Does Human Capital Risk Explain The Value Premium Puzzle?

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Outline

Introduction

Motivation
Literature

Model

Primitives
Planner’s Problem
Equilibrium
Asset Pricing Implications

Results

Fama and French (1996, 1997)
Empirical Evidence and Implications of the Model
Cyclicity and Long Run Risk
The CAPM

Conclusion
Motivation

“Why is relative distress a state variable of special hedging concern to investors? One possible explanation is linked to human capital…” – Fama-French (1996)

<table>
<thead>
<tr>
<th>BE/ME Quintiles</th>
<th>Real Annualized Average Monthly Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>6.36 6.84 8.40 9.12 10.44</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>High</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>BE/ME Quintiles</th>
<th>Univariate Beta’s with Market Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>1.04 0.98 0.92 0.89 0.97</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>High</td>
<td></td>
</tr>
</tbody>
</table>
The Value Premium: Fama and French (1996; 1997; 1998), Rouwenhorst (1999), Liew and Vassalou (2000), ...

Empirical literature on Value Premium, human capital and labor income: Jaganathan and Wang (1996), Jagannathan et al. (1998), Hansson (2004), Veronesi and Santos (2005), ...

Some more theoretical approaches: Veronesi and Santos (2005), Zhang (2005), Bansal, Dittmar and Lundblad (2005), Hansen, Heaton and Li (2008), Garleanu, Kogan, and Panageas (2012), ...

I complement the above research by producing a general equilibrium model with endogenous growth where the Value Premium arises endogenously from risk to human capital.

I draw from the recent literature on asset pricing in a production economy: Kogan (2001, 2003), Eberly and Wang (2009, 2011), ...
Preview

- Model with 2 types of firms where Type A has more human capital risk
  - Reallocation dynamics keep the “Value” (Type A) firms from disappearing
- Simple model mechanism yields:
  - Value premium, relative BE/ME, and price of risk are counter-cyclical.
  - Value firms are more exposed to long-run risk
  - Failure of CAPM when we don’t take human capital returns into account
Evidence: Value firms are more exposed to human capital risk

Possible Explanations:

- Aggr. human capital productivity covary more positively with outcomes of Value firms: “negative shock to a distressed firm [...] implies a negative shock to the value of specialized human capital... Thus, workers avoid the stocks of [all] distressed firms.”–F&F(’96)

- Value firms have relatively more firm-specific human capital and hence are more burdened by their wage bill
Primitives

There is a continuum of identical agents with unit mass and recursive preferences.

The agents maximize

\[ V_t = E_t \int_t^\infty f \{ C_s, V_s \} \, ds \]

\[
\begin{align*}
f \{ C, V \} &= \frac{\beta}{\rho} \left( \frac{C^\rho}{((1 - \gamma)V)^{\frac{\rho}{1 - \gamma}} - (1 - \gamma)V} \right)
\end{align*}
\]

\(\gamma\) and \((1 - \rho)^{-1}\) govern RA and IES respectively.

\(\beta\) is the subjective discount rate.
At date zero, each agent is endowed with human capital $H_0$.

There are 2 types of firms; each with a continuum (with unit mass) of identical firms endowed with physical capital $K^i_0 = K_0$ for $i \in \{A, B\}$.

Type $A$ firms use physical capital ($K^A$) and human capital ($H$) as inputs to production.

Firms of Type $B$ use physical capital ($K^B$).
The human capital at time $t$

$$
\frac{dH_t}{H_t} = \Gamma \ln \left(1 + \frac{l_t^H}{\theta H_t}\right) dt - \delta dt + \sigma_h dZ_t^A
$$

$l_t^H$: investment in human capital on date $t$

$\delta$: depreciation rate

$\theta$ and $\Gamma$: coefficients for adjustment cost

If $\Gamma \equiv \frac{\delta}{\ln(1 + \frac{\delta}{\theta})}$, then as $\theta \to \infty$ we have $E_t\left(\frac{dH_t}{H_t}\right) \to \frac{l_t^H}{H_t} dt - \delta dt$

and as $\theta \to 0$ we have $E_t\left(\frac{dH_t}{H_t}\right) \to -\delta dt$
Process for physical capital is similar to that of human capital.

\[ \frac{dK_t^i}{K_t^i} = \Gamma \ln \left( 1 + \frac{l_t^i}{\theta K_t^i} \right) dt - \delta dt + \sigma dZ_t^i \quad \text{for } i \in \{A, B\} \]

\[ \frac{dH_t}{H_t} = \Gamma \ln \left( 1 + \frac{l_t^H}{\theta H_t} \right) dt - \delta dt + \sigma_h dZ_t^A \]

\[ dZ_t = \{dZ_t^A, dZ_t^B\}': \text{ aggregate uncertainty, Brownian Motion increments} \]

Production of Type A and B firms

\[ Y_t^A = A \left( K_t^A + H_t \right) \]

\[ Y_t^B = AK_t^B \]

Aggregate production

\[ Y_t = Y_t^A + Y_t^B \]

Human capital and physical capital of type A are hit with the same shock, \( dZ_t^A \).
Planner’s Problem

All agents and all firms of each type are identical. The planner simply chooses aggregate quantities.

$$\max \{ I^A_t, I^B_t, I^H_t \} \int_0^\infty \mathbb{E}_0 \int_0^\infty f \{ C_s, V_s \} ds \quad \text{s.t.:}$$

$$\frac{dH_t}{H_t} = \Gamma \ln \left( 1 + \frac{I^H_t}{\theta H_t} \right) dt - \delta dt + \sigma_h dZ_t^A$$

$$\frac{dK^i_t}{K^i_t} = \Gamma \ln \left( 1 + \frac{I^i_t}{\theta K^i_t} \right) dt - \delta dt + \sigma dZ_t^i \quad \text{for } i \in \{ A, B \}$$

$$C_t + I^A_t + I^B_t + I^H_t = Y_t = \mathcal{A} (H_t + K^A_t + K^B_t)$$

Homogeneity $\Rightarrow$ 2 state variables, $x^i_t = \frac{K^i_t}{H_t + K^A_t + K^B_t}$ for $i \in \{ A, B \}$.

