The Race Between Technology and Human Capital

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Abstract

This paper develops a model in which heterogenous firms invest in R&D to improve technology, and heterogeneous workers invest in human capital to increase their earnings. Both investment technologies have stochastic components, and the balanced growth path has stationary, nondegenerate distributions of technology and human capital.

Technology and human capital are complements in production, so the labor market produces assortative matching between firms and workers: firms with higher productivity employ higher quality workers and pay higher wages. Thus, wage differentials across firms have two sources: differences in firm productivity and differences in labor quality.

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1. OVERVIEW

This paper develops a model in which heterogenous firms invest in R&D to improve technology, and heterogeneous workers invest in human capital to increase their earnings. Both investment technologies have stochastic components, and the balanced growth path has stationary, nondegenerate distributions of technology and human capital.

Technology and human capital are complements in production, so the labor market produces assortative matching between firms and workers: firms with higher productivity employ higher quality workers and pay higher wages. Thus, wage differentials across firms have two sources: differences in firm productivity and differences in labor quality.

The heterogeneous firms produce intermediate goods, which are combined with a Dixit-Stiglitz aggregator to produce the single final good. Final goods are used for consumption and three kinds of investment. Incumbent firms invest to improve their productivity, and they die stochastically. Entering firms pay a fixed cost to obtain an initial technology. Workers invest to increase their human capital.

One goal of the model is to examine the interplay between human accumulation and technological change as contributors to long-run growth. From an empirical point of view, the chicken-and-egg issue makes it difficult to distinguish a single “engine” of growth. A theoretical framework that builds in the symbiotic nature of improvements in the two factors may provide insights for assessing, in particular contexts, the role of each.

A second goal is assess the sources of wage inequality. Empirically, it is not easy to distinguish the importance of technology and human capital differences in generating wage differentials across firms. A theoretical framework that incorporates complementarity between the two may be useful in assessing the importance of each.
The setup here builds on the model of technology growth across firms in Luttmer (QJE, 2007), incorporating an active investment decision as Atkeson and Burstein (JPE, 2010). On the human capital side, it develops a similar investment model. From a substantive point of view, the model provides a link between the literature on human capital-based growth, as in Alvarez, Buera and Lucas (2008, 2013), Lucas (1988, 2009), Perla and Tonetti (2014), Perla, Tonetti and Waugh (2014), and others, and the literature on technology-driven growth, as in Atkeson and Burstein (2010), Klette and Kortum (2004), Luttmer (2007), and others. The model also has implications for wage inequality across age cohorts of workers, as documented in Deaton and Paxson (1994), and for productivity dynamics in firms, as documented in Bailey et al. (1992), Bartelsman and Doms (2000), Dunne Roberts and Samuelson (1989), and Hsieh and Klenow (2014). It is also related to the model of technology and wage inequality in Jovanovic (1998).

To start, we will assume that both the average productivity of entering firms and the average initial human capital of new workers grow at a common rate $g$. We will look for a BGP where the cross-sectional distribution of productivities across firms and of human capital across workers are both lognormal, with constant variances. The entering productivities for new firms and workers can then be made endogenous. Specifically, at each date they will be draws from a distribution that depends on the current cross-sectional distribution.

A. Variables

Exogenous:

- population $L$ is exogenous and constant, with birth and death at rate $\delta_L$;
- exit rate for firms, $\delta_F > 0$;
- productivity of entering firms at date $t$ is lognormally distributed, with a fixed
variance and a mean that grows at the constant rate $g$,

$$\ln X_{i\alpha} \sim N \left( \mu_{EX} + gt, \sigma^2_{EX} \right);$$

human capital of entering workers at date $t$ is lognormally distributed, with fixed variance and a mean that grows at the constant rate $g$,

$$\ln H_{j\alpha} \sim N \left( \mu_{EH} + gt, \sigma^2_{EH} \right).$$

Individual decisions:

- productivity $X_{i\alpha}$ at age $\alpha$ of incumbent firm $i$ that entered at date $t$ is a geometric Brownian motion, with fixed variance $\sigma^2_X$.

  The firm’s investment decision is the choice of the drift $g_X$.

- human capital $H_{j\alpha}$ at age $\alpha$ of worker $j$ who entered at date $t$ is a geometric Brownian motion, with fixed variance $\sigma^2_H$.

  The worker’s investment decision is the choice of the drift $g_H$.

Endogenous:

- $N$, the (constant) number of incumbent firms, is determined by free entry.

  $Q_t$ is an aggregate variable, “average productivity.” It grows at rate $g$.

  Its level depends the cross-sectional distribution of productivities $X_{it}$.

  $x_{ita} = X_{ita}/Q_{t+a}$ is firm $i$’s relative productivity at age $\alpha$. On a BGP, $x_i$ has a stationary cross-sectional distribution that depends on $\mu_{EX}, g_X, \sigma^2_{EX}, \sigma^2_X$.

  Firm $i$’s labor demand and profits at date $t + a$ depend on $(x_{ita}, Q_{t+a})$.

  $h_{jta} = H_{jta}/Q_{t+a}$ is worker $j$’s relative human capital. On a BGP, $h_{jta}$ has a stationary cross-sectional distribution that depends on $\mu_{EH}, g_H, \sigma^2_{EH}$ and $\sigma^2_H$.

  $W(h, Q)$ is the wage of a worker with relative human capital $h$ in an economy with average productivity $Q$. 


2. THE STATIC MODEL

In this section we solve for the static equilibrium, given $L, N$, and the distribution functions $F(X), M(H)$.

A. Final good technology

The final good is produced by competitive firms using intermediate goods as inputs. All intermediates enter symmetrically into final good production, but demands for them differ if their prices differ. Specifically, intermediate producers are indexed by their productivity $X > 0$, which determines the price $p(X)$ for their good. Let $f(X)$ denote the density for $X$ across intermediate producers, and let $N$ be the number (mass) of firms. Each final good producer has the CRS technology

$$Y_F = \left[ N \int Y(X)^{(\rho-1)/\rho} f(X)dX \right]^{\rho/(\rho-1)}, $$

where $\rho > 1$ is the substitution elasticity. The final goods sector takes the prices $p(X)$ as given. As usual, the price of the final good is

$$p_F = \left[ N \int p(X)^{1-\rho} f(X)dX \right]^{1/(1-\rho)}, $$

and input demands are

$$Y^d(X) = \left( \frac{p(X)}{p_F} \right)^{-\rho} Y_F, \quad \text{all } X. $$

B. Intermediate producers: choice of labor quality

Intermediate producers use heterogeneous labor, differentiated by its human capital level $H$, as the only input. The output of a firm depends on the size and quality of its workforce, as well as its technology. In particular, if a firm with technology $X$ employs $\ell$ workers with human capital $H$, then its output is

$$Y = \ell \phi(H, X), $$
where $\phi(H, X)$ is the CES function
\[
\phi(H, X) \equiv \left[ \omega H^{(\eta-1)/\eta} + (1 - \omega) X^{(\eta-1)/\eta} \right]^{\eta/(\eta-1)}, \quad \eta, \omega \in (0, 1). \quad (4)
\]
The elasticity of substitution $\eta$ between technology and human capital is assumed to be less than unity, and $\omega$ is the relative weight on human capital. Firms could employ workers with different human capital levels, and in this case their outputs would simply be summed. In equilibrium firms never choose to do so, however, and for simplicity the notation is not introduced.

