Advanced Statistical Mechanics

topics: Theory of Phase Transitions and Renormalization Group


Website: http://home.uchicago.edu/dtson/phys353/
① Second order phase transition.

Phases of water

\[ \text{gas} \quad \text{solid} \quad \text{liquid} \]

Transition between phases with different symmetries cannot occur in a continuous fashion. Example: solid - liquid

ferromagnetism - paramagnetism

We will be mostly concerned with phase transitions with symmetry change:

solid: translational symmetry

liquid: rotational symmetry

ferro-para M: rotational symmetry

One phase is the ordered phase;

the other is disordered phase

[There are situations where one can almost rule out continuous transitions between 2 phases with the same symmetry (topological phase transitions), we shall not talk about these at the end of the quarter of there is time.]

We can imagine situations in which the transition is continuous &
example is the ferro-paramagnetic phase transition
average magnetization = 0 in one phase (para)
≠ 0 in another phase (ferro)
but is continuous function of \( T \) at the critical temperature

\[ T_c \]

Second order

but that does not mean that there is no phase transition
In fact, there must be a phase transition point \( T_c \)
\( \langle M \rangle = 0 \) from \( \langle M \rangle \neq 0 \) and that is where the pt is

The second order pt is different from first order

\[ \langle M \rangle \]

normally associated with discontinuity in thermodynamic function.

It turns out that the second order phase transitions are most interesting. As we will see, there are many aspects of such phase transitions that are insensitive to the details of the model → universality

Great progress understanding universality in 1970s Renormalization Group

We will try to understand it.
The role of Models

In stat. mech. we frequently replace the physical problem (i.e. iron atoms in a lattice) with a simplified model (i.e. spin model of one of the types considered later, ignoring electronic excitation, lattice vibrations) etc.

As in any other subfields of physics, if one chooses models with enough of essential features, one would understand qualitative (sometimes quantitative) features of the physical system

In SM due to universality of the 2nd order PT's the situation is even better. Here it turns out one can obtain all universal properties of the physical system by studying a simple model. For example one cannot accurately predict the Curie temperature $T_c$ but one knows what is the behavior of the critical magnetization near $T_c$: $M(T) \sim (T_c-T)^\beta$ universal.

We thus will start with the simplest model: the Ising model, postponing the discussion of the universality of the obtained result to much later. For a moment, we ask you to believe me that what we obtain will be much useful beyond the model. On another hand you can also enjoy the model in itself.
The Ising model is given by

\[ H = -J \sum_{\langle ij \rangle} S_i S_j \]

\( \langle ij \rangle \) = neighboring sites

Interaction between two sides

\[ J S_i S_j \]

the choice of sign — and \( J > 0 \) favors neighboring spins to point to the same direction.

Note: one can generalize the model in several directions.

1) One can include interaction beyond nearest neighbors, i.e.

\[ H[S] = H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \] and assume for example that

\[ J_{ij} = \frac{1}{|x_i - x_j|^\gamma} \] For \( J_{ij} \) decaying fast enough it will not change the properties of the P.T. much

2) one can consider spins pointing not just along the z axis.

\[ H = -\sum_{\langle ij \rangle} S_i \cdot S_j \quad \vec{S} = (S_x, S_y, S_z), \quad S^2 = 1. \]

This is the Heisenberg model whose behavior may be qualitatively different from the Ising model.

Heuristic considerations:

when \( T < J \): the system is in one of two states \( k_B = 1 \)
all spin up \( \uparrow \uparrow \uparrow \) or all down.
When $T \gg J$ on the other hand the thermal fluctuations have enough energy to randomize the spins, hence the system would be in disordered phase.

Thus we expect a phase transition at some $T_c \sim OCJ$. As we will see this is almost correct, most but not all, the times.

We shall first try to understand the Ising model using a very useful but crude simple, but not exact, method called the mean-field approximation.

The idea of the MF approximation is as followed. We first add

$$H[S_i] = H[-S_i]$$

and then $\langle S_i \rangle = 0$.

$$\langle S_i \rangle = \frac{1}{Z} \sum_{S} e^{-\beta H[S_i]} S_i = \frac{1}{Z} \sum_{S} e^{-\beta H[S_i]} (S_0) = \langle S_i \rangle$$

So what is the meaning of the this happens even at zero temperature. The $\uparrow \uparrow \downarrow \downarrow$ and $\uparrow \downarrow \downarrow \uparrow$ have the same weight $1/4$. Since $\langle S_i \rangle = 0$ on any finite volume, taking $V \rightarrow 0$ limit does not help.

