Interim incentive efficiency and bargaining solutions with incomplete information

General notation on mechanism design with informational incentive constraints:

We restrict our attention to the case where agents' types are independent random variables. $p_i$ is the probability distribution of $i$'s type in $\Theta_i$, $p(\theta) = \prod_{i \in \mathbb{N}} p_i(\theta_i)$, $p_{-i}(\theta_{-i}) = \prod_{j \neq i} p_j(\theta_j)$. $u_i: \mathcal{X} \times \Theta - \mathbb{R}$ is $i$'s utility function.

A direct-revelation mechanism is any mapping from type-profiles to allocations $\mu: \Theta - \mathcal{X}$. $U_i(\mu, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p_{-i}(\theta_{-i}) u_i(\mu(\theta), \theta)$. $\hat{U}_i(\mu, s_i | \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p_{-i}(\theta_{-i}) u_i(\mu(\theta_{-i}, s_i), \theta)$.

$\mu$ is incentive compatible iff $\forall i \in \{1, ..., I\}, \forall \theta_i \in \Theta_i$, $\forall s_i \in \Theta_i$, $U_i(\mu | \theta_i) \geq \hat{U}_i(\mu, s_i | \theta_i)$.

A mechanism $\mu$ is (strongly) interim (Pareto-)dominated by another mechanism $\hat{\mu}$ iff $U_i(\hat{\mu} | \theta_i) > U_i(\mu | \theta_i)$, $\forall i \in \{1, ..., I\}, \forall \theta_i \in \Theta_i$.

$\mu: \Theta - \mathcal{X}$ is (weakly) interim incentive efficient iff $\mu$ is incentive compatible and $\mu$ is not interim Pareto-dominated by any other incentive-compatible mechanism $\hat{\mu}: \Theta - \mathcal{X}$.

So $\mu$ is interim incentive-efficient iff we can find nonnegative weights $\lambda_i(\theta_i)$ for all types $\theta_i$ of all agents $i$ such that: some $\lambda_i(\theta_i) > 0$, and $\mu$ is an optimal solution to the problem:

$$\max_{\mu: \Theta - \mathcal{X}} \sum_{i \in \{1, ..., I\}} \sum_{\theta_i \in \Theta_i} \lambda_i(\theta_i) U_i(\mu | \theta_i) \quad \text{s.t.} \quad U_i(\mu | \theta_i) \geq \hat{U}_i(\mu, s_i | \theta_i), \forall i, \forall \theta_i, s_i \in \Theta_i.$$
The Lagrangean of this problem (with multipliers $\alpha_i(s_i|\theta_i)\geq 0, \forall i, \forall \theta_i, \forall s_i$) is 
$$L(\mu;\alpha) = \sum_{i\in\{1,...,I\}} \sum_{\theta_i\in\Theta_i} \lambda_i(\theta_i) U_i(\mu|\theta_i) + \sum_{i\in\{1,...,I\}} \sum_{\theta_i\in\Theta_i} \alpha_i(s_i|\theta_i)[U_i(\mu|\theta_i) - \hat{U}_i(\mu,s_i|\theta_i)]$$

$$= \sum_{\theta} p(\theta) \sum_i [\lambda_i(\theta_i) u_i(\mu(\theta),\theta) + \sum_{s_i} \alpha_i(s_i|\theta_i) (u_i(\mu(\theta),\theta) - u_i(\mu(\theta_{-i},s_i),\theta))] / p_i(\theta_i)$$

$$= \sum_{\theta} p(\theta) \{\sum_i [\lambda_i(\theta_i) + \sum_{s_i} \alpha_i(s_i|\theta_i)] u_i(\mu(\theta),\theta) - \sum_{s_i} \alpha_i(s_i|\theta_i) u_i(\mu(\theta_{-i},s_i),\theta)} / p_i(\theta_i)$$

