Answers to Assignment 7

1. The game model has two players: 1 and 2. Player 1 has only one type, but player 2 has two possible types: Strong and Weak. Player 1 assigns probability $\alpha$ to the event that player 2 is Strong. Each player has two possible actions: Fight and Yield.

The payoffs $(u_1, u_2)$ depend on their actions and 2's type as follows:

<table>
<thead>
<tr>
<th>P(2's type)</th>
<th>$p_2$(Strong) = $\alpha$</th>
<th>$p_2$(Weak) = $1 - \alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F</td>
<td>Y</td>
</tr>
<tr>
<td></td>
<td>-1, 1</td>
<td>1, 0</td>
</tr>
<tr>
<td></td>
<td>1, 0</td>
<td>F</td>
</tr>
<tr>
<td>Y</td>
<td>0, 1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

If $\alpha < 0.5$, the equilibrium is $\sigma_1 = [F], \sigma_{2,\text{Strong}} = [F], \sigma_{2,\text{Weak}} = [Y]$.

If $\alpha > 0.5$, the equilibrium is $\sigma_1 = [Y], \sigma_{2,\text{Strong}} = [F], \sigma_{2,\text{Weak}} = [F]$.

2. $2$'s Type = A   $2$'s Type = B

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>4,0</td>
<td>0,2</td>
<td>T</td>
<td>4,0</td>
<td>0,4</td>
</tr>
<tr>
<td>B</td>
<td>0,4</td>
<td>2,0</td>
<td>B</td>
<td>0,2</td>
<td>2,0</td>
</tr>
</tbody>
</table>

It is easy to see that there is no equilibrium where 1 uses $T$ for sure or where 1 uses $B$ for sure.

To make player 1 willing to randomize, 1 must think that 2's probability choosing $R$ satisfies

$$4 \times (1 - P(R)) + 0 \times P(R) = EU_1(T) = EU_1(B) = 0 \times (1 - P(R)) + 2 \times P(R),$$

and so $P(R) = 2/3$.

Let $\alpha_2(R | t_2)$ denote the probability of player 2 choosing $R$ given that her type is $t_2$.

So $P(R) = p_2(A) \times \alpha_2(R | A) + p_2(B) \times \alpha_2(R | B)$, where $p_2(t_2)$ is the probability of 2's type being $t_2$.

With increasing differences we can see that, among player 2's types, type A is more inclined toward choosing L, and type B is more inclined toward choosing R.

(a) We assume that $p_2(A) = p_2(B) = 0.5$. So we need $2/3 = 0.5 \times \alpha_2(R | A) + 0.5 \times \alpha_2(R | B)$.

If 2's type A was not willing to choose $R$, then the probability of 2 choosing $R$ could not be more than $p_2(B) = 0.5 < 2/3$.

So there must be a positive probability of 2's type A choosing $R$. But by increasing differences, if 2's type A would be willing to choose $R$, then 2's type B must strictly prefer $R$ over L.

So $\alpha_2(R | B) = 1$. So we need $2/3 = 0.5 \times \alpha_2(R | A) + 0.5 \times 1$. So $\alpha_2(R | A) = 1/3$.

That is, 2's type A would do $(2/3)[L]+(1/3)[R]$, but 2's type B would do $[R]$.

To make 2's type A willing to randomize, player 1's probability of choosing $B$ must satisfy

$$0 \times (1 - P(B)) + 4 \times P(B) = EU_2(L | A) = EU_2(R | A) = 2 \times (1 - P(B)) + 0 \times P(B).$$

So we need $P(B) = 1/3$. That is player 1 must do $(2/3)[T]+(1/3)[B]$.

(b) Now we assume that $p_2(A) = 1/6$ and $p_2(B) = 5/6$. So we need $2/3 = P(R) = (1/6) \alpha_2(R | A) + (5/6) \alpha_2(R | B)$ and $1/3 = P(L) = (1/6) \alpha_2(L | A) + (5/6) \alpha_2(L | B)$.

So there must be a positive probability of 2's type B choosing $L$. But by increasing differences, if 2's type B would be willing to choose $L$, then 2's type A must strictly prefer $L$ over $R$.