Note: in the deterministic model we have $\frac{I^A}{K^A} = \frac{I^B}{K^B} = \frac{I^H}{H}$.
Equilibrium

An equilibrium consists of a set of adapted processes \( \{ C_t, I_t^A, I_t^B, I_t^H \} \forall t \) such that

1. \( \{ I_t^A, I_t^B, I_t^H \} \) solve the Hamiltonian-Jacobi-Bellman equation

2. resource constraint is satisfied: \( C_t + I_t^A + I_t^B + I_t^H = Y_t \) where \( Y_t = A (H_t + K_t^A + K_t^B) \)

3. the LOM for aggregate human and physical capital are satisfied

\[
\begin{align*}
\frac{dH_t}{H_t} &= \Gamma \ln \left( 1 + \frac{I_t^H}{\theta H_t} \right) dt - \delta dt + \sigma_h dZ_t^A \\
\frac{dK_t^i}{K_t^i} &= \Gamma \ln \left( 1 + \frac{I_t^i}{\theta K_t^i} \right) dt - \delta dt + \sigma dZ_t^i \quad \text{for } i \in \{A, B\}
\end{align*}
\]
Asset Pricing

There are 2 risky securities: risky claims on sum of profits of 2 firm types.

Let $\ell^H = \frac{I^H}{H}$, and $\ell^i = \frac{I^i}{K^i}$

$q^i_t$: the value of physical of type $i$  

$\quad q^i = \frac{1}{\Gamma} \left( \ell^i + \theta \right)$  

$p_t$: the value of human capital  

$\quad p = \frac{1}{\Gamma} \left( \ell^H + \theta \right)$  

Stochastic Discount Factor, $\Lambda_t$, follows

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \sigma_{\Lambda,t} \cdot dZ_t$$
**Key Results**

\[ S_t^i = K_t^i q_t^i \quad \text{Value of security } i \in \{A, B\} \]

\[ dR_t^i = \left( r_t + \sigma_{\Lambda,t} \cdot \zeta_t^i \right) dt + \zeta_t^i \cdot dZ_t \]

\[ E_t \left( dR_t^A \right) - E_t \left( dR_t^B \right) = \text{Cov}_t \left( \frac{d\Lambda_t \cdot dq_t^B}{\Lambda_t}, \frac{dq_t^B}{q_t^B} \right) - \text{Cov}_t \left( \frac{d\Lambda_t \cdot dq_t^A}{\Lambda_t}, \frac{dq_t^A}{q_t^A} \right) + \sigma_{\Lambda,t} \cdot \left\{ 1, -1 \right\} \sigma dt \]

\( >0 \text{ on average} \)

Price of risk for shock \( dZ^A \) is larger than that for \( dZ^B \) because negative \( dZ^A \) shocks are more costly to the economy.

Thus, on average physical capital of Type A is less valuable than that of Type B, \( q^A < q^B \), because negative shocks to \( K^A \) imply negative shocks to \( H \).

The BE/ME is: \( \frac{K_t^i}{K_t^i q_t^i} = \frac{1}{q_t^i} \). Thus on average \( \frac{1}{q_t^A} > \frac{1}{q_t^B} \).

However, since \( \partial q_t^i / \partial x_t^i < 0 \), if \( x_t^B \gg x_t^A \) then \( \frac{1}{q_t^A} < \frac{1}{q_t^B} \Rightarrow i_t^B < i_t^A \)

\[ \Rightarrow E_t \left( dR_t^A \right) - E_t \left( dR_t^B \right) < 0 \]
Set $\beta = \tilde{\beta}$, $\delta = 0$, $\Gamma = \theta$, and $\sigma = \sigma_h$.

Search for 7 unknowns $\{\theta, A, c^*, F^*, \gamma, \rho, \sigma\}$ to solve 7 equations:

- In a 1-capital economy: 1) output growth of 2%, 2) $\frac{C}{Y}$ of 90%, 3) resource constraint, 4) FOC for investment, 5) HJB
- 6) risk-free rate of 0.90%, 7) volatility of market portfolio of 16%

**Model fit.** where $dR^A_t$: return on Value stock and $dR^B_t$: return on Growth stock

<table>
<thead>
<tr>
<th>Moment</th>
<th>Data (%)</th>
<th>Model (%)</th>
<th>Moment</th>
<th>Data (%)</th>
<th>Model (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(dR^A_t) - E(dR^B_t)$</td>
<td>4.08</td>
<td>2.06</td>
<td>$\frac{E(dR^A_t - r_t)}{\sigma(dR^A_t)}$</td>
<td>0.56</td>
<td>0.57</td>
</tr>
<tr>
<td>BE/ME (A)</td>
<td>1.38</td>
<td>0.51</td>
<td>$\frac{E(dR^B_t - r_t)}{\sigma(dR^B_t)}$</td>
<td>0.32</td>
<td>0.50</td>
</tr>
<tr>
<td>BE/ME (B)</td>
<td>0.31</td>
<td>0.41</td>
<td>$\frac{E(dR^A_t - dR^B_t)}{\sigma(dR^A_t - dR^B_t)}$</td>
<td>0.39</td>
<td>0.11</td>
</tr>
</tbody>
</table>
From data:

<table>
<thead>
<tr>
<th>BE/ME Quintiles</th>
<th>Low</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Return</td>
<td>6.36</td>
<td>6.84</td>
<td>8.40</td>
<td>9.12</td>
<td>10.44</td>
</tr>
</tbody>
</table>

We can reproduce the above results using additional types of capital with varying levels of \( \text{cov}(\frac{dH}{H}, \frac{dK}{K}) \).

Instead, I use the model's simulations to examine the relationship b/w returns and \( 1/q \)

From the model:

<table>
<thead>
<tr>
<th>BE/ME ((\frac{1}{q})) Quintiles</th>
<th>Low</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Return ((E(dR_t)))</td>
<td>6.72</td>
<td>8.81</td>
<td>10.23</td>
<td>11.29</td>
<td>11.97</td>
</tr>
</tbody>
</table>
On Fama and French (1997)

Substantial variation in the value premium...

Joint Distribution

Expected Excess Returns : $E_t (R_t^A - R_t^B)$

Relative BE/ME : $\frac{1}{q^A} - \frac{1}{q^B}$
Some Empirical Evidence and Model Implications

Key mechanism driving the results: \[ \text{Cov}\left(\frac{dH}{H}, \frac{dK^A}{K^A}\right) > \text{Cov}\left(\frac{dH}{H}, \frac{dK^B}{K^B}\right). \]

⇒ Value firms equity returns covary more positively with aggregate labor income growth.