$$= \sum_{\theta} p(\theta) \{\sum_i [\lambda_i(\theta_i) + \sum_{s_i} \alpha_i(s_i|\theta_i)] u_i(\mu(\theta),\theta) - \sum_{s_i} \alpha_i(\theta_i|s_i) u_i(\mu(\theta),(\theta_{-i},s_i)) / p_i(\theta_i)$$

$$= \sum_{\theta} p(\theta) \sum_i v_i(\mu(\theta),\theta,\lambda,\alpha), \text{ where } v_i \text{ is the } (\lambda,\alpha)\text{-virtual utility function for agent } i,$$

defined to be $$v_i(x,\theta,\lambda,\alpha) = [(\lambda_i(\theta_i) + \sum_{s_i} \alpha_i(s_i|\theta_i)) u_i(x,\theta) - \sum_{s_i} \alpha_i(\theta_i|s_i) u_i(x,(\theta_{-i},s_i))] / p_i(\theta_i).$$

A Lagrange multiplier $\alpha_i(\theta_i|s_i)$ is zero unless its constraint $U_i(\mu|s_i) \geq U_i(\mu,\theta_i|s_i)$ binds, so we must always have complementary slackness: $\alpha_i(\theta_i|s_i)[U_i(\mu|s_i) - U_i(\mu,\theta_i|s_i)] = 0.$

With such an incentive-efficient $\mu$, type $s_i$ jeopardizes $\theta_i$ iff the incentive constraint $U_i(\mu|s_i) \geq \hat{U}_i(\mu,\theta_i|s_i)$ binds and has positive multiplier $\alpha_i(\theta_i|s_i) > 0$ in the Lagrangean for $\mu$.

Virtual utility of any type $\theta_i$ of agent $i$ differs from the actual utility of type $\theta_i$ by exaggerating the differences from $i$'s other types $s_i$ that jeopardize $\theta_i$.

The incentive-efficient $\mu$ appears ex-post efficient in terms of such virtual utilities.

Fact. So an incentive-compatible mechanism $\mu$ is incentive-efficient iff $\exists (\lambda,\alpha)$ such that:

all $\lambda_i(\theta_i) \geq 0, \alpha_i(s_i|\theta_i) \geq 0$; some $\lambda_i(\theta_i) > 0$; $\alpha_i(\theta_i|s_i)[U_i(\mu|s_i) - U_i(\mu,\theta_i|s_i)] = 0, \forall s_i, \forall \theta_i$; and

$$\sum_i v_i(\mu(\theta),\theta,\lambda,\alpha) = \max_{x\in X} \sum_i v_i(x,\theta,\lambda,\alpha), \forall \theta.$$
Example 1, trading: one seller, one buyer, one object. Each knows own private value of the object, which may be $0 (weak) or $80 (strong) for seller, each with probability 1/2, or which may be $100 (weak) or $20 (strong) for buyer, each with probability 1/2. Trade would be mutually beneficial unless both are strong, but what should be the price? A mediator who assists them must plan how the transaction may depend on the information they reveal. This is a mechanism. A natural class of symmetric mechanisms is shown below.

<table>
<thead>
<tr>
<th>Incentive compatibility:</th>
<th>Buyer's value</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.5qy + 0.5(50) \geq 0.5q(100-y)]</td>
<td>[strong] [weak]</td>
</tr>
<tr>
<td>Participation: [y \geq 20]</td>
<td>$20$ $100$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Seller's value</th>
<th>Buyer's value</th>
<th>$P(\text{trade})$, $E(\text{price if trade})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[strong] $80$</td>
<td>&quot;$80&quot;&quot;</td>
<td>0, * q, $100-y</td>
</tr>
<tr>
<td>[weak] $0$</td>
<td>&quot;$0&quot;&quot;</td>
<td>q, $y 1, $50</td>
</tr>
</tbody>
</table>

General symmetric mediation plan or mechanism

Split-the-difference with q=1 and y=10 would give the weak an incentive to claim "strong": 
\[0.5 \times 1 \times 20 + 0.5 \times 1 \times 50 \leq 0.5 \times 0 + 0.5 \times 1 \times 90.\]