So $\alpha_2(L | A) = 1$. So we need $1/3 = (1/6)(1) + (5/6) \alpha_2(L | B)$. So $\alpha_2(L | B) = 1/5$.

That is, 2's type A would do $[L]$, but 2's type B would do $(1/5)[L]+(4/5)[R]$.

To make 2's type B willing to randomize, player 1's probability of choosing $B$ must satisfy

$$0 \times (1 - P(B)) + 2 \times P(B) = EU_2(L | B) = EU_2(R | B) = 4 \times (1 - P(B)) + 0 \times P(B).$$

So we need $P(B) = 2/3$. That is player 1 must do $(1/3)[T]+(2/3)[B]$. 

3. | 2 NotFights | 2 Fights |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1 NotFights</td>
<td>0, 0</td>
<td>0, 2</td>
</tr>
<tr>
<td>1 Fights</td>
<td>V₁, 0</td>
<td>−1, −1</td>
</tr>
</tbody>
</table>

(a) We assume first V₁=3. (([1Fights], [2NotFights]) is an equilibrium. Also ([1NotFights], [2Fights]) is an equilibrium. Also (1/3)[1NotFights]+(2/3)[1Fights], (1/4)[2NotFights]+(3/4)[2Fights]) is an equilibrium. (Notice that, in the randomized equilibrium, 2 fights with higher probability even though she has the lower value for the prize.)

(b) Let P(F) denote the probability that player 1 will fight, which depends on 1's strategy α₁ according to: P(F) = (1/2) α₁(F|V₁=2) + (1/2) α₁(F|V₁=3).

To make player 2 willing to randomize, player 1's probability of fighting must satisfy
0 = EU₂(2NotFights) = EU₂(2Fights) = 2 × (1−P(F)) + −1 × P(F), and so P(F) = 2/3.

So 1's type V₁=2 must be willing to fight. Then by increasing differences, 1's type V₁=3 strictly prefers to fight. So α₁(F|V₁=3) = 1. So 2/3 = (1/2)α₁(F|V₁=2) + (1/2)(1), α₁(F|V₁=2) = 1/3.

That is, 1's type V₁=2 does (1/3)[1Fights]+(2/3)[1NotFights], but 1's type V₁=3 does [1Fights].

To make 1's type V₁=2 willing to randomize, player 2's probability of fighting must satisfy
−1 × P(F) + 2 × (1−P(F)) = EU₁(F|V₁=2) = EU₁(NF|V₁=2) = 0.

So we need P(F) = 2/3. That is, player 2 must do (1/3)[2NotFights]+(2/3)[2Fights].

(c) By increasing differences, player 1 should use a cutoff strategy, Fighting if ℋ₁ > θ and NotFighting if ℋ₁ < θ, for some cutoff θ.

To make player 2 willing to randomize, player 1's probability of fighting must be 2/3, and so P(ℋ₁>θ) = 2/3, which implies that θ = 1/3 (since ℋ₁ is Uniform on the interval 0 to 1).

To make 1's type θ indifferent, player 2's probability of fighting must satisfy
−1 × P(F) + (2+θ) × (1−P(F)) = EU₁(F|ℋ₁=θ) = EU₁(NF|ℋ₁=θ) = 0.

With θ = 1/3, this implies that player 1's P(F) = 0.7. (Note the difference from part (b).)

That is, player 2 must do 0.3[2NotFights]+0.7[2Fights].
Answers to Assignment 6

1. (a) The strategic form has \((u_1,u_2,u_3)\) depending on the actions of the three players as follows:

\[
\begin{array}{ccc}
\text{3 does L} & \text{3 does R} \\
\text{2 does c} & 1,1,1 & 4,4,0 \\
\text{2 does d} & 1,1,1 & 0,0,1 \\
\text{1 does C} & 3,3,2 & 0,0,0 \\
\text{1 does D} & 3,3,2 & 0,0,0 \\
\end{array}
\]

Here \((C,c,R)\) and \((D,c,L)\) are pure-strategy Nash equilibria of this strategic-form game.