Let $W(H)$ denote the wage function. For a firm with technology $X$, the cost of producing one unit of output with labor of quality $H$ is $W(H)/\phi(H, X)$. Optimal labor quality $H^*(X)$ minimizes this expression.

Conjecture that the wage function has the constant elasticity form
\[
W(H) = W_0 H^{1-\varepsilon}, \quad \varepsilon \in (0, 1). \quad (5)
\]
Then optimal labor quality is proportional to $X$, with a constant of proportionality that depends on $\varepsilon$,
\[
H^*(X) = a_H X, \quad (6)
\]
where
\[
a_H \equiv \left( \frac{1 - \varepsilon}{\varepsilon} \frac{1 - \omega}{\omega} \right)^{\eta/(\eta-1)}. \quad (7)
\]
Unit cost is then
\[
\frac{W(a_H X)}{\phi(a_H X, X)} = \frac{a_H^{1-\varepsilon}}{\phi_0} W_0 X^{-\varepsilon}, \quad (8)
\]
where
\[
\phi_0 \equiv \phi(a_H, 1). \quad (9)
\]
For $W(H)$ as in (5), the cost minimization problem is concave if (and only if) $\eta \in (0, 1)$, as assumed here. The quantity of labor hired is proportional to the target level of output,
\[
\ell^*(X; Y_F) = \frac{Y_F}{\phi_0} X^{-1}. \quad (10)
\]
Figure 1 displays isoquants for output and expenditure (total wage bill) in quality-quantity space, for the model parameters

$$\eta = 0.5, \quad \omega = 0.5, \quad W_0 = 1, \quad \varepsilon = 0.5,$$

and the technology, output, and expenditure levels

$$X_1 = 0.5, \quad X_2 = 1, \quad Y_1 = 0.5, \quad Y_2 = 1, \quad E_{11} = 0.7071, \quad E_{12} = 1.4142, \quad E_{21} = 0.5, \quad E_{22} = 1.$$

The dashed curves are output isoquants for the technology level $X_1$, and the broken curves are isoquants for $X_2 > X_1$. The four solid curves are expenditure isoquants, and the small circles indicate the four cost-minimizing input mixes. With $X$ fixed, a higher output level $Y$ increases only the quantity $\ell$ of labor input. With $Y$ fixed, a higher technology level $X$ increases labor quality $H$ and reduces quantity $\ell$. Note that cost minimization by firms implies positively assortative matching: firms with better technologies hire workers with more human capital.

C. Intermediate goods: pricing problem

Suppose the wage function has the conjecture form in (5), so unit cost is as in (8). Given the price $p_F$ for the final good, an intermediate firm with productivity $X$ chooses its price $\hat{p}(X)$ to maximize profits. As usual, the optimal price is a markup $\rho/ (\rho - 1)$ over unit cost. Let $w_0 \equiv W_0/p_F$ denote the scale for the real wage. Then (relative) price, quantity, labor input, and (real) profits for the intermediate firm involve various powers of $X$,

$$\frac{\hat{p}(X)}{p_F} = \frac{\rho}{\rho - 1} \phi_0 \phi_0^{-\varepsilon} p_0 W_0 X^{-\varepsilon} \equiv p_0 w_0 X^{-\varepsilon}, \quad (11)$$

$$\hat{Y}(X) = \left(\frac{\hat{p}(X)}{p_F}\right)^{-\rho} Y_F \equiv (p_0 w_0)^{-\rho} Y_F X^{\rho \varepsilon}, \quad (12)$$
\[
\hat{\ell}(X) = \frac{\dot{Y}(X)}{\phi_0}X^{-1} \equiv \frac{1}{\phi_0} (p_0w_0)^{-\rho} Y_F X^{\rho \varepsilon - 1}, \tag{13}
\]

\[
\frac{\hat{\pi}(X)}{p_F} = \frac{1}{\rho p_F} \dot{\pi}(X) \dot{Y}(X) \equiv \frac{1}{\rho} (p_0w_0)^{1-\rho} Y_F X^{(\rho - 1)\varepsilon}, \tag{14}
\]

where \( Y_F \) is output of the final good, and the constant

\[
p_0 \equiv \frac{\rho}{\rho - 1} \frac{a_{\varepsilon}}{\phi_0},
\]

depends on \( \varepsilon \). The price \( \dot{\pi}(X)/p_F \) depends only on \( w_0 \) and \( X \), while the quantities \( \dot{Y}(X), \hat{\ell}(X), \) and \( \hat{\pi}(X)/p_F \) also depend on \( Y_F \). Note that firms with higher \( X \) have lower prices, higher sales, and higher profits. The effect of productivity on labor input, in (13), depends on \( \varepsilon \), the elasticity of the wage function.

### D. State variables

To analyze investment by firms and households, it is convenient to exploit the fact that on a BGP the means of the \( \dot{X} \)’s and \( \dot{H} \)’s grow at the common, constant rate \( g \), and to look at normalized variables. To this end, define “average productivity”

\[
Q \equiv [E(X^\beta)]^{1/\beta}, \quad \beta \equiv (\rho - 1)/\rho, \tag{15}
\]

and the relative values \( x \equiv X/Q, \ h \equiv H/Q \). On a BGP \( Q \) grows at the constant rate \( g \), and \( x, h \), have stationary distributions. Thus, aggregates depend on \( Q \) and individual choices on \( (x,Q) \) or \( (h,Q) \). It is immediate from the definitions of \( Q \) and \( x \) that

\[
[E(x^\beta)] = 1. \tag{16}
\]

Use (11) in the price index (2) to find that the scale for the real wage is

\[
p_0w_0 = N^{1/(\rho - 1)} \left[ \int X^{\varepsilon(\rho - 1)} f(X) dX \right]^{1/(\rho - 1)},
\]

Conjecture that \( \varepsilon = 1/\rho \), which implies

\[
p_0w_0(Q) = N^{1/(\rho - 1)} Q^{1/\rho}. \tag{17}
\]
From (13), it also implies that the quantity of labor demanded is the same for all technologies. Hence aggregate clearing in the labor market—ignoring heterogeneity across workers—then requires
\[
\frac{L}{N} = \frac{1}{\phi_0} (p_0 w_0(Q))^{-\rho} Y_F,
\]
so output of the final good is
\[
Y_F(Q) = \phi_0 N^{1/(\rho-1)} Q.
\]
(18)

For firms, use (17) and (18) in (11)-(14) to write relative price, output, labor input, and (real) profits as
\[
\frac{p(x)}{p_F} = N^{1/(\rho-1)} x^{-1/\rho},
\]

\[
Y(x, Q) = \phi_0 \frac{L}{N} Q x,
\]

\[
\ell(x) = \frac{L}{N},
\]

\[
\frac{\pi(x, Q)}{p_F} = \frac{\phi_0}{\rho} N^{-1+1/(\rho-1)} Q x^\beta.
\]
(19)

From (19), revenue and profits are proportional to $Q x^\beta$. Since more productive firms employ higher quality workers, the wage paid,
\[
w(x, Q) = w_0(Q) H^*(xQ)^{1-1/\rho}
\]
\[
= p_0^{-1} N^{1/(\rho-1)} Q^{1/\rho} (a_H x Q)^{1-1/\rho}
\]
\[
= p_0^{-1} N^{1/(\rho-1)} Q (a_H x)^{\beta},
\]
is also proportional to $x^\beta$. Over time, average real wages grow at the rate $g$, with part of the growth coming from productivity growth, in $w_0(Q)$, and part from growth in average human capital, in $H = hQ$.