To test this implication, I define

1. Labor income growth: ann. growth in real aggr. income per-capita
2. Value (Growth) portfolio returns: real ann. ret. on Value (Growth) portfolio

<table>
<thead>
<tr>
<th>Aggregate Human Capital</th>
<th>(\beta_{i,h})</th>
<th>P-val</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value Portfolio</td>
<td>1.96</td>
<td>0.03</td>
</tr>
<tr>
<td>Growth Portfolio</td>
<td>0.82</td>
<td>0.38</td>
</tr>
</tbody>
</table>
The model also implies that on average $q^A < q^B$ therefore Value firms have relatively less investment-to-capital ratio and lower asset growth

$$i^A = \Gamma q^A - \theta < i^B = \Gamma q^B - \theta$$

In data:

<table>
<thead>
<tr>
<th>BE/ME Quintiles</th>
<th>Low</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>High</th>
<th>V-G</th>
</tr>
</thead>
<tbody>
<tr>
<td>CapEx</td>
<td>0.077</td>
<td>0.071</td>
<td>0.064</td>
<td>0.059</td>
<td>0.050</td>
<td>-0.024</td>
</tr>
<tr>
<td>Net Investments</td>
<td>0.097</td>
<td>0.085</td>
<td>0.074</td>
<td>0.062</td>
<td>0.041</td>
<td>-0.048</td>
</tr>
<tr>
<td>$\Delta \ln(Asset)$</td>
<td>0.151</td>
<td>0.111</td>
<td>0.090</td>
<td>0.072</td>
<td>0.040</td>
<td>-0.092</td>
</tr>
</tbody>
</table>

In the model:

$$E(i^A) - E(i^B) = -0.015$$

$$E(dK^A/K^A) - E(dK^B/K^B) = -0.007$$
Distribution of State Variables and Impulse Responses

Distributions for State Variables

\[
E(x^A) = 0.31 \\
E(x^B) = 0.37
\]

Expected Excess Return:

\[
E_t(dR^A) - E_t(dR^B)
\]

Relative BE/ME:

\[
\frac{1}{q^A} - \frac{1}{q^B}
\]

Price of Risk:

\[
|\sigma_{t,\lambda}|
\]

IRF Methodology

IRF Methodology

more IRFs
Long-run Risk

Perturbation: \[
\ln(\zeta_t \{\epsilon\}) = \int_0^t -\frac{1}{2} \epsilon^2 \alpha \cdot \alpha ds + \int_0^t \epsilon \alpha \cdot dZ_s
\]

\[
\alpha = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}^\prime
\]

Following Borovicka et al. (2011) and Hansen (2011), the risk-price elasticities for \( i \in \{A, B\} \) are

\[
\pi^i \{x, t\} = \frac{1}{t} \frac{d}{d\epsilon} \ln \left\{ E \left( S_t^i \zeta_t \{\epsilon\} | x_0 = x \right) \right\} \bigg|_{\epsilon = 0} - \frac{1}{t} \frac{d}{d\epsilon} \ln \left\{ E \left( \Lambda_t S_t^i \zeta_t \{\epsilon\} | x_0 = x \right) \right\} \bigg|_{\epsilon = 0}
\]

I quantify the long-run risk with \( \pi^i \{x, \infty\} \) and find

\[
\pi^A \{\bar{x}, \infty\} > \pi^B \{\bar{x}, \infty\}
\]

where \( \bar{x} \) is the mean value of the state variables.
Conditional CAPM

dR_m^t and dR_w^t: returns on the market and total wealth portfolios.

\[
dR^m_t = dR^A_t \omega + dR^B_t (1 - \omega) = \mu^m_t \, dt + \sigma^m_t \cdot dZ_t
\]

\[
dR^w_t = \left( dR^H_t (1 - \omega^A_t - \omega^B_t) + dR^A_t \omega^A_t + dR^B_t \omega^B_t \right) = \mu^w_t \, dt + \sigma^w_t \cdot dZ_t
\]

Consider the regression

\[
E_t (dR^A_t) - E_t (dR^B_t) = \alpha_0 + \alpha_1 \times \sigma^w_t \cdot \Sigma^w_t \left( \beta^A_t - \beta^B_t \right) dt
\]

\[
\beta_{t}^{i,w} = \frac{\text{cov}_t(dR^i_t, dR^w_t)}{\sigma^w_t \cdot \Sigma^w_t} \quad \text{for } i \in \{A, B\}
\]

With Log-Utility, the price of risk is \(\sigma_{A,t} = \sigma_{C,t} = \sigma^w_t\); thus the Conditional CAPM holds: \(\alpha_0 = 0\) and \(\alpha_1 = 1\).

I will run the above regression with \(dR^m\) and \(dR^w\) and compare the pricing errors.
Conditional CAPM Regressions

(1): \[ E_t (dR_t^A) - E_t (dR_t^B) = \alpha_0 + \alpha_1 \times s_t^w \cdot s_t^w (\beta_t^{A,w} - \beta_t^{B,w}) \, dt \]

(2): \[ E_t (dR_t^A) - E_t (dR_t^B) = \alpha_0 + \alpha_1 \times s_t^m \cdot s_t^m (\beta_t^{A,m} - \beta_t^{B,m}) \, dt \]

<table>
<thead>
<tr>
<th>( \gamma = (1 - \rho) = 1 ) (log-utility)</th>
<th>( \gamma &gt; (1 - \rho) \neq 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_0 ) (%)</td>
<td>( (1) )</td>
</tr>
<tr>
<td>0.00</td>
<td>1.1</td>
</tr>
<tr>
<td>t-stat</td>
<td>0.38</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>1.0</td>
</tr>
<tr>
<td>t-stat</td>
<td>( 1.3 \times 10^{16} )</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>1.0</td>
</tr>
<tr>
<td>t-stat</td>
<td>7000.0</td>
</tr>
</tbody>
</table>
Following Jaganathan and Wang (1996):

1) Define $R_t^h$ as the monthly growth in aggregate income.
2) Construct test portfolios (20 BE/ME sorted portfolios).
3) Use Fama-MacBeth (1973) approach; calculate the univariate betas, $\beta_{i,m}$ and $\beta_{i,h}$, and then compare cross-sectional regressions below.
4) Do same for conditional CAPM.

$$E(R_i) = \gamma_0 + \gamma_m \beta_{i,m}$$

$$E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} + \gamma_h \beta_{i,h}$$
Conclusion

I produce a **general equilibrium** model with **endogenous growth** and **Human capital risk** which explains the Value Premium.