So what is the meaning of the notion of phase transition? In order to make this precise, imagine we turn on a finite magnetic field on the system

$$H = -\sum_{ij} S_i S_j - h \sum_i S_i$$
So now for any \( h \) one configuration of spin \( \mathbf{S} \) will be split energetically from the any other spin reversed direction. This allows us to consider the limit of infinite volume at fixed \( h \):

\[
S(h) = \lim_{V \to \infty} \langle S \rangle
\]

and then take the limit \( h \to 0 \). The symmetry \( S_i \rightarrow -S_i \) is now explicitly broken by the magnetic field \( h \). In general \( \langle S \rangle \) configurations of \( \langle S \rangle \) and \( \langle -S \rangle \) would have their energies split by an infinitely large amount (in the \( V \to \infty \) limit).

Thus \( \langle S \rangle \) the magnetization has to be defined as

\[
\langle S \rangle = \lim_{V \to \infty} \lim_{h \to 0} \langle S \rangle
\]

but not vice versa: \( \lim_{h \to 0} \lim_{V \to \infty} \langle S \rangle = 0 \).

Assume that the average magnetization of the system is

\[
\langle S_i \rangle = m_i
\]

Then, we look at one particular spin \( S_i \). It interacts with its nearest neighbors. In the real system the spins are all coupled to each other and their thermal fluctuations cannot be considered independent. But in the MFA we replace all the spin neighboring our chosen spin by the average spin \( \langle S \rangle \) and write a Hamiltonian for just one spin

\[
H = -J \sum_{\langle ij \rangle} S_i S_j - h \sum S_i \Rightarrow H_{\text{spin}} = -qJmS - hS
\]

Now we use \( H_{\text{spin}} \) to compute the thermal expectation value of the spin \( S \)

\[
\langle S \rangle = \frac{e^{\beta (qJm + h)} - e^{-\beta (qJm + h)}}{e^{\beta (qJm + h)} + e^{-\beta (qJm + h)}} = \tanh \left[ \beta (qJm + h) \right]
\]
Self consistency requires that \( \langle S \rangle = m \) hence
\[
m = \tanh \beta (qJm + h)
\]

Consider \( h = 0 \) then here the equation is illustrated by

\[
\tanh \beta qJm
\]

two possibilities are 2

if \( \beta qJ < 1 \) then there is only one solution \( m = 0 \)
if \( \beta qJ > 1 \) then there are 3 solutions \( \) thus one can have spontaneous magnetization here.

This means a phase transition at \( T_\text{c} = \frac{1}{qJ} \)

We now investigate the behavior of \( \langle m \rangle \) near \( T = T_\text{c} \)

We slightly less than \( T_\text{c} \), \( \beta qJ = 1 + \varepsilon \).

In this case the magnetization should be small \( m \ll 1 \) and we can expand \( \tanh x = x - \frac{x^3}{3} \)

\[
m = \beta qJm - \frac{1}{3}(\beta qJ)^3 m^3 \Rightarrow \frac{1}{3}m^3 = \beta qJ - 1
\]

\[
m = \sqrt[3]{3(\beta qJ - 1)} \sim (T_\text{c} - T)^{1/2}
\]

As we will see this behavior is typical for all mean-field theories.

Before going further we reformulate the mean field theory in a different way (this is called the Bragg-Williams approximation).
we thing about the a state in which each spin can be either +1 with probability $\frac{1+m}{2}$ 
-1 with probability $\frac{1-m}{2}$,

and look at the free energy of the system.

The energy is $-J \sum_i \delta_i S_i = -J N q \frac{m^2}{2} \text{ double counting}$
The entropy of each spin is $S = -\sum_i p_i \ln p_i = 
= - \frac{1+m}{2} \ln \frac{1+m}{2} - \frac{1-m}{2} \ln \frac{1-m}{2}$.

[HW] derive that by counting states.

So the free energy is

$$F = E - TS = N \left[ -J q \frac{m^2}{2} + \frac{T}{2} \left( \frac{1+m}{2} \ln \frac{1+m}{2} - \frac{1-m}{2} \ln \frac{1-m}{2} \right) \right].$$

Let us analyze the behavior near $T=0$.

First $F(m)$ is an even function of $m$. $F(m) = F(-m)$.

$$m(\pm m) = m = \frac{m^2}{2} + \frac{m^3}{2} - \frac{m^4}{4} + \ldots$$

$$\frac{F}{N} = e\exp + \frac{1}{2} \left( T - qJ \right) \frac{m^2}{2} + \frac{T}{12} m^4, \quad F = \left( T - qJ \right) m + \frac{T}{3} m^3$$

$T > T_c : \quad T_c = qJ$ depending on $T < T_c$ or $T > T_c$.

