1. Consider a bilateral trading problem where agent 1 is the seller of a unique object and 1’s type may be either H or L, with probabilities \( p_H \) and \( p_L = 1 - p_H \) respectively. Agent 2 is the buyer. The object would be worth $100 to 1 or $190 to 2 if 1’s type is L, but the object would be worth $200 to 1 or $220 to 2 if 1’s type is H.

(a) Find the best separable trading plan for the seller, where the probability of trade \( q_t \) and the expected amount \( w_t \) paid by the buyer depend incentive-compatibly on the seller's type \( t \).

(b) Show that, when the parameter \( p_H \) is less than some critical value \( p \), this separable plan is interim incentive-efficient. Prove this incentive-efficiency by explicitly constructing utility weights and Lagrange multipliers \( (\lambda, \alpha) \), as functions of the parameter \( p_H \), that verify the optimality of this plan. What goes wrong with your construction when \( p_H > p \)?

(c) For the \( p \) that you found in (b), show that the seller's best separable plan is in fact interim pareto-dominated by a pooling contract (in which both types always sell) when \( p_H > p \).

(d) Show that the buyer's optimal trading plan also depends on whether \( p_H \) is above or below \( p \).

(e) For what range of \( p_H \) would a Walrasian market with competitive buyers have more than one equilibrium price that would elicit positive expected supply and competitive demand that clears the market? Show the potentially market-clearing prices, as functions of \( p_H \) in this range, and indicate which is Pareto-superior.

(f) Now suppose that there is a costly public signal that the seller could make before approaching the buyer, and the seller's cost of this signal would be \( c_L > 0 \) or \( c_H > 0 \), depending on the his type. After choosing to make the signal or not, the seller then approaches the buyer and makes a first-and-final offer to sell his object for any selected price, which the buyer can only accept or reject.

[f1] What conditions must \( c_L \) and \( c_H \) satisfy for there to be an equilibrium where the seller would choose to signal when his type is H but not when his type is L?

[f2] What conditions must \( c_L \) and \( c_H \) satisfy for there to be an equilibrium where both types of the seller would choose to signal? (These conditions may depend on the parameter \( p_H \).)

2. Suppose that there are two types of entrepreneurs: "type A" are a 60% fraction of the population, and the rest are "type B". Any type-A entrepreneur has a proposal for a new venture that has probability of success \( p_A = 0.5 \), and its monetary return in case of success is \( R_A = 5 \). Any type-B entrepreneur has a proposal for a new venture that has probability of success \( p_B = 0.25 \), and its monetary return in case of success is \( R_B = 8.4 \). Regardless of the type, any new venture requires an initial labor investment worth \( C = 1 \) from the entrepreneur and also requires an initial financial investment worth \( I = 1 \) from a bank. (The entrepreneur invests \( C \) only if the bank invests \( I \).) Any new venture will return 0 if it is not a success.

Each entrepreneur knows his own type but can misrepresent it when applying for financing from a banker. Contractual terms of financing can specify the debt \( D \) that the entrepreneur must repay the bank if the venture is a success, but nothing will be repaid if the venture is not a success. Suppose that everyone is risk neutral and uses a zero discount rate.

(a) Compute the maximal debt \( D \) that each type of entrepreneur would accept for this investment.

(b) For each type of entrepreneur, compute the minimal debt \( D \) such that the bank's expected returns would cover its cost of investment.

(c) Find an optimal incentive-compatible plan to maximize the bank's expected profit from one entrepreneur.

(d) If many such entrepreneurs were seeking investments from banks with limited funds which can finance only a small fraction of the entrepreneurs, what debt \( D \) would banks require to maximize expected profit?

(e) Suppose now that there are many banks competing to finance a limited collection of such entrepreneurs. Each entrepreneur can enter into an contractual agreement with any bank. Such a contract could give the bank the exclusive right to finance the entrepreneur's new venture, which the bank would do with some (contractually specified) probability \( q \), and the contract could also specify the debt \( D \) that the entrepreneur will repay the bank if the new venture is financed and is successful. These contractual terms can depend on entrepreneur's self-reported type. Find the best separable contract for the entrepreneurs.