(b) The Nash equilibrium \((C,c,R)\) is a sequential equilibrium when 3's belief probability for his left node that follows D is any number \(\alpha\) such that \(\alpha^2 \# (1-\alpha)1\), and so \(\alpha \leq 1/3\). Any such belief probability would be consistent with Bayes rule in this equilibrium, because 3's information set has zero probability in this equilibrium.

(c) The Nash equilibrium \((D,c,L)\) cannot be a sequential equilibrium. When player 3 is expected to choose L, 2's doing c would not be sequentially rational if she got to move, because player 2 would prefer to get 4 from d rather than 1 from c.

2. Let \(\alpha\) denote 2's belief about the probability of 1 being type 1.1 if 2 gets to move; that is, \(\alpha = P(1.1|2\text{moves})\).

When she moves, 2 would be willing to choose a2 if \(\alpha(5) + (1-\alpha)0 \geq \alpha(3) + (1-\alpha)3\), that is \(\alpha \geq 3/5\). She would be willing to choose b2 if \(\alpha \leq 3/5\). She would randomize only when \(\alpha = 3/5\).

(a) For the probability of player 2 moving to equal one, player 1 must be expected to do x1 if he is type 1.1 and z1 if he is type 1.2. Then by Bayes's rule, player 2 must believe that \(\alpha = 2/3\).

With \(\alpha = 2/3\), 2's unique best response is a2, because 2/3 > 3/5. If 2 is expected to do a2, then it is indeed rational for player 1 to do x1 when he is type 1.1 (as 9>5) and to do z1 when he is type 1.2 (as 9>2). So the scenario \(([x1],[z1],[a2])\) forms a sequential equilibrium with \(\alpha = 2/3\).

(b) For the probability of player 2's moving to be zero, player 1 must be expected to do w1 if he is type 1.1 and y1 if he is type 1.2. For player 1 to be willing to behave in this way, he must expect that, if player 2 got to move then her probability of choosing a2 would be less than 2/9. So we need that player 2 must be willing to choose b2 instead. With zero probability, Bayes's rule does not tell us what player 2 should believe about 1's type when she moves, but the requirement that player 2 must be willing to choose b2 implies that her belief probability of her left node (after 1.1) when she moves must be \(\alpha \leq 3/5\).

With any \(\alpha \leq 3/5\), we have a sequential equilibrium with strategies \(([w1],[y1],[b2])\).

(c) If player 2 would surely choose a2 if she moved, then player 1's best response (x1,z1) would make the probability of player 2 moving equal to 1. If player 2 would surely choose b2 if she moved, then player 1's best response (w1,y1) would make the probability of player 2 moving equal to 0. So player 2 must be expected to randomize between a2 and b2 when she moves. To make player 2 willing to randomize, her belief about the probability of type 1.1 when she moves must be \(\alpha = 3/5\). Notice that 3/5 is less than the prior probability of type 1.1, which is 2/3. So player 2's getting to move must be evidence against type 1.1. That is, player 1 must be less likely to let player 2 move when 1's type is 1.1 than when 1's type is 1.1.
So let $\sigma_1(x_1|1.1)$ denote the probability of 1 choosing $x_1$ if his type is 1.1, and let $\sigma_1(z_1|1.2)$ denote the probability of 1 choosing $z_1$ when his type is 1.2.

By Bayes's rule, $\alpha = (2/3)\sigma_1(x_1|1.1)/[(2/3)\sigma_1(x_1|1.1) + (1/3)\sigma_1(z_1|1.2)]$.

To get $\alpha = 3/5 < 1$, we must have $0 < \sigma_1(x_1|1.1) < \sigma_1(z_1|1.2) < 1$. So type 1.1 must randomize.

For type 1.1 to be willing to randomize, the conditional probability of 2 choosing $a_2$ if she moves must be $\sigma_2(a_2|2.3) = 5/9$, because $5 = (5/9)9 + (1 - 5/9)0$.

Against this behavior of player 2, type 1.2 would strictly prefer to choose $z_1$, because $2 < (5/9)9 + (1 - 5/9)0$, and so type 1.2 must choose $z_1$ with probability $\sigma_1(z_1|1.2) = 1$.

Then $3/5 = \alpha = (2/3)\sigma_1(x_1|1.1)/[(2/3)\sigma_1(x_1|1.1) + (1/3)(1)]$

implies that $\sigma_1(x_1|1.1) = 3/4 = 0.75$.