The model has the following features and implications

1) Firm-level and Aggregate Human capital growth covary more positively with assets growth of Value firms. Hence, Value firms are less valuable (lower $q$), greater BE/ME ($1/q$), and greater equity returns. Lower $q \Rightarrow$ lower $\ell$ and lower asset growth for Value firms

2) There is endogenous reallocation due to diversification incentives which leads to co-existence of both Value and Growth firms in equilibrium

3) Counter-cyclical Value premium, relative BE/ME and price of risk. Relatively more long-run risk exposure for Value firms. Failure of the CAPM
\[ \Gamma = \theta \text{ and } \delta = 0 \]

\[ \Gamma = \frac{\delta}{\ln(1 + \frac{\delta}{\theta})} \text{ and } \delta = 0.04 \]
**Model fit.** where $dR^A$: return on Value stock and $dR^B$: return on Growth stock

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data/Targets (%)</th>
<th>Model (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Consumption and Output Growth</td>
<td>2.00</td>
<td>1.94</td>
</tr>
<tr>
<td>Std. Deviation of Cons. and Output Growth</td>
<td>4.00</td>
<td>16.49</td>
</tr>
<tr>
<td>Mean Risk-free Return</td>
<td>0.90</td>
<td>0.22</td>
</tr>
<tr>
<td>Standard Deviation of Risk-free Return</td>
<td>2.00</td>
<td>0.68</td>
</tr>
<tr>
<td>Mean Return of Value Stocks</td>
<td>10.32</td>
<td>10.85</td>
</tr>
<tr>
<td>Standard Deviation of Value Stocks</td>
<td>16.73</td>
<td>18.78</td>
</tr>
<tr>
<td>Sharpe Ratio of Value Stocks</td>
<td>56.31</td>
<td>56.60</td>
</tr>
<tr>
<td>Mean Return of Growth Stocks</td>
<td>6.24</td>
<td>8.79</td>
</tr>
<tr>
<td>Standard Deviation of Growth Stocks</td>
<td>16.62</td>
<td>17.11</td>
</tr>
<tr>
<td>Sharpe Ratio of Growth Stocks</td>
<td>32.13</td>
<td>50.09</td>
</tr>
<tr>
<td>Mean Value premium</td>
<td>4.08</td>
<td>[2.06, 3.12]</td>
</tr>
<tr>
<td>Sharpe Ratio of Value premium</td>
<td>38.50</td>
<td>10.80</td>
</tr>
<tr>
<td>Mean Market Return</td>
<td>7.16</td>
<td>10.09</td>
</tr>
<tr>
<td>Standard Deviation of Market Return</td>
<td>15.45</td>
<td>15.83</td>
</tr>
</tbody>
</table>
Set $\beta = \bar{\beta}$, $\delta = 0$, $\Gamma = \theta$, and $\sigma = \sigma_h$.

Search for 7 unknowns $\{\mathcal{A}, c^*, F^*, \gamma, \theta, \rho, \sigma\}$ to solve 7 equations:

- 1) risk-free rate of 0.90%, 2) volatility of output growth
- And in a 1-capital economy: 3) output growth of 2%, 4) $\frac{C}{Y}$ of 90%, 5) resource constraint, 6) FOC for investment, 7) HJB

### Calibration

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name/Calculation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A}$</td>
<td>Marginal Product of Capital</td>
<td>20.63%</td>
</tr>
<tr>
<td>$c^*$</td>
<td>Consumption to Capital in one-capital economy</td>
<td>18.63%</td>
</tr>
<tr>
<td>$F^*$</td>
<td>Normalized Value Function, $F$, in one-capital economy</td>
<td>0.0791</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Risk Aversion Parameter</td>
<td>59.35</td>
</tr>
<tr>
<td>$\Gamma = \theta$</td>
<td>Adjustment Cost Parameters</td>
<td>2.73%</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Implied IES Parameter $= 2.0025$</td>
<td>0.5006</td>
</tr>
<tr>
<td>$\sigma = \sigma_h$</td>
<td>Standard deviation of Capital Growth</td>
<td>5.50%</td>
</tr>
</tbody>
</table>
Model fit. where $dR^A$: return on Value stock and $dR^B$: return on Growth stock

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data/Targets (%)</th>
<th>Model (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Consumption and Output Growth</td>
<td>2.00</td>
<td>2.02</td>
</tr>
<tr>
<td>Std. Deviation of Cons. and Output Growth</td>
<td>4.00</td>
<td>4.08</td>
</tr>
<tr>
<td>Mean Risk-free Return</td>
<td>0.90</td>
<td>0.90</td>
</tr>
<tr>
<td>Standard Deviation of Risk-free Return</td>
<td>2.00</td>
<td>0.14</td>
</tr>
<tr>
<td>Mean Return of Value Stocks</td>
<td>10.32</td>
<td>10.97</td>
</tr>
<tr>
<td>Standard Deviation of Value Stocks</td>
<td>16.73</td>
<td>4.66</td>
</tr>
<tr>
<td>Sharpe Ratio of Value Stocks</td>
<td>56.31</td>
<td>216</td>
</tr>
<tr>
<td>Mean Return of Growth Stocks</td>
<td>6.24</td>
<td>9.34</td>
</tr>
<tr>
<td>Standard Deviation of Growth Stocks</td>
<td>16.62</td>
<td>4.15</td>
</tr>
<tr>
<td>Sharpe Ratio of Growth Stocks</td>
<td>32.13</td>
<td>204</td>
</tr>
<tr>
<td>Mean Value premium</td>
<td>4.08</td>
<td>[1.64, 1.65]</td>
</tr>
<tr>
<td>Sharpe Ratio of Value premium</td>
<td>38.50</td>
<td>41.42</td>
</tr>
<tr>
<td>Mean Market Return</td>
<td>7.16</td>
<td>10.05</td>
</tr>
<tr>
<td>Standard Deviation of Market Return</td>
<td>15.45</td>
<td>3.92</td>
</tr>
</tbody>
</table>
Transitional Distributions For $x^A$ and $x^B$

Distributions For State Variables

$E(x^A) = 0.31$
$E(x^B) = 0.37$
### Annual data:

<table>
<thead>
<tr>
<th></th>
<th>Portfolio Level Human Capital</th>
<th>Aggr. Human Capital</th>
<th>~Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_{i,h}$</td>
<td>P-val</td>
<td>$\beta_{i,h}$</td>
</tr>
<tr>
<td>Value Portfolio</td>
<td>2.24</td>
<td>0.10</td>
<td>1.96</td>
</tr>
<tr>
<td>Growth Portfolio</td>
<td>1.26</td>
<td>0.53</td>
<td>0.82</td>
</tr>
</tbody>
</table>