$T < T_c$, the function $F(m)$ has

1. minimum at $m=0 \quad T > T_c$

3. extrema at $m = 0$ and $m = \pm \sqrt{3 (qJ - T) \frac{J}{T}}$ at $m = 0 \quad T < T_c$

among which $m=0$ is a maximum and does not correspond to the physical states, and the other minima $(m=0)$
Landau's theory of the second order phase transition

The Landau theory of the 2nd order p.t. can be considered as an extension of the Peierls-Bragg-Williams approximation to the Ising model. It relies on the notion of the free energy as a function of the order parameter. In the case of the Ising model the steps toward constructing the Landau theory would be as follows:

1) The two phases, para- and ferromagnetic, differ from each other by the order parameter magnetization. Thus we identify the order parameter $m$ as the order parameter.

2) We assume there exist a free energy $F$ as function of the order parameter $m$ and the temperature $F(T,m)$.

3) We now impose the symmetry constraints on the form of the function $F(T,m)$. In the case of the Ising model the symmetry is with respect to the sign flip $m \rightarrow -m$ so $F(T,m) = F(T,-m)$.

We now analyze the function $F(T,m)$ by expanding it around $m = 0$:

$$ F(T,m) = F_0(T) + a(T)m^2 + b(T)m^4 + c(T)m^6 + \ldots $$

(Odd powers of $m$ are forbidden by symmetry.) Now suppose $a(T)$ changes sign at some $T$, say from being positive at $T > T_c$ to negative at $T < T_c$.

$$ a(T) = \begin{cases} \infty & \text{for } T > T_c \\ (T - T_c)^2 & \text{for } T < T_c \end{cases} $$

Then the $m = 0$ cannot be a minimum for $T < T_c$. 
Suppose that now near $T = T_c$, $b(T)$ has one well-defined sign. It is reasonable since it would be generally unusual for the coincidence of $b(T)$ to vanish at the same place where $a(T)$ vanishes. In other words, we can tune one "knob" which is $T$ and make one function $a(T)$ to vanish at $T = T_c$, but it would be unreasonable to assume that we can tune two knobs and get both $a(T)$ and $b(T)$ to vanish at the same time. Let $b(T_c) = b_o > 0$. Then the free energy near $T_c$ would be

$$F(T, m) = F_0(T) + a_0(T - T_c) m^2 + b_0 m^4.$$  

For $T > T_c$, the minimum of $F$ is at $m = 0$. For $T < T_c$

$$F(T, m)$$

the minimum is at

$$m = \sqrt{\frac{2b_0}{2a_0}} \sqrt{T_c - T}.$$  

We write:

$$m = (T_c - T) \beta$$  

with $\beta = \frac{1}{2}$.

This is the first "critical exponent" that we encounter.

Let us now compute the thermodynamic free energy. This is the value of $F(T, m)$ at the minimum. Thus

$$T > T_c \quad F(T, m) = F_0(T)$$

$$T < T_c \quad F(T, m) = F_0(T) - \frac{a_0^2}{4b_0} (T_c - T)^2.$$  

although the free energy is continuous at $T = T_c$ as its first derivative. However, the second derivative is discontinuous. Physically, the specific heat has discontinuity. Recall that

$$dF = -S dT \Rightarrow$$

$$C_v = T \frac{dS}{dT} = -T \frac{d^2 F}{dT^2} \Rightarrow C_v \bigg|_{T = T_c + \epsilon} - C_v \bigg|_{T = T_c - \epsilon} = \frac{a_0^2}{2b_0} \frac{1}{T_c}.$$  

(assuming $F_0(T)$ and all its derivatives are continuous.)
this is how $C_v(T)$ behaves as a function of temperature according to mean field theory.

$C_v \sim (T_c - T)^{\alpha}$  \hspace{1cm} $\alpha = 0$ in mean field theory.

Another critical exponent is found in the behavior of the magnetic susceptibility, defined as $\chi = (\frac{\partial m}{\partial h})_{T=0}$.

In the disordered phase ($T>T_c$)

$$F = \alpha_0 (T-T_c) m^2 + k_0 m^4 - h m$$

$$\frac{\partial F}{\partial m} = 2 \alpha_0 (T-T_c) m - h = 0 \Rightarrow m = \frac{h}{2 \alpha_0 (T-T_c)}$$

![Graph]

\[ \chi \sim \left( \frac{1}{T-T_c} \right)^{\delta} \hspace{1cm} \delta = 4 \]

H.W. problem: find the behavior of $\chi$ in the ordered phase.