Incentive compatibility requires \[0.5qy + 0.5(50) \geq 0.5(0) + 0.5q(100-y),\]
and so \[q \leq 25/(50-y).\]
Some interim incentive-efficient mechanisms with $y \leq 20$ and $q \leq 25/(50-y)$:

<table>
<thead>
<tr>
<th>$y=20$, $q=5/6$:</th>
<th>Buyer's value</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Seller's value table" /></td>
<td><img src="image" alt="EU table" /></td>
</tr>
<tr>
<td>P(trade)=2/3 (max!)</td>
<td></td>
</tr>
<tr>
<td>$(.5)(5/6)20 + (.5)50 = (.5)(5/6)80$</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$y=10$, $q=5/8$:</th>
<th>Buyer's value</th>
</tr>
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<tbody>
<tr>
<td><img src="image" alt="Seller's value table" /></td>
<td><img src="image" alt="EU table" /></td>
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</table>

<table>
<thead>
<tr>
<th>$y=0$, $q=1/2$:</th>
<th>Buyer's value</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Seller's value table" /></td>
<td><img src="image" alt="EU table" /></td>
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</table>

Neutral barg soln

The best among these for the strong types is the neutral bargaining solution.
y=0, q=1/2:

<table>
<thead>
<tr>
<th></th>
<th>Buyer's value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$20 [s]</td>
</tr>
<tr>
<td>EU(str)=$5</td>
<td>$80</td>
</tr>
<tr>
<td>EU(wk)=$25</td>
<td>$0</td>
</tr>
</tbody>
</table>

Neutral barg soln

This neutral bargaining solution is equivalent to a randomized dictatorship, where individual each has an equal probability 1/2 of making a first-and-final offer:
each type of seller would offer $100 (or $100−ɛ), each type of buyer would offer $0 (or $ε).
If such a randomized dictatorship is incentive efficient, it is a neutral bargaining solution.

The incentive-efficiency of the above mechanisms can be verified by Lagrangean analysis with \( \lambda_i(s) = 5/8 \), \( \lambda_i(w) = 3/8 \), \( \alpha_i(s|w) = 1/8 \), \( \alpha_i(w|s) = 0 \), for \( i=1 \)(seller), 2 (buyer).
Then an outcome where the probability of trade is \( q \) and the expected payment is \( y \) yields virtual utility \[
\frac{(5/8)(y-80q)-(1/8)(y-0q)}{0.5} = y-100q \text{ for a strong seller,}
\]
\[
(3/8+1/8)(y-0q)/0.5 = (y-0q) \text{ for a weak seller,}
\]
\[
\frac{(5/8)(20q-y)-(1/8)(100q-y)}{0.5} = 0q-y \text{ for a strong buyer,}
\]
\[
(3/8+1/8)(100q-y)/0.5 = (100q-y) \text{ for a weak buyer.}
\]
That is, a strong type's virtual valuation of the object just matches the other's valuation when weak, and so randomizing between trading and not-trading is virtually ex-post efficient.
But the neutral bargaining solution also looks equitable in terms of this virtual utility.
More generally, consider a two-player Bayesian bargaining problem where the disagreement outcome would yield payoff 0 for each type of each player. If $\mu: \Theta \rightarrow X$ is a neutral bargaining solution, then there exist vectors $(\lambda, \alpha, \omega)$ such that:

1. All $\lambda_i(\theta_i) \geq 0$, $\alpha_i(s_i | \theta_i) \geq 0$; some $\lambda_i(\theta_i) > 0$;
2. $U_i(\mu | s_i) \geq U_i(\mu, \theta_i | s_i)$ and $\alpha_i(\theta_i | s_i) [U_i(\mu | s_i) - U_i(\mu, \theta_i | s_i)] = 0$, $\forall s_i$, $\forall \theta_i$, $\forall i \in \{1, 2\}$;
3. $\sum_i v_i(\mu(\theta), \theta, \lambda, \alpha) = \max_{x \in X} \sum_i v_i(x, \theta, \lambda, \alpha)$, $\forall \theta$,

where $v_i(x, \theta, \lambda, \alpha) = [\lambda_i(\theta_i) \sum s_i \alpha_i(s_i | \theta_i)] u_i(x, \theta) - \sum s_i \alpha_i(\theta_i | s_i) u_i(x, (\theta_{-i}, s_i)) / p_i(\theta_i)$;