(f) Can the best separable contract from part (c) be interim Pareto-dominated by a pooling contract, when
the A-types are 60% of the population?
(f) Show that, if competitive banks were expected to offer the pooling contract that is best for the entrepreneurs, then a bank could advertise another contract that would attract only one type of entrepreneur and would generate positive profits for the bank.

3. A firm has a valuable investment opportunity that requires I=100 financing from outside investors. Suppose that the firm must finance this investment by selling new equity shares on the market. The value of the firm with or without this new investment will depend on the firm's technological strength, which could be high or low. If the firm's technological type were high, then the expected value of the firm's future returns would be \(a_H = 150\) and the investment would add \(b_H = 120\) to this expected value. But if the firm's type were low, then the expected value of the firm's future returns would be \(a_L = 50\) and the investment would add \(b_L = 110\) to this expected value. The firm's managers know the firm's type, and they will make financial decisions to maximize the expected returns to the firm's current shareholders. But any new equity shares will be bought by other outside investors who do not know the firm's type. Before learning the firm's decision about whether to seek new financing, outside investors consider these two types to be equally likely (\(p_H = p_L = 1/2\)). If the firm decides to sell shares to raise I=100 for the new investment, then the price of these shares can be described by the value \(V\) of the old owners' shares. Then the old owners will get \(V/(V+I)\) fraction of the expanded firm's returns, and the new investors will get the fraction \(I/(V+I)\) of these returns.

(a) For each type of firm, find the lowest current-owners' value \(V\) at which issuing new equity shares worth I to finance the new investment would be in the interests of the current owners of this type of firm.
(b) For each type of firm, find the highest current-owners' value \(V\) at which new investors could break even (with zero interest) in buying new equity shares worth I to finance the new investment in this type of firm.
(c) Find the highest value \(V\) at which the firm can sell new shares worth I to finance the new investment, given the rational inference that investors should make about the firm's type from the fact that the firm's managers consider this new equity issue to good for the old owners.

4. We are to design an optimal contract for a principal who needs to buy the services of a skilled artisan when the artisan's cost of supplying his services are not known by the principal. If the artisan's cost type is \(t\), then his net utility gain from providing \(q\) hours of service and receiving wage \(w\) would be \(u(w,q|t) = w - tq\). The principal's net benefit from paying \(w\) for \(q\) hours of service from either type would be \(6\sqrt{q} - w\). The number of hours worked (which the principal can observe) and the wage can depend on the artisan's cost type (which the artisan knows), but the artisan could misrepresent his type. The artisan also will not work unless he gets a nonnegative utility gain.

(a) Suppose that the artisan's cost type is either 3 or 5, and the principal thinks that the probabilities are \(P(t=3) = 2/3\) and \(P(t=5) = 1/3\). Find the contract that maximizes the principal's expected net benefit from her business with the artisan. Show the amount of service \(q(t)\) and the wage payment \(w(t)\) for each type \(t\).
(b) If we changed part (a) by assuming that the principal would get \(A_H \sqrt{q} - w\) from paying \(w\) for \(q\) hours of service from a high-type artisan with cost 5, for some parameter \(A_H\) (keeping \(A_L = 6\)), what the bound on \(A_H\) defines the range of parameters such that the binding constraints remain the same for the principal's optimal contract? Compute the optimal contract for \(A_H\) being this bound plus 1.
(c) Now suppose instead that the artisan's cost type is drawn from a Uniform distribution on the interval from 3 to 5. (The principal's gain from trade is \(6\sqrt{q} - w\), as in (a).) Find the contract that maximizes the principal's expected net gain from trade with the artisan. Derive formulas for the amount of service \(q(t)\) and the wage payment \(w(t)\) for each possible type \(t\). Compute the exact numerical values of these formulas at \(t=3\) and \(t=5\); that is, compute \(q(3), w(3), q(5),\) and \(w(5)\).

5. Consider the problem of minimizing the expected cost to the monopsonistic buyer of some service, when one or more possible suppliers exist, but each supplier's cost of providing the service is his own private information. There are \(n+1\) suppliers, numbered 0, 1, 2, ..., \(n\). Exactly one of them must get the contract to supply the service. Supplier 0 has a cost that is drawn from a Uniform distribution on the interval $100 to
$200, and every other supplier (1,2,...,n) has a cost that is independently drawn from a Uniform distribution on the interval $200 to $300.