### Monthly data:

<table>
<thead>
<tr>
<th></th>
<th>Aggregate Human Capital</th>
<th>~Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_{i,h}$</td>
<td>P-val</td>
</tr>
<tr>
<td>Value Portfolio</td>
<td>0.91</td>
<td>0.07</td>
</tr>
<tr>
<td>Growth Portfolio</td>
<td>0.04</td>
<td>0.93</td>
</tr>
<tr>
<td>Value Firms</td>
<td>0.71</td>
<td>0.00</td>
</tr>
<tr>
<td>Growth Firms</td>
<td>0.52</td>
<td>0.00</td>
</tr>
</tbody>
</table>


## Dual-sort Portfolio Returns

<table>
<thead>
<tr>
<th>$\beta_h$ Quintiles</th>
<th>BE/ME Quintiles</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
</tr>
<tr>
<td>Low</td>
<td>$-0.34$</td>
</tr>
<tr>
<td>2</td>
<td>1.06</td>
</tr>
<tr>
<td>3</td>
<td>3.07</td>
</tr>
<tr>
<td>4</td>
<td>4.57</td>
</tr>
<tr>
<td>High</td>
<td>9.33</td>
</tr>
</tbody>
</table>
### $\beta_h$ Quintiles

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Return</td>
<td>$E(dR_t)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.94</td>
<td>2.78</td>
<td>3.39</td>
<td>3.74</td>
<td>6.46</td>
<td></td>
</tr>
</tbody>
</table>

Mean annualized real monthly returns for $\beta_h$-quintile portfolios. $\beta_h$ is the slope from a rolling twelve-month univariate regression of monthly equity returns against monthly (aggregate) labor income growth.
Key mechanism driving the results: \( \text{Cov} \left( \frac{dH}{H}, \frac{dK^A}{K^A} \right) > \text{Cov} \left( \frac{dH}{H}, \frac{dK^B}{K^B} \right) \).

Thus, model predicts that for Value firms labor income \((wH = \mathcal{A}H)\) growth covaries more positively with capital (asset) growth.

\[
\begin{align*}
(1) \quad \ln(\text{Asset})_{i,t} &= a_1 \ln(\text{Wage})_{i,t} + a_2 \ln(\text{Wage})_{i,t} \times \text{ValueDummy}_{i,t} + \varepsilon_{i,t} \\
(2) \quad \Delta \ln(\text{Asset})_{i,t} &= a_1 \Delta \ln(\text{Wage})_{i,t} + a_2 \Delta \ln(\text{Wage})_{i,t} \times \text{ValueDummy}_{i,t} + \varepsilon_{i,t}
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th>Coef.</th>
<th>T-stat</th>
<th>P-val</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) (\ln(\text{Wage}))</td>
<td>1.33</td>
<td>246.71</td>
<td>0.00</td>
</tr>
<tr>
<td>(\ln(\text{Wage}) \times \text{ValueDummy})</td>
<td>0.26</td>
<td>33.80</td>
<td>0.00</td>
</tr>
<tr>
<td>(2) (\Delta \ln(\text{Wage}))</td>
<td>0.48</td>
<td>53.83</td>
<td>0.00</td>
</tr>
<tr>
<td>(\Delta \ln(\text{Wage}) \times \text{ValueDummy})</td>
<td>0.11</td>
<td>7.75</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Below I control for the number (or growth rate) of employees

\[(1) \ln(\text{Asset})_{i,t} = a_1 \ln(\text{Wage})_{i,t} + a_2 \ln(\text{Wage})_{i,t} \times \text{ValueDummy}_{i,t} + a_3 \ln(\text{Emp})_{i,t} + \varepsilon_{i,t}\]

\[(2) \Delta \ln(\text{Asset})_{i,t} = a_1 \Delta \ln(\text{Wage})_{i,t} + a_2 \Delta \ln(\text{Wage})_{i,t} \times \text{ValueDummy}_{i,t} + a_3 \Delta \ln(\text{Emp})_{i,t} + \varepsilon_{i,t}\]

<table>
<thead>
<tr>
<th>Coef.</th>
<th>T-stat</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>\ln(\text{Wage})</td>
<td>1.68</td>
<td>391.53</td>
</tr>
<tr>
<td>\ln(\text{Wage}) \times \text{ValueDummy}</td>
<td>0.11</td>
<td>21.69</td>
</tr>
<tr>
<td>\ln(\text{Emp})</td>
<td>-0.92</td>
<td>-134.67</td>
</tr>
<tr>
<td>\Delta \ln(\text{Wage})</td>
<td>0.30</td>
<td>53.83</td>
</tr>
<tr>
<td>\Delta \ln(\text{Wage}) \times \text{ValueDummy}</td>
<td>0.07</td>
<td>7.75</td>
</tr>
<tr>
<td>\Delta \ln(\text{Emp})</td>
<td>0.47</td>
<td>50.34</td>
</tr>
</tbody>
</table>
## Univariate Betas of Annual Equity Ret. with Annual Ret. on Human Capital:

<table>
<thead>
<tr>
<th>Portfolio Level Human Capital</th>
<th>Aggregate Human Capital</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{i,h}$</td>
<td>$\beta_{i,h}$</td>
</tr>
<tr>
<td>Value Portfolio</td>
<td>2.24</td>
</tr>
<tr>
<td>Growth Portfolio</td>
<td>1.26</td>
</tr>
</tbody>
</table>

First panel uses portfolio-level human capital returns: value-weighted avg wage growth of firms in respective portfolio.

Second panel uses aggregate human capital returns: growth in aggregate per-capita income.