One more critical exponent is obtained by looking at magnetization as a function of the magnetic field at exactly at critical temperature where we have to minimize $F = k_0 m^4 - h m \Rightarrow m = \left( \frac{h}{4 k_0} \right)^{1/3} \sim h^{1/3}$

$$m \sim \left( \frac{h}{4 k_0} \right)^{1/3} \sim h^{1/3} \hspace{1cm} \delta = 3 \text{ for mean-field.}$$

To summarize we have considered 4 critical exponents:

1) Specific heat $C_v \sim (T-T_c)^{\alpha}$  \hspace{1cm} $\alpha = 0$ (3D true results)

2) Order parameter $m \sim (T_c - T)^{\beta} \hspace{1cm} \beta = \frac{1}{2}$ (0.31)

3) Susceptibility $\chi \sim T^{-\gamma} \hspace{1cm} \gamma = 4$ (1.25) $T = T_c - T_c$

4) $T=0$ EOS $\hspace{1cm} m \sim h^{1/\delta} \hspace{1cm} \delta = 3$ (25)
Further notes about the Landau theory of phase transitions.

1) We have assumed that at $T=T_c$ the coefficient $b(T)$ of $m^4$ is positive. If inversely $b(T) < 0$ then the phase transition is first order. Let's see that by assuming the coefficient of $m^6$ is positive. Then

\[ F = \frac{1}{2} b(T) m^4 + c m^6 \]

For $T<T_c$

There are three minima, which correspond to the stable thermal equilibrium. The one(s) with the smallest free energy is a temperature $T_1$ (smaller than $T_c$) where there is a finite jump of the order parameter from a nonzero value to 0. This is a first order phase transition.

2) We can imagine physical system where one can tune not one parameter (like temperature $T$) but two (temperature and pressure $P$, for example). In this case it is possible that at some the nature of the phase transition changes as $P$ changes. For example, we can imagine that at $P<P_0$ the phase transition is first order (i.e. when $a(T)=0$, $b(T)<0$) and for $P>P_0$ the phase transition is second order. Where $a(T)>0$ when $b(T)>0$. 
Then the phase diagram would look something like

![Diagram showing phase transitions](image)

2nd order

1st order

Landau functional & constant

and is called a tricritical point.

HW

3)

Most of the times, it is impossible to rule out a first-order phase transition. In many cases, symmetry considerations allow the phase transition to be second-order, but detailed (which depend on microscopic physics) implies a first-order transition.

We have seen that first-order phase transition can occur when & F is not required by symmetries (i.e., the liquid-gas phase transition).

However, there are cases when one can actually rule out a second-order phase transition from symmetry considerations alone. These are the most interesting cases. We consider here one of these cases.

The case that we consider is the so-called Potts model. We consider a spin that can be in one of three possible states:

\[ S = \hat{S}_1 \text{ or } \hat{S}_2 \]

and the Hamiltonian is

\[ H = -J \sum_{i,j} \hat{S}_i \cdot \hat{S}_j \]

(sometimes this is called the three-state Potts model. Three refers to the number of states each spin can be in).
Develop the molecular-field (Briggs-William) approximation for the three-state Potts model.

We can think about the order parameter as a vector in 2-dim space, or a complex number $z$.

The Landau free energy is then a function of $T$ and $z$: $F = F(T, z)$. There is a $Z_3$ symmetry of the model: $H = -J \sum_j \hat{S}_j \hat{S}_j$ is invariant when we take all spins and rotate by 120°. This corresponds to $z \rightarrow z e^{2\pi i/3}$.

So we have to assume $F(z) = F(z e^{2\pi i/3})$.

The following terms are invariant under the rotations:

$|z|^2, \overline{z}^2 z$ and $|z|^4, (\overline{z}^2 z)^2$ (and any power of $\overline{z} z$).

But there is another invariant $z^3 \overline{z^3}$ (and $\overline{z^3}$).

Therefore, we can write:

$$F(z) = a |z|^2 + b |z|^4 + \text{Re}(z^3 \overline{z^3})$$

This new term is called "cubic invariant" and it is not possible to make it out based on the $Z_3$ symmetry. Therefore generically the coefficient of this term should be non-zero. Now we can show that the temperature at which $a = 0$ is not the point of a 2nd order phase transition.

In fact if $a = 0$, there is a direction in the complex plane where $z^3 \overline{z^3}$ term does not vanish, and the potential looks like this

with minimum shifted from $z = 0$. For a slightly
positive the free energy has the form

\[ \text{with two minima, one at } z=0 \text{ the other at } z \neq 0. \] This means the phase transition from \( z=0 \) to \( z \neq 0 \) is necessarily first-order.

This fact is confirmed in numerical simulation of the three-state Potts model. In three-dimensional cubic lattice, the Potts model has a first-order phase transition. However, in 2D the Potts model has a second-order phase transition instead. Here we see the first indication of the fact that the mean field theory (Landau) is worse at lower dimensions.

One-dimensional Ising chain.

The 1D Ising chain model (Ising chain) can be solved exactly without very much effort (2D is also exactly solvable but the solution is much more involved, we will see). The method used in solving the 1D Ising chain will be useful in 2D case.