3. INTERMEDIATE PRODUCERS: INVESTMENT

Let $X_{ita}$ denote the stochastic process for the productivity of incumbent firm $i$ that enters at date $t$, as function of its age $a$. It is a geometric Brownian motion, with
parameters \((g_X, \sigma_X^2)\), where the firm chooses the drift \(g_X\), which is its investment decision, and the variance \(\sigma_X^2\) is fixed. Since \(Q_t\) grows at the rate \(g\), so the firm’s relative productivity \(x_{ita} \equiv X_{ita}/Q_t\) is also a geometric Brownian motion, with parameters \((g_X - g, \sigma_X^2)\).

A. Investment by incumbents

Consider a firm’s choice of \(g_X\). Recall from (19) that the (real) profit flow for a firm with state \((x, Q)\) is proportional to \(Qx^\beta\). The cost of investment is paid in goods and, we will assume, as in Atkeson and Burstein, that the cost is scaled by current profitability. Thus, the cost of investment for that firm, if it chooses drift \(g_X\), is \(\psi_X(g_X)Qx^\beta\), where the function \(\psi_X\) is strictly increasing and strictly convex.

Since there is no fixed cost of operating, there is no voluntary exit, and the firm operates until the exogenous exit shock hits. Let \(V^F(x, Q)\) denote the value of the firm as a function of the state. The HJB equation for the firm is

\[
(r + \delta) V^F(x, Q) = \max_{g_X} \left\{ \left[ \frac{\phi_0}{\rho} L N^{-1+1/(\rho-1)} - \psi_X(g_X) \right] Q x^\beta 
+ (g_X - g) x V_x^F + \frac{1}{2} \sigma_X^2 x^2 V_{xx}^F + g Q V_Q^F \right\}.
\]

It is straightforward to show that \(V^F\) has the homogeneous form \(V^F(x, Q) = v_F Q x^\beta\), so the HJB equation can be written as

\[
(r + \delta) v_F = \max_{g_X} \left[ \frac{\phi_0}{\rho} L N^{-1+1/(\rho-1)} - \psi_X(g_X) 
+ (g_X - g) \beta v_F + \frac{1}{2} \sigma_X^2 \beta (\beta - 1) v_F + g v_F \right].
\]

(20)

The first-order condition for investment is

\[
\psi'_X(g_X) = \beta v_F,
\]

(21)

so \(g_X\) is independent of \((x, Q)\), and using \(g_X\) in (20) we find that

\[
v_F = \frac{\phi_0 L N^{-1+1/(\rho-1)}/\rho - \psi_X(g_X)}{(r + \delta - g) - \beta (g_X - g) - \beta (\beta - 1) \sigma_X^2/2}.
\]
where

$$g_X \equiv g + \frac{1}{\beta} (r - g + \delta r) - \frac{1}{2} (\beta - 1) \sigma_X^2,$$  \hspace{1cm} (23)

and we require $g_X < \overline{g}_X$ so the firm has finite value. Since $\overline{g}_X > g$, average productivity at incumbent firms could be growing faster than productivity at entrant firms. Nevertheless, incumbents are exiting at a sufficiently rapid rate so that aggregate growth is coming entirely from growth in entrant productivity.

Use (22) in (21) to write the FOC for $g_X$ as

$$\frac{\phi_0}{\rho} LN^{-1+1/(\rho-1)}/ \beta \left( \overline{g}_X - g_X \right) = \psi_X(g_X) + (\overline{g}_X - g_X) \psi'_X(g_X).$$ \hspace{1cm} (24)

Assume that the technology depreciates at a fixed rate $\delta_X \geq 0$ if there is no investment. The following assumption—Inada conditions on $\psi_X$—insures that (24) has a unique solution. For reasons that will be clear later, it restricts the drift $g_X$ with an upper bound $\nu_X$ that is strictly less than $\overline{g}_X$.

ASSUMPTION X: The cost function $\psi_X(g_X)$ is defined on $[-\delta X, \nu_X)$, where $-\delta X \leq 0 < \nu_X \leq \overline{g}_X - (r - g)/\beta$. It is continuously differentiable, strictly increasing, and strictly convex, with $\psi_X(-\delta X) = 0, \psi'_X(-\delta X) = 0, \psi_X(\nu_X) = \overline{\psi}_X$, and $\psi'_X(\nu_X) = +\infty$.

Under Assumption X the RHS of (24) is a strictly increasing function of $g_X$, taking the value zero at $g_X = -\delta X$, and diverging as $g_X \to \nu_X$.

B. The distribution of relative productivity

Since the investment choice $g_X$ is independent of $(x, Q)$, it follows that $X_{ita}$ is a geometric Brownian motion with parameters $(g_X, \sigma_X^2)$. Assume that the initial values for each cohort of entrants are lognormally distributed, with a fixed variance $\sigma^2_{EX}$ and
a mean that grows at the constant rate $g$ over time. Thus, for the cohort of age $a$ at date $t+a$,

$$\ln X_{ita} \sim N(\mu_{EX} + gt + (g_X - \sigma_X^2/2) a, \sigma^2_{EX} + \sigma^2_X a), \quad \text{all } t, a,$$

where $\mu_{EX}$ is the mean of log productivity for entrants at $t = 0$.

Since $Q_t$ grows at the constant rate $g$, relative productivity $x_{ita} = X_{ita}/Q_{t+a}$ for the cohort also has a lognormal distribution. Define $z_{ita} = \ln x_{ita}$ and

$$\mu_0 \equiv \mu_{EX} - \ln Q_0, \quad \text{and} \quad \gamma_z \equiv g_X - \frac{1}{2} \sigma^2_X - g; \quad (25)$$

where $Q_0$ must be determined. Then $z_{ita}$ has a normal distribution that does not depend on $t$.

$$z_{ita} \sim N(\mu_z(a), \Sigma_z^2(a)), \quad \text{all } t, a,$$

where

$$\begin{align*}
\mu_z(a) &= \mu_0 + \gamma_z a, \\
\Sigma_z^2(a) &= \sigma^2_{EX} + \sigma^2_X a.
\end{align*} \quad (26)$$