(a) Characterize an optimal procurement plan that minimizes the buyer's expected cost subject to incentive compatibility and nonnegative-profit constraints for all suppliers.

(b) When n=1, show how supplier 0's probability of getting the job and expected profit depends on his cost.

(c) As n→∞, what is the limiting expected cost for the buyer?

6. A rancher's cattle are forbidden to tresspass on a neighboring farmer's fields unless the farmer sells a right of tresspass to the rancher. Permitting the rancher's cattle to tresspass in these fields would give the rancher some benefit R, but it would cause some cost T for the farmer. The rancher privately knows her benefit R, but the farmer thinks that R was drawn from a Uniform distribution on [0,2]. The farmer privately knows his cost T, but the rancher thinks that T was drawn from a Uniform distribution on [1,3].

A benevolent government official is planning help arbitrate an agreement for the farmer to sell a right of tresspass to the rancher if and only if R ≥ T. If necessary, the government can subsidize or tax the transaction. (That is, the rancher may pay a different amount from what the farmer is paid, the difference being a government subsidy or tax.) But there should be no monetary payments unless the right is actually transferred (which happens in this arbitration only if R ≥ T). Both parties must be given appropriate incentives to reveal their private information.

(a) Compute the expected net total social gains E(max{R−T,0}) that this arbitration can achieve.

(b) Given any cost T in [1,3], show the probability q_{1}(T) that the farmer will grant a right of tresspass to the rancher and show the expected profit U_{1}(T) for the type-T farmer in this arbitration.

(d) Given any benefit R in [0,2], show the probability q_{2}(R) that the rancher will get the right of tresspass and the expected profit U_{2}(R) for the type-R rancher in this arbitration.

(f) Compute the expected net subsidy that the government official must provide in this arbitration.

*(g) Compare the expected net subsidy that would be necessary to achieve the same ex-post efficiency if the initial allocation of rights instead gave a right of trespass to the rancher.

7. Consider a variant of Akerlof's lemons problem. Suppose that a workers' reservation wage θ is drawn from a Uniform distribution on the interval 240 to 1000, and a worker with reservation wage θ would yield earnings worth 1.5θ to a firm that hired him. We assume that a worker knows his type θ, but can misrepresent it, and wages cannot be conditioned on the earnings actually produced.

(a) If a worker could only be hired by one monopsonistic firm, what wage w would maximize the firm's expected profit? What would be the expected productivity of a worker if he accepted this wage?

(b) If all trading were regulated by a Walrasian auctioneer, who sets a wage to make net excess demand equal 0, what equilibrium wage could get markets to clear with a positive expected supply?

*(c) Now consider the system where each worker announces a wage at which he can be hired, and then firms decide whom to hire at the announced wages. First let us consider equilibria where all workers who announce a wage W get hired but nobody who bids more than W gets hired. Characterize the set of W for which such an equilibrium is possible.

*(d) For the worker-bidding game in (c), construct an equilibrium with the following properties: Workers with type θ<400 offer one wage w_{0} and are hired for sure, but all workers with θ>400 offer another wage w_{1} and are hired with some probability q such that 1>q>0; firms would refuse to hire workers at any wage higher than w_{0} other than w_{1}; and firms expect zero net profit from the workers whom they hire at either w_{0} or w_{1}.

*(e) For the worker-bidding game, find a fully separating equilibrium where the wages that workers demand are a strictly increasing function of their productivity types, and the probability of being hired is a decreasing function of their wage demands, but only a worker making the lowest offer would be sure to get employment.