First let us remind ourselves that the Hamiltonian of the 1D chain is

\[ H = -J \sum_{i=1}^{N-1} S_i S_{i+1} - h \sum_{i=1}^{N} S_i \]

and if we can compute the partition function,

\[ Z[T, h] = \sum_{S_1 = \pm 1} \sum_{S_2 = \pm 1} \ldots \sum_{S_N = \pm 1} e^{-\beta H[S_i]} \]

then we would find everything. For example the free energy \( F = -T \ln Z \), entropy \( S = \frac{\partial F}{\partial T} \), specific heat, magnetization \( M = T \frac{\partial}{\partial h} \ln Z \). (Remind yourselves of SM material or any of these formulas are not clear.)
Before computing \( Z \) we are going to make one change to our Ising chain. This change will not affect anything in the limit \( N \to \infty \).

Instead of a chain we will make it a close circle of spins.

In other words, we define spin at \( i = 0 \) and impose a periodic boundary condition \( S_0 = S_N \) at the two ends. This affects only two spins in the chain, so heuristically speaking, if we have a long chain, whatever we do at the ends will have an effect that goes down like \( 1/N \) in the large-\( N \) limit.

The Hamiltonian is

\[
\mathcal{H} = -J \sum_{i=0}^{N-1} S_i S_{i+1} - h \sum_{i=0}^{N-1} S_i \quad S_0 = S_N
\]

we are going to break it into \( N \) pieces, each piece one "link" and one half from each end

\[
\mathcal{H} = \sum_{i=0}^{N-1} \mathcal{H}_i \quad \mathcal{H}_i = -J S_i S_{i+1} - \frac{h}{2} (S_i + S_{i+1})
\]

\[
\mathcal{H}[S_i, S_{i+1}]
\]

Each of \( \mathcal{H}_i \) depends only on two neighboring spins, so for any \( \{S_i\} \)

\[
Z = \sum_{\{S_i = \pm 1\}} e^{\beta \mathcal{H}[\{S_i\}]} = \sum_{\{S_i = \pm 1\}} e^{-\beta \mathcal{H}_0 [S_0, S_1]} e^{-\beta \mathcal{H}_1 [S_1, S_2]} e^{\beta \mathcal{H}_N [S_{N-1}, S_N]} e^{\beta \mathcal{H}_i [S_i, S_{i+1}]} e^{\beta \mathcal{H}_i [S_{i+1}, S_i]} e^{\beta \mathcal{H}_N [S_{N-1}, S_N]}
\]

Let us look at \( T_{S_1 S_2} = e^{-\beta \mathcal{H}[S_1, S_2]} \). This can be thought of as a 2x2 matrix consisting of elements

\[
\begin{pmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{pmatrix}
\]

where 1 corresponds to \( S = +1 \), 2 corresponds to \( S = -1 \).

Then

\[
Z = \sum_{\{S_i\}} T_{S_1 S_2} T_{S_2 S_3} T_{S_3 S_4} \cdots T_{S_{N-1} S_N} T_{S_N S_1} = \operatorname{tr} T^N
\]
Now let's look at $T$. This is called the "transfer matrix".

\[
T = \begin{pmatrix}
  e^{\beta J} & e^{-\beta J} \\
  e^{-\beta J} & e^{\beta (J-h)}
\end{pmatrix}
\]

This is a symmetric matrix; thus there exist a basis in which $T$ is diagonal. The eigenvalues are $\lambda_1, \lambda_2$. Define

\[
T^N = \lambda_1^N + \lambda_2^N,
\]

To find $\lambda_{1/2}$ we have to solve

\[
\det |T-\lambda I| = \begin{vmatrix}
  e^{\beta (J+h)} - \lambda & e^{-\beta T} \\
  e^{-\beta J} & e^{\beta (J-h)} - \lambda
\end{vmatrix} = 0
\]

or

\[
\lambda^2 - 2e^{\beta J} \cosh \beta h \lambda + e^{2\beta J} - e^{-2\beta J} = 0
\]

\[
\lambda = e^{\beta J} \cosh \beta h \pm \sqrt{e^{2\beta J} \cosh^2 \beta h - 2 \sinh(2\beta J)}
\]

Maybe I can transform the expression inside the square root.

\[
e^{2\beta J} \left( e^{\beta h} + e^{-\beta h} \right)^2 - e^{2\beta J} + e^{-2\beta J}
\]

\[
e^{2\beta J} \sinh^2 \beta h + e^{-2\beta J} \lambda = e^{\beta J} \cosh \beta h \pm \sqrt{e^{2\beta J} \sinh^2 \beta h + e^{-2\beta J}}
\]

So let us take the limit $N \to \infty$. $\lambda_+$ dominates.