<table>
<thead>
<tr>
<th>Labor Share $(\frac{\text{LaborExpense}}{\text{LaborExpense} + \text{Profits}})$</th>
<th>Coef.</th>
<th>T-stat</th>
<th>P-val</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value Firms - Growth Firms</td>
<td>0.16</td>
<td>1.71</td>
<td>0.09</td>
</tr>
</tbody>
</table>
### Market Betas

<table>
<thead>
<tr>
<th></th>
<th>$\beta_{t}^{A,m}$</th>
<th>$\beta_{t}^{B,m}$</th>
<th>$\beta_{t}^{A,m} - \beta_{t}^{B,m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.97</td>
<td>0.95</td>
<td>0.02</td>
</tr>
</tbody>
</table>

### Total-Wealth Betas

<table>
<thead>
<tr>
<th></th>
<th>$\beta_{t}^{A,w}$</th>
<th>$\beta_{t}^{B,w}$</th>
<th>$\beta_{t}^{A,w} - \beta_{t}^{B,w}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.05</td>
<td>0.83</td>
<td>0.21</td>
</tr>
</tbody>
</table>
## Mean Returns and Univariate Regression Betas

<table>
<thead>
<tr>
<th>Quantiles of BE/ME</th>
<th>Low</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_{i,h} )</td>
<td>0.04</td>
<td>0.01</td>
<td>0.23</td>
<td>-0.07</td>
<td>0.09</td>
<td>0.03</td>
<td>0.32</td>
<td>0.70</td>
<td>0.24</td>
<td>0.57</td>
</tr>
<tr>
<td>( \beta_{i,m} )</td>
<td>1.08</td>
<td>1.00</td>
<td>1.05</td>
<td>1.09</td>
<td>1.02</td>
<td>0.99</td>
<td>0.98</td>
<td>0.97</td>
<td>1.00</td>
<td>0.91</td>
</tr>
<tr>
<td>( \beta_{i,prem} )</td>
<td>0.39</td>
<td>0.51</td>
<td>0.61</td>
<td>0.58</td>
<td>0.60</td>
<td>0.49</td>
<td>0.58</td>
<td>0.54</td>
<td>0.63</td>
<td>0.66</td>
</tr>
<tr>
<td>( E(R_i) )</td>
<td>0.46</td>
<td>0.53</td>
<td>0.64</td>
<td>0.58</td>
<td>0.59</td>
<td>0.53</td>
<td>0.61</td>
<td>0.61</td>
<td>0.69</td>
<td>0.72</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_{i,h} )</td>
<td>0.68</td>
<td>0.27</td>
<td>0.31</td>
<td>0.82</td>
<td>0.98</td>
<td>0.75</td>
<td>1.12</td>
<td>0.71</td>
<td>0.84</td>
<td>1.11</td>
</tr>
<tr>
<td>( \beta_{i,m} )</td>
<td>0.92</td>
<td>0.91</td>
<td>0.93</td>
<td>0.91</td>
<td>0.94</td>
<td>0.95</td>
<td>1.01</td>
<td>0.93</td>
<td>0.98</td>
<td>1.12</td>
</tr>
<tr>
<td>( \beta_{i,prem} )</td>
<td>0.72</td>
<td>0.61</td>
<td>0.65</td>
<td>0.61</td>
<td>0.70</td>
<td>0.64</td>
<td>0.81</td>
<td>0.82</td>
<td>0.67</td>
<td>1.11</td>
</tr>
<tr>
<td>( E(R_i) )</td>
<td>0.83</td>
<td>0.66</td>
<td>0.69</td>
<td>0.73</td>
<td>0.84</td>
<td>0.81</td>
<td>1.00</td>
<td>0.87</td>
<td>0.78</td>
<td>1.29</td>
</tr>
</tbody>
</table>
### Unconditional CAPM

\[ E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} + \gamma_h \beta_{i,h} \]

<table>
<thead>
<tr>
<th>Coef</th>
<th>T</th>
<th>P-val</th>
<th>Z</th>
<th>P-val</th>
<th>Z (\gamma_0 - R_f)</th>
<th>R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>\gamma_0</td>
<td>0.57</td>
<td>0.83</td>
<td>0.42</td>
<td>0.77</td>
<td>0.61</td>
<td>0.00</td>
</tr>
<tr>
<td>\gamma_m</td>
<td>0.15</td>
<td>0.22</td>
<td>0.83</td>
<td>0.20</td>
<td>0.84</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coef</th>
<th>T-stat</th>
<th>P-val</th>
<th>Z-stat</th>
<th>P-val</th>
<th>Z (\gamma_0 - R_f)</th>
<th>R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>\gamma_0</td>
<td>-0.30</td>
<td>-0.85</td>
<td>0.41</td>
<td>-0.57</td>
<td>-0.80</td>
<td>0.77</td>
</tr>
<tr>
<td>\gamma_m</td>
<td>0.82</td>
<td>2.32</td>
<td>0.03</td>
<td>1.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\gamma_h</td>
<td>0.44</td>
<td>7.63</td>
<td>0.00</td>
<td>4.16</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conditional CAPM

\[ E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} + \gamma_{prem} \beta_{i,prem} + \gamma_h \beta_{i,h} \]

<table>
<thead>
<tr>
<th></th>
<th>Coef</th>
<th>T-stat</th>
<th>P-val</th>
<th></th>
<th>Z-stat</th>
<th>P-val</th>
<th></th>
<th>Z (\gamma_0 - R_f)</th>
<th>R^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>\gamma_0</td>
<td>0.03</td>
<td>0.17</td>
<td>0.87</td>
<td>0.13</td>
<td>0.95</td>
<td>-0.40</td>
<td>0.93</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\gamma_m</td>
<td>-0.12</td>
<td>-0.65</td>
<td>0.52</td>
<td>-0.46</td>
<td>0.64</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\gamma_{prem}</td>
<td>1.25</td>
<td>16.51</td>
<td>0.00</td>
<td>9.28</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Coef</th>
<th>T-stat</th>
<th>P-val</th>
<th></th>
<th>Z-stat</th>
<th>P-val</th>
<th></th>
<th>Z (\gamma_0 - R_f)</th>
<th>R^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>\gamma_0</td>
<td>-0.15</td>
<td>-1.50</td>
<td>0.15</td>
<td>-0.85</td>
<td>0.40</td>
<td>-1.58</td>
<td>0.98</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\gamma_m</td>
<td>0.20</td>
<td>1.83</td>
<td>0.09</td>
<td>1.00</td>
<td>0.32</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\gamma_{prem}</td>
<td>0.93</td>
<td>14.11</td>
<td>0.00</td>
<td>7.94</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\gamma_h</td>
<td>0.16</td>
<td>6.32</td>
<td>0.00</td>
<td>4.43</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conditional CAPM w/o Human Capital

\[ E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} \]

Conditional CAPM w/ Human Capital

\[ E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} + \gamma_h \beta_{i,h} \]
Relative BE/ME:
\[ \frac{1}{q^A} - \frac{1}{q^B} \]

Expected Excess Return:
\[ E_t(dR^A) - E_t(dR^B) \]

Investment-to-Human Capital Ratio: \( t^H \)

Consumption-to-Total Capital Ratio: \( c \)
Investment–to–Physical Capital Ratio: $t^A$

