CAPITALIST INVESTMENT AND POLITICAL LIBERALIZATION

F = (fixed-resource endowment). Additional invested capital is durable and mobile. 
Y(F+k) is net output production (flow) with any investment k ≥ 0. 
Suppose Y(·) is differentiable, strictly concave, Y'(0) = +∞, Y'(+∞) = 0. 
Capitalists' rate of time discounting is r. They must get rent rk on investment k. 
ρ = (rate of time discounting for an authoritarian ruler). 
With investment k, authoritarian ruler gets value  (Y(F+k)−rk)/ρ.  
θ = (fraction of investment that the ruler could expropriate). 
With worst reputational threat, expropriation would yield  θk + Y(F)/ρ. 

Capitalist investment k is feasible without liberalization iff 
(Y(F+k)−rk)/ρ ≥ θk + Y(F)/ρ. 
Equivalently, Y(F+k)−(r+ρθ)(F+k) ≥ Y(F)−(r+ρθ)F.

Let  K_r = argmax_{κ≥0} Y(κ) − rk,
K_{r+ρθ} = argmax_{κ≥0} Y(κ) − (r+ρθ)κ.
\[(Y(F+k) - rk) / \rho \geq \theta k + Y(F) / \rho \quad \text{iff} \quad Y(F+k) - (r + \rho \theta)(F+k) \geq Y(F) - (r + \rho \theta)F.\]

Let \( K_r = \text{argmax}_{\kappa \geq 0} Y(\kappa) - rk \),

\[K_{r+\rho \theta} = \text{argmax}_{\kappa \geq 0} Y(\kappa) - (r + \rho \theta)\kappa.\]

**Thm 1.** \( \exists h_0 \) such that \( k \) is feasible without liberalization iff \( 0 \leq k \leq h_0 \).

In this interval, the optimum for ruler is \( k = \min\{h_0, K_r - F\} \).

If \( F \geq K_{r+\rho \theta} \), then \( h_0 = 0 \).

If \( F < K_{r+\rho \theta} \), then \( F+h_0 > K_{r+\rho \theta} \) and \( Y(F+h_0) - (r + \rho \theta)(F+h_0) = Y(F) - (r + \rho \theta)F. \)

When \( F < K_{r+\rho \theta} \), a small increase in \( F \) would cause both \( h_0 \) and \( F+h_0 \) to decrease.

**Cor 1.** Given \((Y(), r, \rho, \theta)\), if \( Y(K_r) - (r + \rho \theta)K_r < Y(0) \) then the ideal investment \( K_r - F \) is not feasible without liberalization for any fixed endowment \( F < K_r \).

But if \( Y(K_r) - (r + \rho \theta)K_r \geq Y(0) \), then there exists some \( f_0 \) such that the ideal investment \( K_r - F \) is feasible without liberalization iff \( F \leq f_0 \).

This bound \( f_0 \) satisfies \( Y(f_0) - (r + \rho \theta)f_0 = Y(K_r) - (r + \rho \theta)K_r \) and \( f_0 < K_{r+\rho \theta} < K_r \).

(So \( F \) near \( K_{r+\rho \theta} \) always makes ideal \( K_r \) infeasible without liberalization.)

**Cor 2.** If the production function is \( Y(\kappa) = A\kappa^\alpha \), for some \( A > 0 \) and \( 0 < \alpha < 1 \), then the ideal investment \( K_r - F \) is feasible without liberalization for \( F = 0 \) iff \( \alpha \leq r / (r + \rho \theta) \).

(So even at \( F = 0 \), ideal \( K_r \) is infeasible without liberalization when \( \alpha > r / (r + \rho \theta) \).)
Liberalization $\lambda$ is probability of ruler losing power if he tried to expropriate capital. False-alarm scandals occur at rate $\psi$.

Ruler's present discounted value is $V(k,\lambda) = (Y(F+k)-rk)/(\rho+\psi\lambda)$.

Trying to expropriate would yield $W(k,\lambda) = (1-\lambda)(\theta k + Y(F)/\rho)$.

Optimal regime $(k,\lambda)$ maximizes $V(k,\lambda)$ s.t. $V(k,\lambda) \geq W(k,\lambda)$, $k \geq 0$, $0 \leq \lambda \leq 1$.