The distribution of $z$ across firms of all ages is a mixture of normals. In particular, since the exit rate $\delta_F > 0$ is fixed, the cohort of age $a$ gets weight $\delta_F e^{-\delta_F a}$, all $a \geq 0$, and

$$F(z) = \int_0^\infty \delta_F e^{-\delta_F a} \Phi \left( z; \mu_0 + \gamma_z a, \sigma^2_{EX} + \sigma^2_X a \right) da, \quad (27)$$

where $\Phi(z; m, s^2)$ is a normal cdf with parameters $(m, s^2)$. Hence the mean of the mixed distribution is

$$\bar{\mu}_z = \int_0^\infty \delta_F e^{-\delta_F a} \mu_z(a) da$$

$$= \mu_0 + \gamma_z \frac{1}{\delta_F}. \quad (28)$$

The variance for the cohort of age $a$ grows like $a^2$. Since the exit rate is exponential in $a$, the variance of the mixed distribution is finite.
C. Entry

Entry costs are also paid in goods. At date \( t \), a potential entrant can invest \( I_E Q_t \) units of goods and obtain a new product. Hence the entry condition is

\[
I_E Q_t \geq E[V_F (X_{i0}/Q_t, Q_t)] = v_F Q_t E\left[ x_{i0}^\beta \right],
\]

with equality if firms enter. Entry is strictly positive on the BGP, and from (26), the distribution of relative productivity for entrants is constant. Hence the entry condition is

\[
I_E = v_F E\left[ x_{i0}^\beta \right]. \tag{29}
\]

4. HOUSEHOLDS

Individuals, who are finite-lived, are organized into infinitely-lived dynastic household, with each household comprising a representative cross-section of the population. Individual members of a dynasty pool their earnings, and the dynasty allocates family income to consumption and investment in human capital. There is a continuum of identical households of total mass one.

A. Consumption

Individual household members die at a constant rate \( \delta_L \), and are replaced by an equal inflow of new members, so the size of each household, \( L \), is constant. Each household member supplies one unit of labor inelastically, so \( L \) is also aggregate labor supply.

All household members share equally in consumption, and the household has the usual constant-elasticity preferences

\[
U = \int_0^\infty e^{-\beta t} \frac{1}{1-\theta} c(t)^{1-\theta} dt,
\]

13
where \( \hat{r} > 0 \) is the pure rate of time preference and \( 1/\theta > 0 \) is the elasticity of intertemporal substitution. On the BGP, per capita consumption grows at the rate \( g \), so the real interest rate is

\[
r = \hat{r} + \theta g.
\]

(30)

Household income also grows at the rate \( g \), so its PDV is finite if and only if \( r > g \). The following restriction ensures that this is so.

**Assumption G:** Assume

\[
\hat{r} > (1 - \theta) g.
\]

**B. Investment in human capital**

New entrants into the workforce at date \( t \) have initial human capitals \( H_{t_0} \) that are lognormally distributed with a mean that grows at the rate \( g \) over time. That is, \( \ln H_{t_0} \sim N(\mu_{EH} + gt, \sigma^2_{EH}) \). Each individual then makes investments continuously over his lifetime to maximize the expected (net) discounted value of lifetime earnings. The investment process is like the one for firms. Specifically, the individual chooses the drift \( g_H \) for his human capital, and pays the associated cost. The variance \( \sigma^2_H \) for the process is fixed.

Recall the definition of relative human capital, \( h \equiv H/Q \). The pair of state variables \( (h, Q) \) is convenient for analyzing the individual’s investment problem. Recall from (5) and (17) that the individual’s (real) wage rate is proportional to \( Qh^\beta \). Assume the cost of investment is scaled like the wage, so the cost for an individual with state \( (Q, h) \) who chooses drift \( g_H \) is \( \psi_H(g_H)Qh^\beta \), where the function \( \psi \) is strictly increasing and strictly convex.

Let \( V^L(h^\beta, Q) \) denote the expected discounted value of earnings over the rest of this individual’s life, if he follows an optimal investment plan. Since \( Q \) grows at the rate \( g \) and \( h \) is a geometric Brownian motion with parameters \( (g_H - g, \sigma^2_H) \), the HJB
equation is

\[(r + \delta_L) V^L(h, Q) = \max_{g_H} \left\{ \left[ p_0^{-1} N^{1/(\rho - 1)} - \psi_H(g_H) \right] Q h^\beta + (g_H - g) h V^L_h + \frac{1}{2} \sigma_H^2 h V^L_{hh} + g Q V^L_Q \right\} \].

Again, it is easy to show \( V^L \) has the homogeneous form \( V^L(h, Q) = v_L Q h^\beta \), so the HJB equation can be written as

\[(r + \delta_L) v_L = \max_{g_H} \left[ \frac{1}{p_0} N^{1/(\rho - 1)} - \psi(g_H) \right. \]
\[+ (g_H - g) \beta v_L + \frac{1}{2} \sigma_H^2 \beta (\beta - 1) v_L + g v_L \].

The FOC for optimal investment is

\[\psi'_H(g_H) = \beta v_L, \] (32)

so \( g_H \) is independent of \((h, Q)\), and using \( g_H \) in (31) we find that

\[v_L = \frac{p_0^{-1} N^{1/(\rho - 1)} - \psi(g_H)}{(r + \delta_L - g) - \beta (g_H - g) - \beta (\beta - 1) \sigma_H^2 / 2} \]
\[= \frac{1}{\beta} \frac{p_0^{-1} N^{1/(\rho - 1)} - \psi(g_H)}{\overline{g}_H - g_H} \] (33)

where

\[\overline{g}_H \equiv g + \frac{1}{\beta} (r + \delta_L - g) - \frac{1}{2} (\beta - 1) \sigma_H^2. \] (34)

We require \( g_H < \overline{g}_H \), so expected net earnings are finite. Since \( \overline{g}_H > g \), wage growth for experienced workers could be faster than overall wage growth. Nevertheless, older workers are retiring at a sufficiently rapid rate so this effect is not contributing to aggregate wage growth.

Use (33) in (32) to write the first order condition as

\[p_0^{-1} N^{1/(\rho - 1)} = \psi_H(g_H) + (\overline{g}_H - g_H) \psi'_H(g_H). \] (35)
Assume human capital depreciates at a fixed rate $\delta_H \geq 0$, if there is no investment. As before, it is convenient to put Inada conditions on $\psi_H$.

**Assumption H:** The cost function $\psi_H(g_H)$ is defined on $[-\delta_H, \overline{g}_H]$, where $\delta_H \geq 0$. It is continuously differentiable, strictly increasing, and strictly convex, with $\psi_H(-\delta_H) = 0$, $\psi'_H(-\delta_H) = 0$, and $\lim_{g_H \to \overline{g}_H} \psi_H(g_H) = +\infty$.

Under Assumption H the RHS of (35) is a strictly increasing function of $g_H$, taking the value zero at $g_H = -\delta_H$, and diverging as $g_H \to \overline{g}_H$. Hence, given $N$, there is a unique value $g_H$ satisfying (35).

### C. The distribution of relative human capital

Since the investment choice $g_H$ is independent of $(h, Q)$, the human capital $H_{jta}$ of an individual $j$ born at date $t$, as function of his age $a$, is a geometric Brownian motion with parameters $(g_H, \sigma_H^2)$. The initial values for each cohort of newborns are lognormally distributed, with a fixed variance $\sigma_{EH}^2$ and a mean that grows at the constant rate $g$. Thus,

$$\ln H_{jta} \sim N(\mu_{EH} + gt + (g_H - \sigma^2/2) a, \sigma_{EH}^2 + \sigma_H^2 a),$$

where $\mu_{EH}$ is the mean of log human capital for new entrants to the workforce at $t = 0$.