So our sequential equilibrium has type 1.1 doing 0.25[w1]+0.75[x1], type 1.2 doing [z1], player 2 doing (5/9)[a2]+(4/9)[b2], and player 2 believing $\alpha = P(1.1|2$ moves) = 3/5 when she moves.

(d) The strategic form is

<table>
<thead>
<tr>
<th></th>
<th>a2</th>
<th>b2</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1y1</td>
<td>4, 6</td>
<td>4, 6</td>
</tr>
<tr>
<td>w1z1</td>
<td>6.33, 4</td>
<td>3.33, 5</td>
</tr>
<tr>
<td>x1y1</td>
<td>6.67, 5.33</td>
<td>0.67, 4</td>
</tr>
<tr>
<td>x1z1</td>
<td>9, 3.33</td>
<td>0, 3</td>
</tr>
</tbody>
</table>

For type 1.1 to be willing to randomize, the conditional probability of 2 choosing $a_2$ if she moves must be $\sigma_2(a_2|2.3) = 5/9$, because $5 = (5/9)9 + (1 - 5/9)0$.

Against this behavior of player 2, type 1.2 would strictly prefer to choose $z_1$, because $2 < (5/9)9 + (1 - 5/9)0$, and so type 1.2 must choose $z_1$ with probability $\sigma_1(z_1|1.2) = 1$.

Then $3/5 = \alpha = (2/3)\sigma_1(x_1|1.1)/[(2/3)\sigma_1(x_1|1.1) + (1/3)(1)]$

implies that $\sigma_1(x_1|1.1) = 3/4 = 0.75$.

So our sequential equilibrium has type 1.1 doing 0.25[w1]+0.75[x1], type 1.2 doing [z1], player 2 doing (5/9)[a2]+(4/9)[b2], and player 2 believing $\alpha = P(1.1|2$ moves) = 3/5 when she moves.

(b) The normal representation in strategic form is

<table>
<thead>
<tr>
<th></th>
<th>a2</th>
<th>b2</th>
<th>c2</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1y1</td>
<td>4.7</td>
<td>4.7</td>
<td>4.7</td>
</tr>
<tr>
<td>w1z1</td>
<td>7.4</td>
<td>3.5</td>
<td>5.7</td>
</tr>
<tr>
<td>x1y1</td>
<td>2.8</td>
<td>1.5</td>
<td>5.4</td>
</tr>
<tr>
<td>x1z1</td>
<td>6.5</td>
<td>0.3</td>
<td>6.4</td>
</tr>
</tbody>
</table>

(c) There is a Nash equilibrium (w1y1, b2).
Answers to Assignment 5:

1. In state 1, as long as nobody deviates, they get $U_1(b_1,a_2) = 2$, $U_2(b_1,a_2) = 1$ every period, and so their expected discounted average values in state 1 are $V_1(1) = 2$, $V_2(1) = 1$. In state 2, as long as nobody deviates, they get $U_1(a_1,b_2) = 1$, $U_2(a_1,b_2) = 2$ every period, and so their expected discounted average values in state 2 are $V_1(2) = 1$, $V_2(2) = 2$. In state 1, if player 1 deviates to $a_1$ then he expects $(1 - \delta)\delta V_1(2) + \delta = (1 - \delta)\delta V_2(1) + \delta$. In state 1, if player 2 deviates to $b_2$ then she expects $(1 - \delta)\delta V_1(2) + \delta = (1 - \delta)\delta V_2(1) + \delta$. So to deter deviations in state 1, we need $2 \geq (1 - \delta)\delta V_1(2) + \delta$ and $1 \geq (1 - \delta)\delta V_2(1) + \delta$. The first inequality is satisfied when $\delta \geq 6/7$ and the second is satisfied for all $\delta$ between 0 and 1. So nobody wants to deviate in state 1 if $\delta \geq 6/7$.

Similarly, nobody wants to deviate in state 2 when $\delta \geq 6/7$ (but now it is player 2 who has to be deterred from deviating to $a_2$ which would increase her payoff to 8 now but would cause a switch back to state 1, which is worse for her). So we have an equilibrium when $\delta \geq 6/7$.