\[
\mathcal{F} = \lambda_+ \Rightarrow \mathcal{F}(T,h) = -NT \ln \lambda_+
\]

\[
\mathcal{F}(T,h) = -NT \ln \left[ e^{\beta J} \cosh \beta h \pm \sqrt{e^{2\beta J} \sinh^2 \beta h + e^{-2\beta J}} \right]
\]

Let us look at small $h$ regime:

\[
\mathcal{F}(T,h) = -NT \ln \left[ e^{\beta J} \left(1+\frac{\beta h^2}{2}\right) \pm \sqrt{\beta h e^{-\beta J} \left(1+\frac{1}{2} e^{\beta J} (\beta h)^2\right)} \right]
\]

So at small $h$ it's $O(h^2)$ ($M=0$ at small $h$): no symmetry breaking at any temperature!
It's look at small temperature $\beta J \gg 1, \ T \ll J$

then if $\beta h \gg e^{-\beta J}$, we have to be very large $h$. because $e^{-\beta J}$ is exponentially small, then

$$F(T, h) = -NT \ln \left( e^{\beta J} \cosh \beta h + e^{\beta J} \sinh \beta h \right), \ h > 0$$

$$= -NT \ln \left( e^{\beta J}h \right) = -N(J + h)$$

then $\langle M \rangle = -\frac{\partial F}{\partial h} = N$ which is maximal magnetization

there symmetry is not broken (no spontaneous magnetization) at any non-zero temperature

but for small temperature $\beta h \ll 1, \ T \ll J$, then an exponentially small magnetic field

larger than $Te^{-\beta J}$ already make the system completely magnetized.

Recall that in our system, at zero temperature the system has to be in the ground state, and the ground states there are only $2i$ $\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$... Why does an arbitrarily small temperature completely disorder our spin system?

The answer is that in 1D you can have a large change in the total spin with a finite energy cost. Imagine the following configuration:

```
\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow
```

This config is only $2J$ in energy higher than the ground state. This is called a "kink". At very small temperatures we can instead of spins think about a gas of kinks. When $T \ll J$ the kink gas is very dilute, since each kink comes with an Boltzmann suppression factor $e^{-2\beta J}$.

The average distance between kinks is $e^{2\beta J}$, between 2 kinks the spin is all up or all down. So in fact in
a 1-dimensional system there cannot be symmetry breaking (ordered phase) at any finite temperature.

Returning to the fact that at low temperature
the distance between two domain walls is of order $\exp(\beta J) \gg 1$. 
That means when we turn on a small magnetization $H$, it acts on the whole chain of spins between two links

$L \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \rightarrow$

so the energy gain is $e^{2\beta J}$ when that $\sim O(T)$ these effects of the magnetic field on the spin is expected supposed to be substantial.

Now we can see why the Ising model in dimensions from 2 to higher higher have phase transition occurring at a nonzero temperature while in 1D the phase transition, in essence, happens at zero temperature. In 2D there is no finite-energy configuration that can destroy destroy the magnetic order, say if we have to have in 2D a large puddle of spin $\downarrow$ in a sea of spin $\uparrow$,

the energy cost of a puddle of size $L$ is $JL$. In 2D a "puddle" would cost exactly the energy of the 2 ends, where the spins are $\uparrow$ or $\downarrow$. (puddle has scientific name: domain)

One can understand the phase transitions in 2D as a competition between the energy and the entropy. The critical temperature is the temperature above which the entropy wins and it is favorable, from the free-energy perspectives, to have domain walls of arbitrarily large size. Consider a domain wall with length $L$. The energy of the domain wall of course is $2JL$ (each domain link breaks a pair of spin from $\uparrow\uparrow$ to $\downarrow\downarrow$). Now we have to count (to find the entropy) how many domains are there of size $L$. If we have a completely random walk then the number would be $\sim 2^L$ (at each vertex there are 3 ways to turn).
Of course a large fraction of these random walks do not correspond to any domain wall, since they may not come back to the original point, or the line may intersect with itself. The problem of counting non-intersecting closed curves with length $L$ is actually difficult one, but it is reasonable to assume that it is exponential in $L/c$ with some $c$. Then the free energy cost of nucleating a domain with boundary of leng $L$ is

$$\Delta F = E - TS = 2J L - T \ln c^L = (2J - T \ln c) L$$

for $T > \frac{2J}{\ln c}$. Then it is advantageous to nucleate large domains and disorder the system, but for smaller $T$ it is not.

Returning to the 1D Ising chain, we now compute the correlation function between two spins $S_m, S_n$, separated by a distance $N - M$ lattice spacings. The correlation function is the average of the product of the spins $\langle S_m S_n \rangle$ and tells us how likely the sign of one spin ($S_m$) is correlated with the sign of the other spin ($S_n$). If there is perfect correlation then $\langle S_m S_n \rangle = 1$, if knowing one spin, one knows nothing about the other (i.e., the other spin can be $+1$ or $-1$ with equal probability, then $\langle S_m S_n \rangle = 0$).