Investment–to–Human Capital Ratio: $t^H$

Consumption–to–Total Capital Ratio: $c$

$\Delta Z^A + \Delta Z^B$
Price of Risk:

\[ \sigma_{t,\lambda}^{(1)} \]

\[ \text{Price of Risk: } \sum_{t,\lambda} \left( 1 + \alpha dZ^A + dZ^B \right) \]
Exp. Capital Growth: \[ E \left( \frac{dK}{K} \right) = \Gamma \ln \left( 1 + \frac{\iota}{\theta} \right) dt \]

Typical Simulation

Investment–to–Capital Ratio: \( \iota \)

Value  Growth
BE/ME: \( \frac{1}{q} = \frac{\Gamma}{\iota + \theta} \)

Typical Simulation

Investment-to-Capital Ratio: \( \iota \)

<table>
<thead>
<tr>
<th>BE/ME Ratio: ( \frac{1}{q} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.55</td>
</tr>
<tr>
<td>0.50</td>
</tr>
<tr>
<td>0.45</td>
</tr>
<tr>
<td>0.40</td>
</tr>
<tr>
<td>0.35</td>
</tr>
</tbody>
</table>

Value  Growth
To calculate the returns on the Value and Growth stocks I merge monthly returns data from CRSP with fundamentals data from Compustat for the years 1963-2012. Following the approach of Fama and French (1993 and 1996), I form Value and Growth portfolios using the top thirtieth and the bottom thirtieth percentiles of BE/ME distributions with the BE/ME cut-offs from the Kenneth R. French Data Library.

BE is the sum of book equity, deferred taxes, and investment tax credit, minus the book value of preferred stock for fiscal year $t - 1$. ME is the value of common equity at the end of year $t - 1$. I then calculate returns from July of year $t$ through June of year $t + 1$.

The mean return shown in the tables for the Value and Growth portfolios is the annualized average monthly returns. I multiply the average monthly return by twelve and the standard deviation by the square-root of twelve.

I adjust the security returns for inflation using the GDP deflator. I use a two-year rolling geometric average of the GDP deflator. I do so because I calculate the security returns in data from July through June as done in Fama and French (1993; 1996).
The model is consistent with a setting where agents can invest in human capital by acquiring more schooling.

The adjustment cost for human capital may reflect some opportunity cost of time spent on schooling (as a means of investing in human capital), psychic costs, or a reduction of the time spent on leisure activities that are valuable to agents.

The adjustment cost for physical capital may reflect some frictions to capital reallocation, installation costs, or more general forms of capital illiquidity.

In the model adj. costs are necessary for time-varying BE/ME and to ensure that both types of firms co-exist in equilibrium.
Theorem 2. The model with processes for effective units of capital is equivalent to one where the total factor productivity (TFP) shocks are modeled as separate variables, $a^i_t$, for $i \in \{A, B\}$. That is, the model can be re-written as

$$\frac{dh_{j,t}}{h_{j,t}} = \Gamma \ln \left(1 + \frac{l^h_{j,t}}{\theta a^A_t h_{j,t}}\right) dt - \delta dt$$

$$\frac{dk^i_{j,t}}{k^i_{j,t}} = \Gamma \ln \left(1 + \frac{l^i_{j,t}}{\theta a^i_t k^i_{j,t}}\right) dt - \delta dt \quad \text{for} \ i \in \{A, B\}$$

$$da^i_t = a^i_t \sigma dZ^i_t$$

$$Y_t = A \left(a^A_t h_t + a^A_t k^A_t + a^B_t k^B_t\right)$$

where $h_t = \int_j h_{j,t} dj$, $k^i_t = \int_j k^i_{j,t} dj$ and the prices of human and physical capital are defined as $a^A_t p_t$ and $a^i_t q^i_t$. Due to scale invariance, the value function is unaffected.

To preserve scale invariance, $a_t$ appears in the adjustment function. "The fact that adjustment costs are higher for high $a_t$ can be justified by the fact that high TFP economies are more specialized." – Brunnemeir and Sannikov (2011)
1) We can re-interpret the current specification of the model as follows: Type A firms use $K^A$ in their production process and Type B firms use $K^B$. There is a third type of competitive firms that rent $H$ from the agents. With CRS the third type makes zero profits so the value of these firms is zero.

2) We can also think of “Value” firms as being comprised of a mix for Type A firms and the third type of firms. And “Growth” firms as being comprised of a mix for Type B firms and the third type of firms

3) We can obtain similar results if Type A firms use $K^A$ and $H^A$ in their production process and Type B firms use $K^B$ and $H^B$. All we need is

$$\text{Cov} \left( \frac{dH^A}{H^A}, \frac{dK^A}{K^A} \right) > \text{Cov} \left( \frac{dH^B}{H^B}, \frac{dK^B}{K^B} \right)$$
There are 2 state variables, \( x_t^i = \frac{K_t^i}{H_t + K_t^A + K_t^B} \) for \( i \in \{A, B\} \).

We can write the value function as

\[
V \{H + K^A + K^B, x^A, x^B\} = \frac{1}{1 - \gamma} \left( (H + K^A + K^B) F \{x^A, x^B\} \right)^{1-\gamma}
\]

The Hamiltonian-Jacobi-Bellman (HJB) equation is

\[
0 = \max_{I^A, I^B, I^H} f \{C, V\} \, dt + V_H E(dH) + V_A E(dK^A) + V_B E(dK^B) + \frac{1}{2} \left( V_{AA} (dK^A)^2 + V_{BB} (dK^B)^2 + V_{HH} (dH)^2 + 2 V_{AH} dH dK^A \right)
\]

\( dH, dK^A, dK^B \) are given by the pre-specified LOM’s \( E(dH), E(dK^A), \) and \( E(dK^B) \) are the drifts and \( C_t = Y_t - (I_t^A + I_t^B + I_t^H) \)
Let $c = C_{H+K^A+K^B}$, $\ell_i = \frac{i^i}{K^i}$, and $\ell^H = \frac{i^H}{H}$. Following Eberly and Wang (2009; 2011), $c$, $\ell_i$, $\ell^H$ and $F$ jointly solve

Resource constraint: $c = A - x^A \ell^A - x^B \ell^B - (1 - x^A - x^B)\ell^H$

FOC's for $I^A, I^B, I^H$: $c = F \left( \frac{\Gamma (-x^A F_A - x^B F_B + F)}{\beta (\ell^H + \theta)} \right)^{\frac{1}{\rho-1}}$

$c = F \left( \frac{\Gamma (- (x^A - 1) F_A - x^B F_B + F)}{\beta (\ell^A + \theta)} \right)^{\frac{1}{\rho-1}}$

$c = F \left( \frac{\Gamma (-x^A F_A - (x^B - 1) F_B + F)}{\beta (\ell^B + \theta)} \right)^{\frac{1}{\rho-1}}$

where $F \{x^A, x^B\}$ also solves the PDE in the 2 states obtained by simplifying the HJB; with boundary conditions $F \{1, 0\} = F \{0, 1\} = \bar{F}$ and $F \{0, 0\} = \hat{F}$. 