Since average productivity $Q_t$ grows at the constant rate $g$, relative human capital $h_{ita} = H_{ita}/Q_{t+a}$ also has a lognormal distribution. Define $\zeta_{ita} = \ln h_{ita}$, and

$$\mu_{\zeta 0} \equiv \mu_{EH} - \ln Q_0, \quad \text{and} \quad \gamma_{\zeta} \equiv g_H - \frac{1}{2} \sigma_H^2 - g.$$  \hspace{1cm} (36)

Then

$$\zeta \sim N(\mu_{\zeta}(a), \Sigma_{\zeta}^2(a)),$$

all $t, a$,

where

$$\mu_{\zeta}(a) = \mu_{\zeta 0} + \gamma_{\zeta} a.$$  \hspace{1cm} (37)
\[ \Sigma^2(a) = \sigma_{E}^2 + \sigma_{H}^2 a. \]

As with technology, the distribution of \( h \) in the whole population is a mixture of normals. Since the exit rate is \( \delta_L > 0 \),

\[ M(\zeta) = \int_0^\infty \delta_L e^{-\delta_L a} \Phi \left( \zeta; \mu_{\zeta} + \gamma_{\zeta} a, \sigma_{E}^2 + \sigma_{H}^2 a \right) da, \quad (38) \]

where \( \Phi \) is a normal cdf. The mixed distribution has mean

\[ \overline{\mu}_{\zeta} = \mu_{\zeta} + \gamma_{\zeta} \frac{1}{\delta_L}, \quad (39) \]

and the variance is finite.

5. THE BALANCED GROWTH PATH

To complete the description of a BGP, we must impose market clearing for every level of human capital and determine the values for various endogenous constants.

a. Market clearing for labor

Proposition 1 shows that if the cdf’s \( F(z) \) and \( M(\zeta) \) are the mixtures of normals in (27) and (38), then a constant elasticity wage function clears the market for labor at every human capital level if \( \varepsilon = 1/\rho \) and the parameters of the stationary distributions conform in a certain sense. The resulting labor demand in (13) is constant across firms.

**Proposition 1:** Suppose the distributions of relative technology and relative human capital (in logs), the cdf’s \( F(z) \) and \( M(\zeta) \), are the mixtures of normals in (27) and (38), and let \( N, L \), be the mass of firms and the size (mass) of the workforce. Then the wage function in (5) clears the market for every kind of labor if

\[ \varepsilon = 1/\rho, \quad (40) \]
and

\[
\begin{align*}
\mu_{z0} &= \mu_{z0} + \ln a_H, \\
\sigma_{EH}^2 &= \sigma_{EX}^2,
\end{align*}
\] (41)

where \(a_H\) is defined in (7).

**Proof:** For the wage function in (5), a firm with relative productivity \(z\) chooses labor with relative human capital

\[
\zeta^*(z) = z + \ln a_H.
\]

As shown in section 2D, for \(\varepsilon = 1/\rho\), every firm demands the same quantity of labor, which in equilibrium must be \(L/N\). Hence market clearing for all levels for human capital requires

\[
LM(z + \ln a_H) = N \int_{-\infty}^{z} \frac{L}{N} f(z) dz, \quad \text{all } z,
\]

or

\[
M(z + \ln a_H) = F(z), \quad \text{all } z. \quad (42)
\]

Note that (41) implies \(\gamma \zeta /\delta_L = \gamma z /\delta_F\). Use this fact and the change of variable \(\delta_L a = \delta_F b\) to write \(M\) in (38) as

\[
M(z + \ln a_H) = \int_{0}^{\infty} \delta_F e^{-\delta_F b} \Phi \left( z; \mu_{z0} + \gamma z, \sigma_{EX}^2 + \sigma_X^2 b \right) db, \quad \text{all } z,
\]

so the required condition holds. \(\blacksquare\)

**b. Levels**

With \(\varepsilon = 1/\rho\), the constants \(a_H\) and \(\phi_0\) in (7) and (9) are determined. The interest rate \(r\) in (30) depends on the exogenous parameters \(g, \hat{r}, \text{ and } \theta\). It remains to determine \(N, Q_0, g_X, v_F, g_H\) and \(v_H\), where \(Q_0\) determines \(\mu_{z0}\). The conditions for optimal
investment by firms, together with the free entry condition and the equilibrium condition in (16) together determine $N, \mu_0, g_X, v_F$. Then $g_H$ is determined by (41) and $v_L$ by (33).

For any $N$, Assumption X ensures that the FOC (24) has a unique solution $g_X(N)$, with $\lim_{N \to 0} g_X(N) = \nu_X$, $\lim_{N \to \infty} g_X(N) = -\delta_X$, and

$$g_X'(N) = -\frac{\rho - 2 \phi_0 L}{\rho - 1} \frac{N^{-2+1/(\rho-1)}}{g_X - g_X(N)}.$$  \hspace{1cm} (43)

Assume that $\rho > 2$, so $g_X' < 0$. The normalized value of the firm in (22) then depends on $N$ directly and also indirectly, through $g_X$. Call this value $v_F(N)$,

$$v_F(N) \equiv \frac{1}{\beta} \left( \frac{\phi_0 L}{\rho} N^{-(\rho-2)/(\rho-1)} - \psi_X[g_X(N)] \right).$$  \hspace{1cm} (44)

For $\nu_X < \tilde{g}_X$, the denominator in (44) is positive and bounded. As $N \to 0$, the term $N^{-(\rho-2)/(\rho-1)}$ in the numerator diverges, while $\psi_X$ remains finite, so $\lim_{N \to 0} v_F(N) = +\infty$. As $N \to +\infty$, both terms in the numerator go to zero, so $\lim_{N \to +\infty} v_F(N) = 0$.

Recall that $g_X$ is chosen to maximize $v_F$, and apply the envelope theorem to (44) to find that

$$v_F'(N) = -\frac{\rho - 2}{\rho - 1} \frac{\phi_0 L}{\beta} \frac{N^{-2+1/(\rho-1)}}{g_X - g_X(N)} < 0.$$  \hspace{1cm}

Use $v_F(N)$ and the expressions for $\mu_\alpha(0)$ and $\Sigma_\alpha^2(0)$ in (26) in the free entry condition (29) to define the function $m_A(N)$ by

$$I_E = v_F(N) \exp \left[ \beta m_A(N) + \frac{1}{2} \beta^2 \sigma_{EX}^2 \right].$$  \hspace{1cm} (45)

Since $v_F(N)$ is strictly decreasing, $m_A(N)$ is strictly increasing. Moreover, since $v_F(N)$ takes values in $(0, +\infty)$, $m_A$ takes values in $(-\infty, +\infty)$. Equilibrium requires $\mu_\alpha = m_A(N)$, putting one joint restriction on $(N, \mu_\alpha)$.