In the one-stage game $b_1$ and $b_2$ are strongly dominated strategies, so the unique equilibrium is $(a_1,a_2)$, yielding payoffs $(8,8)$.

2. (a) In the one-period randomized equilibrium, each player $i$ uses the randomized strategy $2/3[a_i]+1/3[b_i]$, and each player gets expected payoff equal to 2, because $2 = (2/3)3 + (1/3)0 = (2/3)5 + (1/3)(-4)$. Let state 0 be "cooperating", and let state 1 be "randomizing". They start in state 0. In state 0, each player $i$ should play $a_i$. They continue in state 0 as long as both do $a_i$, but they change next to state 1 if anyone does $b_i$. In state 1, each player $i$ randomizes according to $(2/3)[a_i]+(1/3)[b_i]$ each period. Once in state 1, they always continue in state 1. The values $V_i(\theta)$ for player $i$ of being in state $\theta$ satisfy:

$V_i(0) = (1 - \delta)3 + \delta V_i(0)$, so $V_i(0) = 3$ for $i=1,2$.

$V_i(1) = (1 - \delta)2 + \delta V_i(1)$, so $V_i(1) = 2$, for $i=1,2$.

Player $i$'s discounted average value of deviating to $b_i$ in state 0 is $(1 - \delta)5 + \delta 2$. So for equilibrium, we need $3 \geq (1 - \delta)5 + \delta 2$, and so $\delta \geq 2/3$.

(b) We consider an equilibrium with 3 states: state 0 = "cooperate", state 1 = "player 1 acts superior" and state 2="player 2 acts superior". In state 0 they should play $(a_1,a_2)$. In state 1 they play $(b_1,a_2)$. In state 2 they play $(a_1,b_2)$. They start in state 0. In state 0, if they do $(b_1,a_2)$ then they switch to state 2, but if they do $(a_1,b_2)$ then they switch to state 1. Once in state 1 or state 2, they remain in the same state forever. As long as equilibrium predictions are fulfilled, the state is expected to always stay the same. So the discounted average value $V_i(\theta)$ for each player $i$ in each state $\theta$ are:

$V_1(0) = 3$, $V_2(0) = 3$, $V_1(1) = 5$, $V_2(1) = 0$, $V_1(2) = 0$, $V_2(2) = 5$.

In state 0, for each player $i$, the discounted average value of deviating is $(1 - \delta)5 + \delta 0$. So for equilibrium, we need $3 \geq (1 - \delta)5 + \delta 0$, and so $\delta \geq 2/5$.

States 1 and 2 each involve repeating forever a one-period equilibrium ($(b_1,a_2)$ or $(a_1,b_2)$), and so a player can never gain by unilaterally deviating from the equilibrium in state 1 or state 2.
3. First we must analyze the randomized equilibrium of the one-stage game.

The randomized equilibrium of the one-stage game is \((p[a_1]+(1-p)[b_1], q[a_2]+(1-q)[b_2])\) where 

\[Eu_1 = q(0)+(1-q)(2) = q(8)+(1-q)(0)\] and \[Eu_2 = p(8)+(1-p)(0) = p(0)+(1-p)(2),\]

and so \(q=0.2,\ p = 0.2,\ \text{and } Eu_1 = 1.6,\ Eu_2 = 1.6.\)

The equilibrium of the repeated game has three social states:

In state 1 they play \((a_1,a_2)\). In state 2 they play \((b_1,a_2)\). In state 3 they play the randomized equilibrium of the one-stage game: \((0.2[a_1]+0.8[b_1], 0.2[a_2]+0.8[b_2])\).

From state 1, if \((a_1,a_2)\) is played then next period they go to state 2, but otherwise they go to state 3.

From state 2, if \((b_1,a_2)\) is played then next period they go to state 1, but otherwise they go to state 3.

From state 3, they stay in state 3 forever regardless of what anyone does.

