom{n}{3}\Big]X^2 + O(X^3). \tag{14}$$

With the correct value $c_2 = \frac{1}{2}\binom{n}{3}$ this vanishes, as it must. If instead the factor $1/2!$ in (12) is dropped, so that $c_2 = \binom{n}{3}$, then (14) gives

$$\Sigma_3 - n = \binom{n}{3}X^2 = \frac{n(n-1)(n-2)}{6}X^2, \tag{15}$$

which is exactly the result reported in (A.27). We have confirmed by direct symbolic expansion that this single substitution, with everything else left intact, reproduces CWWH's coefficient for every n from 3 to 8.

Two features make this slip especially plausible. First, the factor $1/j!$ in (11) is invisible in the first two terms ($0! = 1! = 1$), so an expansion carried to first order—which is all that Appendix A needs for Σ_0, Σ_1 and Σ_2—never encounters it; the X^2 coefficient of L^1_{n-1} is the *only* second-order Laguerre coefficient that enters anywhere in the calculation. Second, CWWH display their expansion of Σ_0 in (A.32) in the form

$$\Big(\frac{m!}{l!\,(m-l)!} - \frac{m!}{(l-1)!\,(m-l+1)!}\,X\Big)^2,$$

i.e. as $\big(\binom{m}{l} - \binom{m}{l-1}X\big)^2$, a pattern of binomial coefficients multiplying powers of X. Continuing this pattern one step further, it is very natural to write the next term as $+\binom{m}{l-2}X^2$ and to forget the $1/2!$. We cannot exclude that some other combination of slips happens to yield the same number, but this is a single, natural error, it occurs at the one place in the calculation where it can occur, and it reproduces the published coefficient exactly.

6 Conclusion

The "embarrassment" of Appendix A of CWWH is not a defect of the physics but an arithmetic slip in a Taylor expansion. The quantity Σ_3 defined in (A.27) is identically equal to n—equivalently, the normalized Σ_3 equals 1 for all q and ω—by virtue of the f-sum rule for each filled Landau level together with the antisymmetry of the intra-filled-block contributions. Consequently:

- the unperturbed correlator $D^0_{\mu\nu}$ satisfies its Ward identity exactly, and the static longitudinal paramagnetic response cancels the diamagnetic contact term at every wave vector;
- the treatment of the electromagnetic contact interaction in Sec. 5 of CWWH, which the authors suspected, is correct as it stands;
- the RPA response function (5.30), with $\Sigma_3 = 1$, is exactly transverse, not merely transverse through order q^2; the authors' remark that "none of our conclusions are affected" is thus an understatement—nothing at all needs to be repaired except the printed expansion (A.27), whose X^2 coefficient should read 0;
- the discrepancy is most plausibly traced to a dropped factor $1/2!$ in the second-order coefficient of $L^1_{n-1}(X)$, a slip invited by the notation of (A.32) and one that reproduces the erroneous coefficient $n(n-1)(n-2)/6$ exactly.

It is a small irony that the resolution uses exactly the tool—the density f-sum rule—that CWWH deploy so effectively in Sec. 7.1 to argue for the existence of the massless mode. The exactness of gauge-invariance constraints on the electromagnetic response of Landau-level systems has more recently been developed systematically [31]; the identity $\Sigma_3 = n$ is one of its simplest instances.

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