Let $Q(k) = [Y(F+k)-rk]/[\theta k + Y(F)/\rho]$, and $q(\lambda) = (\rho+\psi\lambda)(1-\lambda)$.

So $V(k,\lambda) \geq W(k,\lambda)$ iff $Q(k) \geq q(\lambda)$.

Let $\Lambda(k)$ denote the smallest $\lambda \geq 0$ such that $Q(k) \geq q(\lambda)$.

Optimal $k$ maximizes $V(k,\Lambda(k))$ over $k \geq 0$.

Notice $q(\lambda) = \rho+(\psi-\rho)\lambda - \psi\lambda^2$, which is maximized at $(1-\rho/\psi)/2$,
and $q(0) = \rho = q(1-\rho/\psi)$, $q(1)=0$.

So if $Q(k) \geq \rho$ then $\Lambda(k) = 0$.
If $Q(k) < \rho$ then $\Lambda(k) = \{\psi-\rho + [(\psi-\rho)^2 + 4\psi(\rho-Q(k))]^{0.5}\}/(2\psi)$.

**Thm 2.** If $(k,0)$ is optimal, then $k=\min\{h_0, K_r-F\}$, as in Thm 1.

If $(k,\lambda)$ is optimal with $\lambda>0$, then $Q(k) = q(\lambda)$, $Y'(F+k) = r + \theta \psi (1-\lambda)^2$,
$\lambda = \Lambda(k) > 1-\rho/\psi$, and $k > h_0$. 
Recall that the worst endowment without liberalization was $F = K_{r+\rho \theta}$. Incentives for liberalization may be greatest with such intermediate endowments.

**Cor 3.** If $F = K_{r+\rho \theta}$ and $\psi < \rho$ then the optimal regime has $\lambda > 0$.

Strong liberalization becomes optimal in cases where investments yield approximately constant returns for national output. Approximate linearity implies that, when $Y' > r$, large investments can yield surplus returns, but then strong liberalization is needed to protect these investments.

**Theorem 3.** Consider production functions of the form $Y(F+k) = A(F+k)^\alpha$, where $A > 0$ and $0 < \alpha < 1$, so that the parameters of our models are $(r, \rho, \theta, \psi, F, A, \alpha)$. Consider a sequence of models where $(r, \rho, \theta, \psi, A, \alpha)$ converge to finite positive limits, and the fixed endowments $F$ satisfy $\lim Y'(F)/r > 1$. If $\lim \alpha = 1$ then the optimal $(k, \lambda)$ satisfy $\lim Y'(F+k)/r = 1$ and $\lim \lambda = 1$. 
Consider adding fixed revenue for the government, independent of invested capital. Adding a fixed revenue means adding a positive constant $z>0$ to the production function, changing it from $Y(\kappa)$ to $\dot{Y}(\kappa) = Y(\kappa) + z$ for all $\kappa \geq 0$, keeping all other parameters unchanged.

**Thm 4.** Adding a fixed revenue $z>0$ would allow greater investments to be feasible with any given liberalization $\lambda$ such that $\max\{0,1-\rho/\psi\} < \lambda < 1$.

But when the optimal regime has positive liberalization $\lambda$, adding a fixed revenue would decrease both the optimal liberalization and the optimal investment.

When the optimal regime involves no liberalization ($\lambda=0$), adding a fixed revenue would not change the optimal regime.

**Proof:** Adding $z>0$ to $Y$ increases $V$ by $z/(\rho+\psi \lambda)$ and increases $W$ by $z(1-\lambda)/\rho$.

With $\lambda > \max\{0,1-\rho/\psi\}$, we get $(1-\lambda)(\rho+\psi \lambda) < \rho$ and $z/(\rho+\psi \lambda) > z(1-\lambda)/\rho$.