Remember the canonical ensemble we have

$$\langle S_m S_n \rangle = \frac{1}{Z} \sum_{\{S_i\}} e^{-\beta \hat{H}[\{S_i\}]} S_m S_n$$

but $\hat{H}[\{S_i\}] = \hat{H}[S_1, S_2] + \hat{H}[S_2, S_3] + \ldots + \hat{H}[S_{N-1}, S_N]$
\[
\langle S_m S_n \rangle = \frac{1}{Z} \sum_{\{S\}} e^{-\beta H[S_s, S_n]} e^{-\beta H[S_{m+1}, S_m]} \ldots e^{-\beta H[S_1, S_0]} S_m e^{-\beta H[S_m, S_{m+1}]}
\]

where \( T \) is the transfer matrix. We will set the magnetic field to \( h = 0 \) for simplicity, then

\[
T = \begin{pmatrix}
 e^{\beta J} & e^{-\beta J} \\
 e^{-\beta J} & e^{\beta J}
\end{pmatrix}
\]

The matrix is diagonalized by an orthogonal matrix \( O \),

\[
O = \frac{1}{\sqrt{2}} \begin{pmatrix}
 1 & 1 \\
 1 & -1
\end{pmatrix} \quad O^T O^{-1} = \frac{1}{2} \begin{pmatrix}
 1 & 1 \\
 1 & -1
\end{pmatrix}
\]

We insert \( O O^{-1} \) into the expression

\[
\text{tr}(S_z^L S_z^T N^{-L}) = \text{tr}(O S_z^L O^T C O^{-1})^N \cdot \text{tr}(S_z^L O^{-1} (O T O^{-1})^N)
\]

\[
S_z O^{-1} = \begin{pmatrix}
 2 \text{ch} \beta J & 0 \\
 0 & 2 \text{sh} \beta J
\end{pmatrix} \quad \text{ch}_n = \cosh n \quad \text{sh}_n = \sinh n
\]

and \( O S_z O^{-1} = -S_z \)

Then the calculation becomes straightforward matrix algebra and we find

\[
\langle S_m S_n \rangle = \frac{(2 \text{sh} \beta J)^N + (2 \text{ch} \beta J)^N}{(2 \text{ch} \beta J)^N + (2 \text{sh} \beta J)^N} \cdot (2 \text{ch} \beta J)^{N-L} + (2 \text{ch} \beta J)^{N-L}
\]

Now we keep \( L \) fixed and take \( N \to \infty \) sinoe \( \text{sh} \beta J < \text{ch} \beta J \).
only the first terms in the numerators and denominators survive, and
\[ \langle S_m S_n \rangle = (\tanh^2 \beta J)^L \]

For large \( L \) the correlations of spins decrease exponentially with the distance (recall that \( \tanh \beta J \ll 1 \)).

It is instructive to consider the small temperature regime \( T \ll J \) or \( \beta J \gg 1 \). Then
\[ \langle S_m S_n \rangle \approx (1 - 2 e^{2 \beta J})^L \leq \exp(-2L e^{2 \beta J}) = e^{-L/\xi} \]

where the correlation length \( \xi \) is
\[ \xi = \frac{\beta J}{2} \]

and increases exponentially when \( T \to 0 \). This is in consistence with the diluteness of the Ising model— the distance between the kinks is roughly the correlation length of the spins.

Fluctuations in the theory of phase transitions.

So far we have presented Landau's theory of phase transitions as a theory of the homogeneous, disordered order parameter. Of course in a thermal ensemble we do not expect the order parameter to have a fixed value, it must have thermal fluctuations, but in general thermal fluctuations do not have to be uniform in space, in fact, uniform fluctuations are very rare for large volume. Thus, one we take into account thermal fluctuations of the order parameter, one needs to make the fluctuations spatially dependent: \( m = m(x) \).

Now one may worry that when we make \( m \) a function of \( x \), all the gain that we have made by going to Landau's theory is lost—what is the difference between \( m = m(x) \) and the original spin model in which \( \mathcal{S} \) is a function of the position of the spin \( S = S_i \)?
The answer is that near a 2nd order phase transitions the most important thermal fluctuations are long-wavelength fluctuations. What we will see is that the correlation length near a 2nd order phase transition diverges, similarly to the correlation length in the 1D Ising chain (although the divergence is not generally exponential as in 1D). That means we don't have to keep the information about thermal fluctuations at very short distances, all we need to keep track of are fluctuations at the long wavelengths, much longer than the distance spacing. So the theory that includes thermal fluctuations is more complicated than the Landau theory, it is still infinitely simpler than the original spin model.