So the discounted average values \(V_i(\theta)\), for each player \(i\) in each state \(\theta\), must satisfy:

\[V_1(1) = (1-\delta_1)(0) + \delta_1 V_1(2),\ \text{and } V_1(2) = (1-\delta_1)(8) + \delta_1 V_1(1).\]

So \((1-\delta_1)^2)V_1(2) = (1-\delta_1)(8)\) and so \([\text{with } (1-\delta_1)^2=(1-\delta_1)(1+\delta_1)]\)

we get \(V_1(2) = 8/(1+\delta_1)\) and \(V_1(1) = 6.4/ (1+\delta_1).\)

Similarly, \(V_2(1) = (1-\delta_2)(0) + \delta_2 V_2(2),\ \text{and } V_1(2) = (1-\delta_2)(0) + \delta_2 V_2(1).\)

and so we get \(V_2(1) = 8/(1+\delta_2)\) and \(V_2(2) = 6.4/(1+\delta_2).\)

\(V_1(3) = (1-\delta_1)(1.6)+\delta_1 V_1(3),\) and so \(V_1(3) = 1.6.\) Similarly, \(V_2(3) = 1.6.\)

For an equilibrium we need:

\[\delta_1(8)/(1+\delta_1) = V_1(1) \geq (1-\delta_1)(8) + \delta_1 V_1(3) = (1-\delta_1)(8) + \delta_1(1.6)\]

which implies \(6.4\delta_1^2 + 6.4\delta_1 - 8 \geq 0,\)

\[8/(1+\delta_1) = V_1(2) \geq (1-\delta_1)(0) + \delta_1 V_1(3) = (1-\delta_1)(0) + \delta_1(1.6)\]

which implies \(0 \geq 1.6\delta_1^2 + 1.6\delta_1 - 8,\)

\[8/(1+\delta_2) = V_2(1) \geq (1-\delta_2)(0) + \delta_2 V_2(3) = (1-\delta_2)(0) + \delta_2(1.6),\]

which implies \(0 \geq 1.6\delta_2^2 + 1.6\delta_2 - 8,\)

\[\delta_2(8)/(1+\delta_2) = V_2(2) \geq (1-\delta_2)(2) + \delta_2 V_2(3) = (1-\delta_2)(2) + \delta_2(1.6),\]

which implies \(0.4\delta_2^2 + 6.4\delta_2 - 2 \geq 0.\)

Notice, all of these inequalities are satisfied when \(\delta_1\) and \(\delta_2\) are very close to 1.

The middle two are satisfied for all \(\delta_i\) between 0 and 1, so only the first and last matter.

The first is satisfied when \(\delta_1 \geq (-6.4+(6.4^2+4*6.4*8)*0.5)/(2*6.4) = 0.7247.\)

The last is satisfied when \(\delta_2 \geq (-6.4+(6.4^2+4*0.4*2)*0.5)/(2*0.4) = 0.3066.\)
Answer to assignment 4, question 2.

Player 1 chooses a1 such that 0 ≤ a1 ≤ 1, and player 2 chooses a2 such that 0 ≤ a2 ≤ 1.

Their payoffs (u1, u2) depend on the chosen numbers (a1,a2) and a known parameter γ as follows:
\[ u_1(a_1,a_2) = \gamma a_1a_2 - (a_1)^2, \quad u_2(a_1,a_2) = 2a_1a_2 - a_2. \]

Then \( \partial u_1 / \partial a_1 = \gamma a_2 - 2a_1 \).

So in the independent-move game, for any given \( a_2 \geq 0 \) and \( \gamma \geq 0 \), 1's best response for choosing \( a_1 \) between 0 and 1 is \( b_1(a_2) = \min\{ (\gamma/2)a_2, 1 \} \).

(When \( (\gamma/2)a_2 > 1 \), we get \( \partial u_1 / \partial a_1 > 0 \) for any \( a_1 < 1 \).)

\( \partial u_2 / \partial a_2 = 2a_1 - 1 \).

So for any given \( a_1 \), 2's best response for choosing \( a_2 \) between 0 and 1 is \( b_2(a_1) = 0 \) if \( a_1 < 0.5 \), \( b_2(a_1) = 1 \) if \( a_1 > 0.5 \), and \( b_2(0.5) \) is the set of all numbers between 0 and 1.

Thus, to search for Nash equilibria of the game where they choose their actions independently, we have three possible cases to consider:

The first possible case is when \( a_2 = 0 \) and \( a_1 = b_1(0) \), which will be an equilibrium if \( b_2(b_1(0)) = 0 \).