4. $\omega_i(\theta_i) \leq U_i(\mu | \theta_i)$, $\forall \theta_i$, $\forall i$;

and $[\lambda_i(\theta_i) \sum s_i \alpha_i(s_i | \theta_i) \omega_i(\theta) - \sum s_i \alpha_i(\theta_i | s_i) \omega_i(\theta_{-i}, s_i)] / p_i(\theta_i)$

$$= \sum_{\theta_{-i}} p_{-i}(\theta_{-i}) \sum_j v_j(\mu(\theta), \theta, \lambda, \alpha) / 2, \forall \theta_i, \forall i.$$

In fact, we will get $\omega_i(\theta_i) = U_i(\mu | \theta_i)$ unless $\lambda_i(\theta_i) = 0$.

This virtual-utility characterization extends the weighted-utility characterization of Nash's solution: $\exists (\lambda_1, \lambda_2)$ such that all $\lambda_1 > 0$, $\lambda_2 > 0$; $\sum_{i \in \{1, 2\}} \lambda_i u_i(\mu) = \max_{x \in X} \sum_{i \in \{1, 2\}} \lambda_i u_i(x)$, and $\lambda_1 u_1(\mu) = \lambda_2 u_2(\mu) = \sum_j \lambda_j u_j(\mu) / 2$.

Example 2. A firm is bargaining with a worker over the terms of employment $x = (q, y)$ where $q = \text{(quantity of labor supplied)}$, $y = \text{(total wages paid to worker)}$.

The firm's type is equally likely to be low L or high H, and so $p_2(L) = 1/2 = p_2(H)$.

The payoffs are $u_1(q, y) = y^{0.5} - q$, $u_2(q, y|L) = q^{0.5} - y$, $u_2(q, y|H) = 2q^{0.5} - y$.

In welfare analysis, we may let worker's weight be $\lambda = 1$. So an interim incentive-efficient mechanism $(q, y) = (q_L, q_H; y_L, y_H)$ must, for some $\lambda = (\lambda_H, \lambda_L)$:

maximize $(q, y)$ \[ \lambda_L[q_L^{0.5} - y_L] + \lambda_L[2q_H^{0.5} - y_H] + [0.5(y_H^{0.5} - q_H) + 0.5(y_H^{0.5} - q_H)] \]

subject to $q_L^{0.5} - y_L - (q_H^{0.5} - y_H) \geq 0$ \[[\text{Lagrange multiplier } \alpha_{|L}] \]

and $2q_H^{0.5} - y_H - 2q_L^{0.5} - y_L \geq 0$ \[[\text{Lagrange multiplier } \alpha_{|H}] \].

The solution $(q, y)$ can be computed from $\lambda$ and $\alpha$ by the first-order conditions

$0 = \frac{\partial}{\partial y_L} = 0.25/y_L^{0.5} + \alpha_{|H} - \alpha_{|L} - \lambda_L,

0 = \frac{\partial}{\partial y_H} = 0.25/y_H^{0.5} + \alpha_{|L} - \alpha_{|H} - \lambda_H,

0 = \frac{\partial}{\partial q_L} = (0.5/q_L^{0.5})(\lambda_L + \alpha_{|L} - 2\alpha_{|H}) - 0.5,

0 = \frac{\partial}{\partial q_H} = (0.5/q_H^{0.5})(2\lambda_H + 2\alpha_{|H} - \alpha_{|L}) - 0.5.

From the above, we can derive equations to compute $(\lambda, \alpha)$ from $(q, y)$:

$\alpha_{|L} = q_H^{0.5} - 0.5/y_H^{0.5}, \quad \alpha_{|H} = 0.25/y_L^{0.5} - q_L^{0.5}$,

$\lambda_L = 0.25/y_L^{0.5} + \alpha_{|H} - \alpha_{|L}, \quad \lambda_H = 0.25/y_H^{0.5} + \alpha_{|L} - \alpha_{|H}$.