*(f) For the worker-bidding game, find an equilibrium such that all workers with type θ<400 offer wage w_{0} from (d) and are hired for sure, but workers with θ>400 offer a wage that is a strictly increasing function of their type and have positive probabilities of getting hired. Show that this equilibrium interim Pareto-dominates the equilibrium from (e).
ECONOMICS 30300b ASSIGNMENT 1: Moral-hazard exercises

1. Observable revenue $y$ will be drawn from a Normal probability distribution with mean $m_H$ if the agent chooses high effort, but with mean $m_L$ if the agent chooses low effort. The variance will be $\sigma^2$ regardless of the agent's effort. The agent's monetary cost of effort is $c_H$ if his effort is high, or $c_L$ if his effort is low. Suppose $c_H > c_L$ and $m_H > m_L$. The agent has a constant risk tolerance $T$. Let $w_o$ denote the risk-free wage that the agent could get from other employers (with zero effort costs).

(a) Suppose that agent's wage must depend on observable profit $y$ only by a linear formula $w(y)=\alpha+\beta y$. We discussed in class how to define the optimal linear wage formula to minimize a risk-neutral principal's expected wage cost subject to the constraint of giving the agent an incentive to participate with high effort. Given the same participation and moral-hazard constraints, show how might the principal's optimal wage plan for the principal change if the principal instead had some constant risk tolerance $S$.

(b) For a risk-neutral principal who wants to maximize expected net profit $E(y-w(y))$, write down the inequalities that these parameters must satisfy for the principal to prefer motivating high effort (with optimal wages) over motivating low effort or not hiring the agent at all (and consequently getting zero revenue).

(c) Now consider the special case of $c_H=1$, $c_L=0$, $m_H=2$, $m_L=0$, $\sigma=2$, $w_0=0$, $T=2$. Compute the minimal expected wage cost and the expected net profit for the risk-neutral principal with the optimal linear wage for motivating high effort.

(d) For the parameters in (c), instead of using a linear formula to determine the wage as a function of the revenue $y$, suppose that the principal will use a three-wage rule, paying some (low) wage $w_1$ if $y < -1$, some (middle) wage $w_M$ if $-1 < y < 1$, and some (high) wage $w_H$ if $y > 1$. Find the three-wage rule $(w_L,w_M,w_H)$ that minimizes the expected wage cost for the principal subject to the participation constraint and the incentive constraint that the agent is be willing to work here with high effort.

First show a system of equations that specify both how these three wages would depend on the Lagrange multiplier of the moral-hazard incentive constraint and how this Lagrange multiplier can be computed. Then, with a computer, numerically estimate this Lagrange multiplier, compute the optimal three-wage rule, and compute the principal's expected costs under this optimal three-wage rule.

[Hint: A standard normal cumulative distribution $\Phi$ has $\Phi(-1.5)=0.0668, \Phi(-0.5)=0.3085, \Phi(0.5)=0.6915$]

2. An agent will choose efforts $a_1$ and $a_2$ to put into tasks 1 and 2. He can choose any nonnegative numbers $a_1 \geq 0$ and $a_2 \geq 0$, and then his cost of effort will be $C(a_1,a_2) = 2(a_1+a_2)^2 + (a_1)^2 + (a_2)^2$ per unit time. The principal can observe two outcomes $Y_1$ and $Y_2$ which are independent Normal random variables, each with the variance $\sigma^2 = 1$, and with means $E(Y_1) = a_1$, $E(Y_2) = a_2$.

The principal will get income $R(Y_1,Y_2) = 24y_1 + 24y_2$. Following Holmstrom and Milgrom's advice, the principal will pay the agent a wage of the form $w(Y_1,Y_2) = A + B_1Y_1 + B_2Y_2$. Then the principal's profit will be $R(Y_1,Y_2) - w(Y_1,Y_2)$. The agent's utility function $u(w-C)$ has constant risk tolerance $T=2$, and the agent's outside option pays $w=0$.

(a) Express the agent's certainty equivalent payoff as a function of his actions $(a_1,a_2)$, and the wage-plan's parameters $(A,B_1,B_2)$.

(b) Show how the agent's optimal actions $(a_1,a_2)$ would depend on the principal's $(A,B_1,B_2)$. (Remember that these efforts $a_1 \geq 0$ and $a_2 \geq 0$ cannot be negative numbers.)

(c) The principal wants to choose an incentive plan $(A,B_1,B_2)$ that maximizes his expected profit $R-w$, subject to a moral-hazard constraint that the agent's actions $(a_1,a_2)$ be optimal for the agent, and an ex-ante participation constraint that the agents certainty equivalent is not less than $w=0$. Noting the obvious symmetry in this problem, let us consider plans where $B_1=B_2$.