The second possible case is when \( a_2 = 1 \) and \( a_1 = b_1(1) \), which will be an equilibrium if \( b_2(b_1(1)) = 1 \).

In the third possible case is when \( a_1 = 0.5 \), which can be an equilibrium with any \( a_2 \) such that \( b_1(a_2) = 0.5 \).

(a) With \( \gamma = 1.5 \), there are three Nash equilibria when players choose their actions independently.

First, we find an equilibrium where \( a_2 = 0 \) and \( a_1 = b_1(0) = 0 \), because \( b_2(0) = 0 \).

Second, we find an equilibrium where \( a_2 = 1 \) and \( a_1 = b_1(1) = \gamma/2 = 0.75 \), because \( b_2(0.75) = 1 \).

Third, we have an equilibrium with \( a_1 = 0.5 \) and \( a_2 = 2/3 \), because \( b_2(2/3) = (1.5/2)(2/3) = 0.5 \).

(c) With \( \gamma = 0.8 \), there is only one equilibrium when the players choose their actions independently.

We still have an equilibrium where \( a_2 = 0 \) and \( a_1 = b_1(0) = 0 \), because we still have \( b_2(0) = 0 \).

There is no equilibrium with \( a_2 = 1 \) because \( b_2(b_1(1)) = b_2(0.4) = 0 \neq 1 \).

There is no equilibrium with \( a_1 = 0.5 \) because \( 0.5 = b_1(a_2) = (0.8/2)a_2 \) would require \( a_2 = 1.25 \), which is not feasible (1.25 > 1).

In a game where player 2 moves after observing player 1, there are two cases to consider:

If player 1 chooses \( a_1 \leq 0.5 \) then \( a_2 = 0 \) is a best response for player 2.

If player 1 chooses \( a_1 \geq 0.5 \) then \( a_2 = 1 \) is a best response for player 2.

When \( a_1 = 0.5 \), player 1 would prefer that player 2 should choose \( a_2 = 1 \) for any \( \gamma > 0 \). So we can look for a subgame-perfect equilibrium where 2 uses the strategy: \( \beta_2(a_1) = 0 \) if \( a_1 < 0.5 \), \( \beta_2(a_1) = 1 \) if \( a_1 \geq 0.5 \).

The optimal choice for 1 in the subgame perfect equilibrium is the action \( a_1 \) that maximizes \( u_1(a_1, \beta_2(a_1)) \) subject to the feasibility condition \( 0 \leq a_1 \leq 1 \).

To find this maximum, we should separately consider each side of the discontinuity at \( a_1 = 0.5 \).

For any \( \gamma \), \( u_1(a_1,0) = -(a_1)^2 \) is maximized over \( a_1 < 0.5 \) by \( a_1 = 0 \), which would yield \( u_1 = 0 \).

So the optimum should be found by maximizing \( u_1(a_1,1) = \gamma a_1 - (a_1)^2 \) over \( a_1 \) such that \( 0.5 \leq a_1 \leq 1 \). (Notice that \( \gamma x - x^2 \) is maximized over all \( x \) by \( x = \gamma/2 \).)

(b) With \( \gamma = 1.5 \), \( 1.5a_1 - (a_1)^2 \) is maximized by \( a_1 = \gamma/2 = 0.75 \), which satisfies our constraint \( 0.5 \leq a_1 \leq 1 \) and yields \( u_1(a_1,1) = 1.5(1)(0.75) - (0.75)^2 = 9/16 > 0 = u_1(0,0) \). So we have a subgame perfect equilibrium in which 1 chooses \( a_1 = 0.75 \), knowing that 2 will apply the strategy \( \beta_2 \) described above, yielding an outcome where \( a_2 = \beta_2(0.75) = 1 \).

(d) With \( \gamma = 0.8 \), \( u_1(a_1,1) = 1.5a_1 - (a_1)^2 \) would be maximized by \( a_1 = \gamma/2 = 0.4 \), but that would violate the constraint \( 0.5 \leq a_1 \leq 1 \) that is required to keep \( \beta_2(a_1) = 1 \).