For interim incentive efficiency, these must satisfy: $\lambda_L \geq 0$, $\lambda_H \geq 0$, $\alpha_{|L} \geq 0$, $\alpha_{|H} \geq 0$,

$q_L^{0.5} - y_L \geq q_H^{0.5} - y_H$, $2q_H^{0.5} - y_H \geq 2q_L^{0.5} - y_L$, and

$\alpha_{|L}[(q_L^{0.5} - y_L) - (q_H^{0.5} - y_H)] = 0$, and $\alpha_{|H}[(2q_H^{0.5} - y_H) - (2q_L^{0.5} - y_L)] = 0$. 
Payoffs are $u_1(q,y) = y^{0.5} - q$, $u_2(q,y|L) = q^{0.5} - y$, $u_2(q,y|H) = 2q^{0.5} - y$.

Even if incentive compatibility could be ignored, there would be basic qualitative difference between bargaining ex-ante (before firm learns its type) or ex post (after both learn types).

Even if incentive compatibility could be ignored, there would be basic qualitative difference between bargaining ex-ante (before firm learns its type) or ex post (after both learn types).

(Green for high-type firm, blue for low-type firm, red for worker.

Solid lines: isopreference at solution. Dashed lines: isopreference at 0. Dotted lines: ex-post efficient $(q,y)$ pairs.)

In ex-ante Nash bargaining solution without incentive constraints, the worker gets full income insurance, but the firm gets more labor when productivity is higher, and so the low-type firm would strictly prefer to act high.

So ex-ante analysis suggests low type jeopardizes high (to win on worker's insurance).
Ex-post Nash bargaining solution in each state, not IC.

In ex-post Nash bargaining solutions, the high firm pays more for labor and so strictly prefers to act low. 
So ex-post analysis suggests that high jeopardizes low (for stronger bargaining position).
With IC, still, low jeopardizes high ex ante, but high jeopardizes low in interim solutions.

Ex-ante Nash bargaining solution with incentive compatibility. Interim neutral bargaining solution, with incentive compatibility.

(Solid lines: isopreference at solution. Dashed lines: isopref at 0. Dotted lines: ex-post efficient.)

Bargaining solutions shown in utility diagram: dark squares are interim, triangles are ex ante; blue are for low-type, green are high-type; large symbols are for equal bargaining-ability, small are for bargaining ability on one side only (high pairs are from workers-dictating, low pairs are from firm-dictating).
Interim and ex-ante bargaining solutions with incentive compatibility:

(Dark squares are interim solutions, light triangles are ex ante solutions; blue are payoffs when low, green are when high; large symbols are for equal bargaining ability, small are for bargaining ability on one side only (high pairs are from workers-dictating, low pairs are from firm-dictating).
The firm's best ex ante \( (q,y) \) maximizes \( 0.5U_2(q,y | L) + 0.5U_2(q,y | H) \) subject to incentive compatibility and an ex-ante participation constraint for the worker: \( U_1(q,y) \geq 0 \).

The solution is \( q_L = 0.351, \ y_L = 0.178, \ q_H = 0.738, \ y_H = 0.444 \), which yields payoffs \( u_1(q_L,y_L) = 0.071, \ U_2(q,y | L) = u_2(q_L,y_L | L) = 0.414, \ u_1(q_H,y_H) = -0.071, \ U_2(q,y | H) = 1.273, \ U_1(q,y) = 0.5(0.071) + 0.5(-0.071) = 0 \).

The low type jeopardizes high in this mechanism. It is supported as incentive-efficient by \( \lambda_L = \lambda_H = 0.484, \ \lambda_1 = 1, \ \alpha | _L = 0.109, \ \alpha | _H = 0 \) (or \( \lambda_L = \lambda_H = 0.5, \ \lambda_1 = 1.034, \ \alpha | _L = 0.112, \ \alpha | _H = 0 \)).