For another restriction, recall from (16) that in equilibrium $E_F [x^\beta] = 1$. For each age cohort $\alpha$, the conditional expectation of $x^\beta$ is

$$E \left[ x^\beta | \alpha \right] = e^{\beta \mu_\alpha + \beta^2 \sigma_{EX}^2 / 2} e^{(\beta \gamma_\alpha + \beta^2 \sigma_X^2 / 2) \alpha}. $$

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Integrate over age cohorts, use the definition of $\gamma_z$, and impose the equilibrium condition to define $m_B(N)$ by

$$1 = e^{\beta m_B(N) + \beta^2 \sigma_X^2 / 2} \int_0^\infty e^{\{\beta [g_X(t) - g] + \beta (\beta - 1) \sigma_X^2 / 2\} a} e^{-\gamma X} \delta F \, da$$

$$= e^{\beta m_B(N) + \beta^2 \sigma_X^2 / 2} \int_0^\infty e^{-\{\beta \gamma X - g_X(t)\} - (r - g)} \delta F \, da$$

where Assumption X ensures that the integral is finite. Hence

$$e^{\beta m_B(N) + \beta^2 \sigma_X^2 / 2} = \frac{1}{\delta F} \{\beta \gamma X - g_X(t)\} - (r - g)\}, \quad (46)$$

Since $g_X$ is decreasing in $N$, $m_B(N)$ is increasing, and since $g_X \in [-\delta_X, \nu_X]$, the range for $m_B(N)$ is bounded above and below.

Thus, the functions $m_A$ and $m_H$ are both strictly increasing. The former has range $(-\infty, +\infty)$ and the latter has a bounded range, so there is at least one intersection. Any point where the two cross defines a pair $(N, \mu_{x_0})$ associated with a BGP.

Given $(N, \mu_{x_0})$, (24) and (22) determine $g_X$ and $v_F$, and (41) determines $\mu_{x_0}$ and $g_H$. Given $g_H$, the cost function $\psi_H$ must be reverse engineered so that the FOC (35) holds. The normalized value of the wage stream $v_L$ is then given by (33). The requirement $g_H \in (-\delta_H, \overline{\gamma}_H)$ holds if and only if

$$\frac{-\delta_H - g}{\delta_L} \leq \frac{-\delta_X - g}{\delta_F}, \quad \text{and} \quad \frac{\overline{\gamma}_H - g}{\delta_L} \geq \frac{\nu_X - g}{\delta_L}.$$  

The first condition is an additional restriction on the parameters. Since Assumption X requires $\nu_X < \overline{\gamma}_X$, the second condition holds.

Clearing in the final goods market determines the level of consumption, $C$, as a residual: output minus investment by incumbent firms, entrants, and workers,

$$C(t) = Y_F(t) - I_X(t) - I_N(t) - I_H(t)$$

$$= [\phi_0 L N^{1/(\rho - 1)} - N \psi_X(g_X) E (x^\beta) - \delta F NI_E - L \psi_H(g_H) E (h^\beta)] Q(t)$$

$$\equiv c_0 Q(t), \quad \text{all } t.$$
Hence consumption grows like $Q$, at rate $g$. Consumption can also be written as the sum of profit and labor income, net of investment and entry costs,

$$c_0 = \left[ \frac{\phi_0}{\rho} LN^{1/(\rho-1)} - N\psi_X(g_X) \right] - N\delta_F I_E + \left[ \beta \phi_0 N^{1/(\rho-1)} - \psi_H(g_H) a^\beta_H \right] L.$$  

Since the free entry condition implies that expected profits of future entrants are exactly offset by entry costs, net profit income is the annuitized value of incumbent firms, $N\nu_F$. Consequently

$$c_0 = (r - g) N\nu_F + \left[ \beta \phi_0 N^{1/(\rho-1)} - \psi_H(g_H) a^\beta_H \right] L \tag{48}$$

where the second line uses (33).

### Uniqueness

The solution is unique if $m_B(N)$ is everywhere flatter than $m_A(N)$. Differentiate (45) and (46) to find that

$$\beta m'_A(N) = -\frac{v'_F(N)}{v_F(N)}$$

$$= -g'_X(N) \frac{\psi''_X(g_X(N))}{\psi'_X(g_X(N))} > 0,$$

$$\beta m'_B(N) = \frac{-\beta g'_X(N) \delta_F}{\delta_F - \beta [g_X(N) - g] + \beta (1 - \beta) \sigma^2_X} > 0.$$  

Thus, the solution is unique if the cost function is sufficiently convex, if

$$\frac{\psi''_X(g_X)}{\psi'_X(g_X)} > \frac{1}{\delta_F/\beta - [g_X(N) - g] + (1 - \beta) \sigma^2_X/2}, \quad \text{all } g_X \in [-\delta_X, \nu_X].$$

### 6. EFFICIENCY

#### A. Allocation of labor

The allocation of labor at each date is efficient. Since the labor market is perfectly competitive and labor is supplied inelastically, this conclusion is not surprising. Each
producer has an incentive to reduce output, to exploit his market power, tending to reduce labor demand and wages. But since labor is inelastically supplied, wages fall enough to ensure full employment.

To see that allocation of labor across producers is efficient, first note that because better technologies and higher human capital are complements in production, efficiency clearly requires assortative matching.

Given the distribution functions $F(x)$ and $G(h)$ for relative productivity and human capital, and the total masses $L$ and $N$ of labor and firms, function allocating $\ell(x)$ units of labor to firm $x$ leads to the mapping $\hat{h}(\cdot)$ if

$$N \int x \ell(v)f(v)dv = LG(h(x)), \quad \text{all } x,$$

or

$$N\ell(x)f(x) = Lg(h(x))h'(x), \quad \text{all } x. \quad (49)$$

An efficient allocation of labor chooses $\ell(x)$, or equivalently $h'(x)$, to maximize total output $Y$. Thus, it solves the calculus of variations problem

$$\max \left\{ \ell(x) \right\} \int N [\phi(h(x), x)\ell(x)]^\beta f(x)dx, \quad \text{s.t. (49)},$$

or, since $N$ and $L$ are fixed,

$$\max \int [\phi(h(x), x)g(h(x))h'(x)]^\beta (f(x))^{1-\beta} dx \equiv \max \int \Omega [x, h(x), h'(x)] dx.$$

The usual Euler equation

$$\frac{\partial \Omega}{\partial h} = \frac{d}{dx} \frac{\partial \Omega}{\partial h'}, \quad \text{all } x,$$

here implies

$$\beta (\phi gh')^\beta f(x)^{1-\beta} \left( \frac{\phi_h}{\phi} + \frac{g'}{g} \right) = \frac{d}{dx} \left[ \beta (\phi g)^\beta f(x)^{1-\beta} (h')^{\beta-1} \right],$$

or

$$h' \left( \frac{\phi_h}{\phi} + \frac{g'}{g} \right) = \beta \left( \frac{\phi_h}{\phi} h' + \frac{\phi_x}{\phi} + \frac{g'}{g} h' \right) + (1 - \beta) \left( \frac{f'}{f} - \frac{h''}{h} \right),$$

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or

\[
\frac{h''}{h} + \frac{\phi_h h'}{\phi g} + \frac{g'}{g} h' - \frac{f'}{f} = \frac{\beta}{1 - \beta} \frac{\phi_x}{\phi}, \quad \text{all } x. \tag{50}
\]

The resource constraint (49) for the labor market implies

\[
\frac{\ell'(x)}{\ell(x)} + \frac{f'}{f} = \frac{g'}{g} h' + \frac{h''}{h'}, \quad \text{all } x.
\]

Use this fact in (50) to get

\[
\frac{\ell'(x)}{\ell(x)} = \frac{1}{\phi} \left[ \frac{\beta}{1 - \beta} \phi_x - \phi_h h' \right], \quad \text{all } x. \tag{51}
\]

Consider linear allocation functions, \( h(x) = ax \). Since \( \phi_X(a, 1)/\phi_H(a, 1) = a^{1/\eta} (1 - \omega)/\omega \), the term on the right in (51) vanishes for the value \( a = a_H \) defined in (7), and (51) holds for any uniform labor allocation, \( \ell(x) = \overline{\ell} \). The required level is determined by (49),

\[
\ell(x) = \overline{\ell} = \frac{L}{N}, \quad \text{all } x,
\]

which agrees with (19).