What plan $(A,B_1,B_1)$ would maximize the principal's expected profit subject to the moral-hazard and participation constraints? What is the principal's expected profit?

(d) Now let us consider plans where $B_2=0$, so the agent is encouraged to specialize in task 1. What plan $(A,B_1,0)$ would maximize the principal's expected profit subject to the moral-hazard and participation constraints? What is the principal's expected profit?
3. The revenue from a project will be a random variable in some finite set \( Y \) of possible values, but the probability distribution over \( Y \) will depend on the unobservable effort of an agent. The agent's effort could be high or low. For each \( y \) in \( Y \), let \( p(y|H) \) denote the probability of revenue being \( y \) when effort is high, and let \( p(y|L) \) denote the probability of revenue being \( y \) when effort is low. The agent's wage can be a function \( w(y) \) depending on the revenue \( y \), but wages must always be nonnegative (limited liability): \( w(y) \geq 0, \forall y \in Y \).

The agent gets a positive unobservable benefit \( B \) from choosing low effort (no cost of high effort). Let \( w_o \) denote the payoff that the agent could get from alternative employment.

Both the agent and his employer are risk neutral. The employer's optimal wage plan should minimize the expected wage cost subject to the constraints that the agent should be willing to participate in this project with high effort, and should get a nonnegative wage.

(a) Suppose first that \( w_o \) is 0 or close to 0. Show that an optimal wage pays a strictly positive wage only for an outcome \( y_1 \) in \( Y \) that minimizes the likelihood ratio \( p(y|L)/p(y|H) \). To prove this, show the Lagrange multipliers of the participation constraint and of the moral-hazard constraint (as formulas that depend on the above parameters) that verify the optimality of this solution.

(b) What is the highest value (as a function of the other parameters) of \( w_o \) such that the solution in part (a) remains optimal? If \( w_o \) were strictly larger, then what would be the Lagrange multipliers of the participation and moral-hazard constraints?

(Hint: The Lagrange multiplier is 0 for a constraint when it is a strict inequality in any optimal solution.)

(c) Now suppose that the employer can specify both a nonnegative wage \( w(y) \geq 0 \) and a nonnegative punishment \( z(y) \geq 0 \) for any observable outcome \( y \) in \( Y \). The worker's net utility will be \( w(y) - z(y) \) but the employer's cost will be just \( w(y) \). Again, suppose that \( w_o \) is 0 or close to 0.

Show that there is an optimal wage/punishment plan in which the agent would get a strictly positive wage only for the outcome \( y_1 \) that minimizes the likelihood ratio \( p(y|L)/p(y|H) \), and the agent would get a strictly positive punishment only for the outcome \( y_2 \) that maximizes this likelihood ratio. Show formulas for the Lagrange multipliers of the participation and moral-hazard constraints.

4. A firm has an opportunity for a potentially valuable investment project, but it must be managed by an agent who is subject to moral-hazard temptations. The project requires an initial investment costing \( I = 20 \).

(a) Show the optimal wage plan for minimizing the firm's expected wage cost while giving the agent an incentive to participate and behave well in managing this project.

(b) Find the lowest return \( R \) for which the firm should invest in the project and pay the agent to behave well.

(c) How would your answers to (a) and (b) change if the agent had assets worth \( A = 3 \) that he could invest in the project, but only if he can get an expected wage that is worth at least this amount?

(d) Now suppose that the firm has two similar project opportunities. Successes in the two projects are independent events, but each depends in the same way on the behavior of its managing agent.

The firm is considering a plan in which, if the first project succeeds then it would invest in the second project and put the same agent in charge of it, with all wage payment deferred until the end of the second project. Describe an optimal wage plan for minimizing the expected cost of paying the agent while giving the agent an incentive to participate and behave well both in managing the first project and in managing the second project if it is undertaken. What is the lowest return \( R \) for which the firm should be willing to do this?

Consider the case of \( R = 50 \) and explain why the firm would be willing to undertake the second project if the first project succeeds but not if it fails, even though the two projects' chances of success are independent.