The ex-ante Nash bargaining solution maximizes the ex-ante Nash product \( U_1(q,y)[U_2(q,y | L) + 0.5U_2(q,y | H)] \) subject to incentive compatibility.

The solution is \( q_L = 0.187, \ y_L = 0.334, \ q_H = 0.490, \ y_H = 0.601 \), which yields the utility payoffs \( u_1(q_L,y_L) = 0.391, \ U_2(q,y | L) = 0.0984, \ u_1(q_H,y_H) = 0.286, \ U_2(q,y | H) = 0.798, \ U_1(q,y) = 0.5(0.391) + 0.5(0.286) = 0.338 \).

The low type jeopardizes the high in this mechanism, and it is supported as incentive-efficient by \( \lambda_L = \lambda_H = 0.377, \ \lambda_1 = 1, \ \alpha | _L = 0.0551, \ \alpha | _H = 0 \).

The worker's best ex ante \( (q,y) \) maximizes \( U_1(q,y) \) subject to incentive compatibility and an ex-ante participation constraint for the firm \( 0.5U_2(q,y | L) + 0.5U_2(q,y | H) \geq 0 \).

The solution is \( q_L = 0.103, \ y_L = 0.605, \ q_H = 0.322, \ y_H = 0.851 \), which yields the utility payoffs \( u_1(q_L,y_L) = 0.674, \ U_2(q,y | L) = -0.284 < 0 (!), \ u_1(q_H,y_H) = 0.601, \ U_2(q,y | H) = 0.284, \ U_1(q,y) = 0.5(0.674) + 0.5(0.601) = 0.638 \).

The low type jeopardizes the high in this mechanism, and it is supported as incentive-efficient by \( \lambda_L = \lambda_H = 0.296, \ \lambda_1 = 1, \ \alpha | _L = 0.0249, \ \alpha | _H = 0 \).
The low firm would not participate with $U_2(q,y|L) < 0$ when it bargains knowing its type. So worker's best interim $(q,y)$ maximizes $U_1(q,y)$ subject to incentive compatibility and two interim participation constraints for the firm: $U_2(q,y|L) \geq 0$ and $U_2(q,y|H) \geq 0$.

The solution is $q_L = 0.057$, $y_L = 0.239$, $q_H = 0.295$, $y_H = 0.847$, which yields the utility payoffs $u_1(q_L,y_L) = 0.432$, $U_2(q,y|L) = 0$, $u_1(q_H,y_H) = 0.625$, $U_2(q,y|H) = 0.239$, $U_1(q,y) = 0.5(0.432) + 0.5(0.625) = 0.529$.

The high type jeopardizes the low in this mechanism, and it is supported as incentive-efficient by $\lambda_H = 0$, $\lambda_L = 0.783$, $\lambda_1 = 1$, $\alpha|L = 0$, $\alpha|H = 0.783$.

The firm's best best safe $(q,y)$ maximizes any weighted average of $U_2(q,y|L)$ and $U_2(q,y|H)$ subject to incentive compatibility and ex-post participation constraints for the worker: $u_1(q_L,y_L) \geq 0$ and $u_1(q_H,w_H) \geq 0$.

(Informational incentive constraints and ex-post participation constraints together imply that each type of firm's expected payoff is maximized by the same best safe mechanism.)

The best safe solution for the firm here is $q_L = 0.397$, $y_L = 0.157$, $q_H = 0.630$, $y_H = 0.397$, which yields the utility payoffs $u_1(q_L,y_L) = 0$, $U_2(q,y|L) = 0.472$, $u_1(q_H,y_H) = 0$, $U_2(q,y|H) = 1.191$, $U_1(q,y) = 0.5(0.071) + 0.5(-0.071) = 0$.

Both incentive constraints are satisfied as strict inequalities. This best safe mechanism is incentive-efficient, supported by $\lambda_1 = 1$, $\lambda_L = 0.630$, $\lambda_H = 0.397$, $\alpha|L = 0 = \alpha|H$.