Note that normalized output is

\[
y = \left[ \int N \left[ \phi(a_H x, x) \frac{L}{N} \right]^\beta f(x) dx \right]^{1/\beta} \\
= N^{1/\beta} \phi_0 \frac{L}{N} E \left[ x^\beta \right]^{1/\beta} \\
= \phi_0 L N^{1/(\rho - 1)} \tag{52}
\]

where the last line uses (16).

**B. Investment**

To assess whether investment is efficient, consider a joint perturbation to investments in technology and human capital for which (42) holds at every date, preserving the constant elasticity wage function.
Let $\tilde{X}(t)$ denote the perturbed technology, let $\tilde{x}(t) = \tilde{X}(t)/Q(t)$ denote the perturbed technology relative to the original aggregate variable $Q(t)$, and let $\tilde{z} = \ln \tilde{x}$. Define $\tilde{H}, \tilde{h}, \tilde{\zeta}$ analogously, and call the perturbed distributions

$$
\tilde{F}(\tilde{z}; t) = \int_0^\infty \Phi(\tilde{z} + \mu \zeta_0 + \gamma \zeta - \chi \zeta(b; t), \sigma_{EX}^2 + \sigma_X^2 b) e^{-\delta_F b} \delta_F db,
$$

$$
\tilde{M}(\tilde{\zeta}; t) = \int_0^\infty \Phi(\tilde{\zeta} + \mu \zeta_0 + \gamma \zeta + \chi \zeta(a; t), \sigma_{EH}^2 + \sigma_H^2 a) e^{-\delta_L a} \delta_L da,
$$

all $t$, where $\chi \zeta(b; t)$ and $\chi \zeta(a; t)$ denote the cumulative perturbations to growth for the cohorts of firms of age $b$ and workers with experience $a$, both at date $t$.

Suppose the perturbation to technology growth is constant over time and across firms. Then $\chi \zeta(b, t) = \varepsilon_X \min \{b, t\}$, all $b, t \geq 0$, where $\varepsilon_X > 0$. Use (41) and the change of variable $\delta_F b = \delta_L a$ to find that $\tilde{F}(\tilde{z}; t) = \tilde{M}(\tilde{z} + \ln a_H; t)$, all $\tilde{z}, t$, if and only if

$$
\chi \zeta(a; t) = \varepsilon_X \min \left\{ \frac{\delta_L a}{\delta_F b}, t \right\}, \quad \text{all } a, t \geq 0. \quad (53)
$$

Since $\chi \zeta$ is the cumulative perturbation, clearly (53) holds for $t = 0$ or $a = 0$. Suppose $\delta_L/\delta_F < 1$, which is the empirically relevant case. Then (53) holds if and only if the flow perturbations to human capital growth are

$$
\varepsilon_H(a; t) = \begin{cases} 
\varepsilon_X, & \text{if } t \delta_F/\delta_L < a, \\
\varepsilon_X \delta_L/\delta_F, & \text{otherwise.} 
\end{cases} \quad (54)
$$

At any date $t$, sufficiently old workers, those with age $a > t \delta_F/\delta_L$, get the perturbation $\varepsilon_X$, while all others get the (smaller) perturbation $\varepsilon_X \delta_L/\delta_F$. In the long run virtually all workers get the smaller perturbation, to offset the fact that workers have a lower exit rate than firms.

If $\delta_L/\delta_F \geq 1$, reverse the roles of $X$ and $H$ in the argument. Note that the perturbation does not affect the long run growth rate $g$.

To calculate the effect of these changes on output and investment costs, define

$$
\tilde{Q}(t) \equiv \left[ E \left( \tilde{X}^\beta(t) \right) \right]^{1/\beta} = Q(t) \left[ E \left( \bar{x}^\beta(t) \right) \right]^{1/\beta}.
$$
Since $N,L$ are unchanged, and the labor allocation across technologies still satisfies $H = a_H \tilde{X}$, (18) implies that the change in output is
\[
\Delta Y_F(t) = Y_F(t) \frac{\Delta Q(t)}{Q(t)}, \quad \text{all } t.
\]
On the perturbed path, the total cost of investment in technology at $t$ is
\[
\tilde{I}_X(t) = \psi_X(g_X + \varepsilon_X)N\tilde{Q}(t),
\]
so for $\varepsilon_X$ small, the change is
\[
\Delta I_X(t) \approx N\psi_X'(g_X)Q(t)\varepsilon_X + I_X(t)\frac{\Delta Q(t)}{Q(t)}, \quad \text{all } t.
\]
The change in entry costs is simply
\[
\Delta I_N(t) \approx I_N(t)\frac{\Delta Q(t)}{Q(t)}, \quad \text{all } t.
\]
The change in the cost of investment in human capital is more complex, since older workers get a larger perturbation and average human capital varies across cohort.

Define
\[
k_0 \equiv e^{\beta\mu_0 + \beta^2 \sigma^2_{EX}/2},
\]
\[
\kappa \equiv \frac{1}{\delta_F} \left( \beta \gamma_z + \beta^2 \frac{1}{2} \sigma^2_X \right) = \frac{1}{\delta_L} \left( \beta \gamma_z + \beta^2 \frac{1}{2} \sigma^2_H \right),
\]
where the second line uses the fact that Proposition 1 requires $\gamma_z/\delta_L = \gamma_z/\delta_F$. Then use (28) and (39) to find that on the (unperturbed) BGP, the conditional expectations of $h^\beta$ and $x^\beta$, given age, are
\[
E_{M|a}[h(a)^\beta|a] e^{-\delta_L a} = a_H^\beta k_0 e^{-\delta_L (1-\kappa)a},
\]
\[
E_{F|b}[x(b)^\beta] e^{-\delta_F b} = k_0 e^{-\delta_F (1-\kappa)b}.
\]
Since (16) implies
\[
1 = E_F[x^\beta] = \int_0^\infty E_{F|b}[x(b)^\beta] e^{-\delta_F b} \delta_F db
\]
\[
= k_0 \int_0^\infty e^{-\delta_F (1-\kappa)b} \delta_F db = \frac{k_0}{1-\kappa},
\]
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it follows that $k_0 = 1 - \kappa$. Use (54) to find that the change in human capital investment cost at date $t$ is

$$
\Delta_{IH}(t) \approx L\psi'_H(g_X)Q(t)\varepsilon_X \left[ \frac{\delta_L}{\delta_F} \int_{t_0}^{t_0} \mathbb{E}[h(a)^\beta | a] e^{-\delta_L a} \delta_L da \\
+ \int_{t_0}^{t_0} \mathbb{E}[h^\beta(a) | a] e^{-\delta_L a} \delta_L da \right] + I_H(t) \frac{\Delta Q(t)}{Q(t)}
$$

$$
= L\psi'_H a_H^\beta Q(t)\varepsilon_X \left[ \frac{\delta_L}{\delta_F} \int_{t_0}^{t_0} e^{-\delta_L (1-\kappa) a} (1 - \kappa) \delta_L da \\
+ \int_{t_0}^{t_0} e^{-\delta_L (1-\kappa) a} (1 - \kappa) \delta_L da \right] + I_H \frac{\Delta Q}{Q}, \quad \text{all } t.
$$