The theory is called the Ginzburg-Landau theory. We now assume the existence of a free energy that depends on the function \( m = m(\vec{x}) \): the free energy is now a functional

\[
F = F[m(\vec{x})]
\]

and the partition function should now be a sum over all possible configurations \( m = m(\vec{x}) \), i.e., a path integral

\[
Z = \int Dm(\vec{x}) \, e^{-\beta F[m(\vec{x})]}
\]

[the free energy \( F[m(\vec{x})] \) should be thought of as]

\[
F = E[m(\vec{x})] - T S[m(\vec{x})], \quad \text{where} \quad S[m(\vec{x})] \text{'s log of the # of \textit{all} microscopic configs which correspond to } m(\vec{x}) \text{ grouped into one config } m(\vec{x}).
\]

If you have studied field theory you will recognize the similarity between Eq. (1) with the path integral of the partition function in field theory

\[
Z = \int D\phi(\vec{x}) \, e^{iS(\vec{x})}
\]
except in field theory the exponent is a phase \( e^{iS} \). However, if one perform a rotation from real to imaginary time (the Wick rotation) one obtain \( e^{-S_E} \) where \( S_E \) is the so-called Euclidean action.

Thus classical statistical mechanics is formally described by an Euclidean field theory, where the field \( m^t = m(x) \) lives in space only; there is no time in equilibrium.

Extending our previous discussions, we now assume that near phase transitions we can expand \( F[m(x)] \) in powers of \( m \) and powers of spatial derivatives

\[
F[m(x)] = \int d^d \{ \frac{1}{2} a(x) m^2(x) + \frac{1}{4} b(x) m^4(x) + \frac{1}{2} f(x) |\nabla m(x)|^2 + \ldots \}
\]

Still with a few terms in \( F \), one cannot evaluate the path integral exactly. How can we proceed? We will first use a familiar approximation in field theory, the perturbation theory. In the language of path integral, it corresponds to the Saddle-point approximation. We will discuss the validity of this approximation later.

Let's assume we have \( T > T_c, \; \alpha(T) > 0 \). Then the minimum of \( F[m(x)] \) is at \( m(x) = 0 \). To the quadratic order over fluctuations, we can ignore the \( m^4 \) term. Thus

\[
Z = \int D m(x) \exp \left\{ -\beta \int d^d \left\{ \frac{\beta}{2} |\nabla m(x)|^2 + \alpha m^2(x) \right\} \right\}
\]

we decompose \( m(x) \) into Fourier components

\[
m(x) = \frac{1}{\sqrt{W}} \sum_k e^{i \vec{k} \cdot \vec{x}} m_k.
\]
Here, \( \mathcal{B} m_k = \int d(Re m_k) \, d(Im m_k) \)

\[
\int dz^* dz \, e^{-A z^* z} = \int d(Re z) \, d(Im z) \, e^{-A (Re z)^2 + (Im z)^2}
\]

\[
= \sqrt{\frac{\pi}{A}} \sqrt{\frac{\pi}{A}} = \frac{\pi}{A}
\]
then \( F [m(x)] = \frac{1}{2} \sum_k (a + f k^2) m_k m_{-k} \) \( m_{-k} = m_k \)
using the formula,
\[
\int d^2z \, e^{-\frac{1}{2} A \bar{z} \bar{z}} = \frac{\pi A}{\sqrt{4a + f k^2}} \quad \text{but we have to avoid doubly}
\text{counting } m_k \text{ and } m_{-k} \text{ term}
\]
we then find \( Z = \text{const.} \prod_k \frac{A}{\sqrt{4a + f k^2}} \quad F = -\ln Z \)
(the constant will transform to a smooth regular contribution to the free energy after taking log). Thus we have
\[
F = F_0 (1) + \frac{1}{2} \sum_k \ln (a + f k^2)
\]
\[
= F_0 (1) + \frac{1}{2} \int \frac{dk}{(2\pi)^2} \ln (a(k) + f k^2).
\]
The integral over \( k \) diverges at large \( k \), but we have said that we will keep only long wavelength fluctuations of \( m(x) \), thus the integral has to be cut off; the upper limit is some \( \Lambda \) \( \Lambda \) is such that \( \frac{1}{\Lambda} \) is still much larger than the lattice spacing.

This is the symmetric (disordered) phase. In the symmetry broken phase, the minimum of the functional
\( F [m] \) is located at \( m = m_0 = \sqrt{-a/b} \), we expand
\[
m(x) = m_0 + \delta m(x)
\]
\[
F(m) = \iint d^2z \left\{ \frac{f}{2} (\delta m)^2 + (-a) (\delta m)^2 + \# (\delta m^2) + \# (\delta m^4) \right\}
\]
we neglect these in the quadratic saddle point approximation.