In the range where \( \beta_2(a_1) = 1 \), we find that \( \partial u_1(a_1,1) / \partial a_1 = 0.8 - 2a_1 < 0 \) because \( a_1 \geq 0.5 \) in this range.

Thus, the best choice for player 1 is \( a_1 = 0.5 \), which yields the outcome \( a_2 = \beta_2(0.5) = 1 \) and \( u_1(0.5,1) = 0.8(0.5)(1) - (0.5)^2 = 0.15 > 0 \).
Answer to assignment 4, question 3: Find all Nash equilibria of the following 2x3 game.

\[
\begin{array}{ccc}
\text{Player 1} & \text{L} & \text{M} & \text{R} \\
\text{T} & 0, 4 & 5, 6 & 8, 7 \\
\text{B} & 2, 9 & 6, 5 & 5, 1 \\
\end{array}
\]

**Answer**

If player 1 chooses T for sure, 2's best response is R, and T is best for 1 against R, and so we have an equilibrium (T, R) with payoffs (u_1, u_2) = (8, 7).

If player 1 chooses B for sure, 2's best response is L, and B is best for 1 against L, and so we have an equilibrium (B, L) with payoffs (u_1, u_2) = (2, 9).

Now let's search for equilibria where player 1 randomizes over the support \{T, B\}, using a randomized strategy \(p[T] + (1 - p)[B]\).

**Guess:** 2's support is \{L, M\}. This support cannot give us an equilibrium because B is strictly better for 1 against both L and M, and so no randomization over L and M by player 2 could ever make player 1 willing to randomize over T and B. So there is no equilibrium with this support.

\((p[T] + (1 - p)B)\) would make 2 indifferent between L and M only when \(p = 2/3\), but \(q[L] + (1 - q)[M]\) would make 1 indifferent between T and B only if \(q = -1\), which is impossible.

**Guess:** 2's support is \{L, R\}. Notice \(q[L] + (1 - q)[R]\) makes 1 indifferent between T and B only when \(q 0 + (1 - q)8 = q 2 + (1 - q)5\), and so \(q = 3/5\).

\(p[T] + (1 - p)B\) makes 2 indifferent between L and R only when \(p 4 + (1 - p)9 = p 7 + (1 - p)1\), and so \(p = 8/11\). Then \(\text{Eu}_2(L) = (8/11)4 + (3/11)9 = 59/11 = \text{Eu}_2(R) = (8/11)7 + (3/11)1\).

But \(\text{Eu}_2(M) = (8/11)6 + (3/11)5 = 63/11 > 59/11\), and so there is no equilibrium with this support.

**Guess:** 2's support is \{M, R\}.

\(q[M] + (1 - q)[R]\) makes 1 indifferent between T and B only when \(q 5 + (1 - q)8 = q 6 + (1 - q)5\), and so \(q = 3/4\), which yields \(\text{Eu}_1(T) = (3/4)5 + (1/4)8 = 23/4 = \text{Eu}_1(B) = (3/4)6 + (1/4)5\).

\(p[T] + (1 - p)B\) makes 2 indifferent between M and R only when \(p 6 + (1 - p)5 = p 7 + (1 - p)1\), and so \(p = 4/5\). Then \(\text{Eu}_2(M) = (4/5)6 + (1/5)5 = 29/5 = \text{Eu}_2(R) = (4/5)7 + (1/5)1\), and \(\text{Eu}_2(L) = (4/5)4 + (1/5)9 = 25/5 < 29/5\), and so we have an equilibrium with this support.

This equilibrium is \((.8[T] + .2[B], .75[M] + .25[R])\), and it yields expected payoffs \((5.75, 5.8)\).

There cannot be any equilibrium where 2 randomizes over the whole support \{L, M, R\}, because, as we have seen, \(p[T] + (1 - p)B\) could make 2 indifferent between M and R only when \(p = 4/5\), but it could make 2 indifferent between L and R only when \(p = 8/11 \neq 4/5\).

(The L and M lines intersect at \(p = 2/3 = 0.667\). The L and R lines intersect at \(p = 8/11 = 0.727\), below the M line. The M and R lines intersect at \(p = 4/5 = 0.8\).)