Summing the changes in output and investment costs, and integrating over time, we find that the net gain from the perturbation is

$$
J \equiv \int_{0}^{\infty} e^{-rt} \left\{ [Y_F(t) - I_X(t) - I_H(t) - I_N(t)] \frac{\Delta Q(t)}{Q(t)} \\
- Q(t)\varepsilon_X \left[ N\psi'_X + L\psi'_H a_H^\beta \left[ \frac{\delta_L}{\delta_F} + \left(1 - \frac{\delta_L}{\delta_F}\right) e^{-\delta_F (1-\kappa) t} \right] \right] \right\} dt.
$$

The first line in (55) is the net gain from increasing $Q(t)$, which raises output but also increases investment costs. The second line is the direct cost of the increased investment.

Recall from (47) that the term in brackets in the first line of (55) is simply consumption, $C(t) = c_0 Q(t)$, where $c_0$ is in (48). Hence the first line in (55) is

$$
J_1 = \left[ (r - g) N v_F + \beta (\bar{\gamma}_H - g_H) L v_L a_H^\beta \right] \int_{0}^{\infty} e^{-rt} \Delta Q(t) dt.
$$

To calculate $\Delta Q(t)$, let $\tilde{x}(b, t)$ denote the relative technology for firms of age $b$ at date $t$ on the perturbed path. Then for each $(b, t)$, $\tilde{x}(b, t)$ is lognormally distributed, with the same variance as $x(b)$ and a mean that is larger by $\varepsilon_X \min \{b, t\}$. Define

$$
G(\varepsilon_X, t) \equiv \left[ \int_{0}^{\infty} \mathbb{E}_{\tilde{F}|b}[\tilde{x}(b, t)^\beta] e^{-\delta_F b} \delta_F db \right]^{1/\beta}
$$
\[
\left[ \delta_F (1 - \kappa) \int_0^\infty e^{\beta \varepsilon X \min\{b, t\} e^{-\delta_F (1 - \kappa) b} db} \right]^{1/\beta}, \quad \text{all } t,
\]

with \( G(0, t) = 1 \), and
\[
G_{\varepsilon X}(0, t) = \delta_F (1 - \kappa) \int_0^\infty \min \{b, t\} e^{-\delta_F (1 - \kappa) b} db, \quad \text{all } t.
\]

Use a Taylor series approximation to find that
\[
\Delta Q(t) = \left[ G(\varepsilon_X, t) - 1 \right] Q(t)
\approx \varepsilon_X Q(t) \delta_F (1 - \kappa) \int_0^\infty \min \{b, t\} e^{-\delta_F (1 - \kappa) b} db, \quad \text{all } t.
\]

Since \( Q(t) \) grows at the rate \( g \),
\[
\int_0^\infty e^{-rt} \Delta Q(t) dt \approx \varepsilon_X Q_0 \int_0^\infty e^{-(r-g)t} \delta_F (1 - \kappa) \left[ \int_0^t be^{-\delta_F (1 - \kappa) b} db + \int_t^\infty e^{-\delta_F (1 - \kappa) b} db \right] dt
\]
\[
= \varepsilon_X Q_0 \int_0^\infty e^{-(r-g)t} \left[ \frac{1 - e^{-\delta_F (1 - \kappa) t}}{\delta_F (1 - \kappa)} - t e^{-\delta_F (1 - \kappa) t} + t e^{-\delta_F (1 - \kappa) t} \right] dt
\]
\[
= \varepsilon_X Q_0 \frac{1}{\delta_F (1 - \kappa)} \left[ \frac{1}{r - g} - \frac{1}{r - g + \delta_F (1 - \kappa)} \right]
\]
\[
= \varepsilon_X Q_0 \frac{1}{r - g \beta (g_X - g_X)}
\]

Hence
\[
J_1 = \frac{Q_0 \varepsilon_X}{r - g} \left[ \frac{r - g}{\beta (g_X - g_X)} N v_F + \frac{g_H - g_H}{g_X - g_X} b_L a_H^2 \right]. \quad (57)
\]

Next, recall from (21) and (32) that \( \psi'_X(g_X) = \beta v_F \), and \( \psi'_H(g_H) = \beta v_L \). Hence the second line in (55) is
\[
J_2 \equiv \frac{Q_0 \varepsilon_X}{r - g} \int_0^\infty e^{-(r-g)t} \left\{ N \beta v_F + L \beta v_L a_H^\beta \left[ \frac{\delta_L}{\delta_F} + \left( 1 - \frac{\delta_L}{\delta_F} \right) e^{-\delta_F (1 - \kappa) t} \right] \right\} dt
\]
\[
= \frac{Q_0 \varepsilon_X}{r - g} \left\{ N \beta v_F + L \beta v_L a_H^\beta \left[ \frac{\delta_L}{\delta_F} + \left( 1 - \frac{\delta_L}{\delta_F} \right) \frac{r - g}{r - g + \delta_F (1 - \kappa)} \right] \right\}
\]
\[
= \frac{Q_0 \varepsilon_X}{r - g} \left[ N \beta v_F + L \beta v_L a_H^\beta \frac{r - g + \delta_L (1 - \kappa)}{r - g + \delta_F (1 - \kappa)} \right]
\]
\[
= \frac{Q_0 \varepsilon_X}{r - g} \left[ \beta N v_F + \beta \frac{g_H - g_H}{g_X - g_X} b_L a_H^2 \right]. \quad (58)
\]
Use (57) and (58) in (55) to find that the perturbation is welfare improving for 
\(\varepsilon_X > 0\) if and only if

\[
\left[ \frac{r - g}{\beta} - \beta (\bar{g}_X - g_X) \right] N v_F + (1 - \beta) (\bar{g}_H - g_H) L v_L a_H^\beta > 0.
\]

(59)

Since \(1 - \beta = 1/\rho > 0\) and \(g_H < \bar{g}_H\), the second term is positive, and the weight
\(L v_L a_H^\beta\) on this term is the EDV of earnings over their remaining lifetimes of members
of the current workforce. The first term could have either sign, and the weight \(N v_F\)
on this term is EDV of profits of incumbent firms.

7. CONCLUSIONS

[To be completed]
REFERENCES