When the best safe mechanism is incentive efficient, it is a strong solution for the firm as informed principal at the interim stage. (See Myerson, Econometrica (1983).)
Payoffs are \( u_1(q,y) = y^{0.5} - q \), \( u_2(q,y|L) = q^{0.5} - y \), \( u_2(q,y|H) = 2q^{0.5} - y \).

The interim neutral bargaining solution is \( q_L = 0.156 \), \( y_L = 0.122 \), \( q_H = 0.409 \), \( y_H = 0.611 \), which yields the utility payoffs \( u_1(q_L,y_L) = 0.193 \), \( U_2(q,y|L) = u_2(q_L,y_L|L) = 0.273 \), \( u_1(q_H,y_H) = 0.372 \), \( U_2(q,y|H) = 0.669 \), \( U_1(q,y) = 0.5(0.193)+0.5(0.372) = 0.283 \).

This mechanism is incentive compatible, with the high-type firm jeopardizing the low type: \( U_2(q,y|L) > u_2(q_H,y_H|L) = 0.029 \), \( U_2(q,y|H) = u_2(q_L,y_L|H) = 0.669 \).

It is supported as incentive-efficient by \( \lambda_1 = 1 \), \( \lambda_L = 1.035 \), \( \lambda_H = 0 \), \( \alpha_H = 0.320 \), \( \alpha_L = 0 \). Then the firm's virtual utilities are
\[
\begin{align*}
    v_2(q,y,L) &= [1.035(q^{0.5} - y) - 0.32(2q^{0.5} - y)]/0.5 = 1.43(0.553q^{0.5} - y), \\
    v_2(q,y,H) &= 0.32(2q^{0.5} - y)/0.5 = 0.64(2q^{0.5} - y),
\end{align*}
\]
and the worker's virtual utility is the same as actual utility \( v_1(q,y) = y^{0.5} - q \).

In each state, the \((q,y)\) allocation maximizes the sum of virtual utilities \( v_1+v_2 \), yielding:
\[
\begin{align*}
    v_1(q_L,y_L)+v_2(q_L,y_L,L) &= 0.193+0.138 = 0.331, \\
    v_1(q_H,y_H)+v_2(q_H,y_H,H) &= 0.372+0.428 = 0.800. 
\end{align*}
\]
Even with welfare-weight \( \lambda_H=0 \), the high-type firm gets more virtual utility (0.428) than it gives the worker (0.372).

The warranted claims of the two types of firm are \( \omega_2(L) = U_1(q,y|L) = 0.273 \), and \( \omega_2(H) = 0.625 < U_2(q,y|H) = 0.669 \), which correspond to equitable virtual utilities
\[
\begin{align*}
    [1.035(0.273) - 0.32(0.625)]/0.5 &= 0.165 = (1/2)\times0.331 \text{ (half the low-state virtual utilities)}, \\
    0.32(0.625)/0.5 &= 0.4 = (1/2)\times0.800 \text{ (half the high-state virtual utilities)}. 
\end{align*}
\]
Some references:

http://home.uchicago.edu/~rmyerson/research/interim.pdf
\[ u_1(q,y) = y^{0.5} - q, \quad u_2(q,y|L) = q^{0.5} - y, \quad u_2(q,y|H) = 2q^{0.5} - y. \quad p_2(L) = 1/2 = p_2(H). \]

(\text{Red for worker. Blue for low-type firm. Green for high-type firm. Dotted: ex-post efficient \((q,y)\).})

Ex ante Nash bargaining solution without IC.
Low (blue) type of firm would report high (green) type.

Ex ante Nash bargaining solution with incentive compatibility.

Ex post Nash bargaining solution in each state, without IC.
High (green) type of firm would report low (blue) type.

Interim neutral bargaining solution, with IC.
Worker's best with IC, firm interim participation (IR).

Firm's best safe (IC, worker participation with each type).

Interim neutral bargaining solution, with IC.

Firm's best ex ante (IC, worker participation only ex ante).