Let $K(\lambda) = \Lambda^{-1}(\lambda)$. Adding $z>0$ increases $K(\lambda)$.

$$K'(\lambda) = q'(\lambda)/Q'(K(\lambda)) = (\psi - \rho - 2\psi \lambda) (\theta k + Y(F)/\rho)/[Y'(F+K(\lambda)) - r - \theta q(\lambda)].$$

$$\frac{d}{d\lambda} \ln(V(K(\lambda),\lambda)) = \frac{1}{W} \frac{d}{d\lambda} [(1-\lambda)(\theta K(\lambda) + Y(F)/\rho)]$$

$$= \theta (2\psi \lambda + \rho - \psi)/[r + \theta q(\lambda) - Y'(F+K(\lambda))] - 1/(1-\lambda),$$

decreasing in $z$ (thru $K$). So increasing $z$ decreases optimal $\lambda$ and $k$, with $Y'(F+k) = r + \theta \psi (1-\lambda)^2$. 


\[
\max_{k \geq 0, \lambda \in [0,1]} \quad V(k, \lambda) = \frac{(Y(F+k) - rk)}{(\rho + \psi \lambda)} \quad \text{s.t.} \quad V(k, \lambda) \geq (1 - \lambda)(\theta k + Y(F)/\rho).
\]

**Example.** Consider an island with production \( Y(F+k) = (F+k)^{0.4} \), investors' discount rate \( r=0.05 \), authoritarian-ruler's discount rate \( \rho=0.1 \), expropriatable fraction \( \theta=1 \), and scandal rate \( \psi=0.1 \).

The ideal investment \( K_r=32 \) (with \( Y(K_r)=4 \)) is not feasible without liberalization. With \( F=0 \), optimum is \( k = 23.61, \lambda = 0 \); then \( Y(F+k) = 3.54, \; V(k,\lambda) = 23.61 \).

With \( F=K_r + \rho \theta = 5.128 \), optimum is \( k=0, \lambda=0 \); then \( Y(F+k) = 1.92, \; V(k,\lambda) = 19.23 \).

The curse of natural resources even harms the ruler here.

\( \lambda=0 \) is optimal \( \forall F \) when \( \psi=0.10 \) here.

But reducing the scandal rate to \( \psi=0.05 \) makes positive liberalization \( \lambda \) optimal for intermediate endowments \( 1.9 < F < 10.1 \).
Example. Consider an archipelago where production is \((F+k)^{0.4}L^{0.5}\).
Each island has labor \(L=1\), fixed capital \(F=5\), \(r=0.05\), \(\rho=0.1\), \(\theta=1\), \(\psi=0.1\).
When \(L\) are serfs, we get \(Y=(F+k)^{0.4}\), optimum \(k=0.258\), \(\lambda=0\); \(Y=1.94\), \(V=19.29\).
A free labor fringe has marginal product \(0.5Y/L = 0.971\).

Suppose one island frees its serfs, extensively recruits mobile free labor at wage \(w\):
\(Y(F+k) = \max_{L \geq 0} (F+k)^{0.4}L^{0.5} - wL = (F+k)^{0.8}/(4w)\), with \(L = (F+k)^{0.8}/(2w)^2\).
With \(w=0.971\), we get \(Y(F+k) = 0.2574(F+k)^{0.8}\), optimum is \(k=1126\), \(\lambda=0.931\), with \(L=73.5\), yielding \(Y = 71.36\), \(V(k,\lambda) = 77.95\).
The possibility of matching capitalist investments with additional labor creates a strong incentive for liberalization \((77.95 > 19.29)\).
But 73.5>1. If all free serfs and compete for free labor, the wage must increase.

With \(w=1.777\), we have \(Y(F+k) = 0.1407(F+k)^{0.8}\) and get two optimal regimes:
liberal optimum \(k=47.73\), \(\lambda=0.903\), with \(L=1.89\), \(V=5.10\);
nonliberal optimum \(k=0\), \(\lambda=0\), with \(L=0.29\), \(V=5.10\). \((K_r-F=57.8-5=52.8.)\)
Average matches supply \(L=1\) when 44% of islands are liberal, 56% nonliberal.

Reducing \(\psi\) can decrease optimal \(\lambda\). An island with \(\psi=0.05\) has \(k=48.44\), \(\lambda=0.874\);
\(\lim_{\psi \to 0} \lambda = 1 - [Y(K_r) - r(K_r - F)]/[Y(F) + \rho \theta(K_r - F)] = 0.832\).
Figure 2. Required liberalization $\Lambda(k)$ and ruler's value $V(k,\Lambda(k))$, for $Y(F+k)=0.1407(F+k)^{0.8}$ with $F=5$, $r=.05$, $\rho=.1$, $\psi=.1$, $\theta=1$; from $w=1.777$. 