- PDE
- Boundary Conditions
- Projection Method
- back to Implementation
1. Re-write the state variables as $\{\tilde{x}^A, \tilde{x}^B\}$, functions of Chebyshev nodes.

2. Approximate $F\{x^A, x^B\}$ with a complete Chebyshev polynomial, $\tilde{F}\{\tilde{x}^A, \tilde{x}^B\}$. The approximation function will be composed of complete orthogonal basis functions.

3. Define the residual function, $R$, as the PDE where we plug in the approximation $\tilde{F}$ and the as well as the nodes $\{\tilde{x}^A, \tilde{x}^B\}$.

4. Using the collocation approach, the vector of polynomial coefficients $\alpha$, is chosen to solve $R\{\alpha\} = 0$ on the grid $\{\tilde{x}^A, \tilde{x}^B\}$. First choose the size of the $n \times n$ grid. Then start with low order polynomials and solve for $\alpha$.

5. Use an interpolation method to obtain the solution for $F\{x^A, x^B\}$ over the continuous state space $\{x^A \in [0, 1], x^B \in [0, 1 - x^A]\}$. Plug this function, $F\{x^A, x^B\}$, back into the PDE and examine the size of the PDE errors over the continuous state space.

6. Steadily increase the degree of the polynomial and repeat the procedure until the PDE errors are minimized.
Let $c = \frac{C}{H+K^A+K^B}$, $i^i = \frac{l^i}{K^i}$, and $i^H = \frac{l^H}{H}$. Plugging the conjecture for $V$ in the HJB and simplifying a bit yields

$$0 = \frac{\beta}{\rho} \left( \frac{A - x^A l^A - x^B l^B - (1 - x^A - x^B) l^H}{F} \right) + \phi \{ x^A, x^B \}$$

$$+ \ln \left( \frac{l^H + \theta}{\theta} \right) (x^A + x^B - 1) \left( \frac{\Gamma x^A F_A}{F} + \frac{\Gamma x^B F_B}{F} - \Gamma \right)$$

$$+ \ln \left( \frac{l^B + \theta}{\theta} \right) x^B \left( - \frac{\Gamma x^A F_A}{F} - \frac{\Gamma (x^B - 1) F_B}{F} + \Gamma \right)$$

$$+ \ln \left( \frac{l^A + \theta}{\theta} \right) x^A \left( - \frac{\Gamma (x^A - 1) F_A}{F} - \frac{\Gamma x^B F_B}{F} + \Gamma \right)$$
There are 3 boundary cases: 
\[ \{x^A = 1, \quad x^B = 1, \quad 1 - x^A - x^B = 1\} \].

I first solve the model for these 3 cases. I then use Projections Methods from Judd (1998) and approximate \( F \{x^A, x^B\} \) with a complete Chebyshev polynomial of degree 20, \( \tilde{F} \), in \( \{x^A, x^B\} \).

I solve for the coefficients of the polynomial which jointly satisfy the following conditions:

1) \( \tilde{F}\{x^A, x^B\} \) solves the HJB,
2) \( \tilde{F}\{x^A, x^B\} \) satisfies the 3 boundary cases and,
3) FOC’s for \( \ell^A, \ell^B, \ell^H \) and resource the constraint are satisfied.
Proposition 1

We can decentralize the planner’s problem as follows.

1) Agent endowed with \(H_0\), takes wage rate \(\omega_t\), price of human capital \(p_t\), and initial financial wealth \(\tilde{W}_0\) as given. Agent has access to a risk-less bond with return \(r_t\) and a risky claims. Risky security prices: \(S_t = \{S_t^{(A)}, S_t^{(B)}\}'\) follows

\[
dS_t = \left(\mu_t \text{diag}(S_t) - D_t\right) dt + \begin{pmatrix} S_t^{(A)} & S_t^{(B)} \\ S_t^{(B)} & S_t^{(B)} \end{pmatrix} dZ_t
\]

The agent solves

\[
\max_{\{c_{j,t}, l_{j,t}^H, H_{j,t}, \omega_{j,t}\}} E_0 \int_0^\infty f(C_{j,t}, V_{j,t}) dt \quad \text{s.t.:}
\]

\[
d\tilde{W}_{j,t} = (\tilde{W}_{j,t} \omega_t + \omega_{j,t} \cdot \tilde{W}_{j,t}(\mu_t - 1r_t) - c_{j,t} - l_{j,t}^H + \omega_t H_{j,t}) dt + \omega_{j,t}' \tilde{W}_{j,t} \zeta_t dZ_t
\]

\[
dH_{j,t}/H_{j,t} = \Gamma \ln \left(1 + \frac{l_{j,t}^H}{\theta H_{j,t}}\right) dt - \delta dt + \sigma_h dZ_t^A
\]

\(\omega_{j,t} = \{\omega_{j,t}^A, \omega_{j,t}^B\}'\): fraction of \(\tilde{W}_{j,t}\) invested in risky securities

and \(\zeta_t\) is a 2 \times 2 matrix.
2) Let $\Lambda_t$ denote the SPD, firms of Types $A$ and $B$ (respectively) own physical capital $K^i_t$ for $i \in \{A, B\}$, take the wage rate $\omega_t$, and the price of physical capital $q^i_t$ as given and solve

$$\max_{K^A_{j,t} , H_{j,t}, I^A_{j,t}} \int_0^\infty \Lambda_t \left( A \left( K^A_{j,t} + H_{j,t} \right) - \omega_t H_{j,t} - I^A_{j,t} \right) dt \quad \text{s.t.:}$$

$$dK^A_{j,t} / K^A_{j,t} = \Gamma \ln \left( 1 + \frac{I^A_{j,t}}{\theta K^A_{j,t}} \right) dt - \delta dt + \sigma dZ^A_t$$

$$\max_{K^B_{j,t} , I^B_{j,t}} \int_0^\infty \Lambda_t \left( A K^B_{j,t} - I^B_{j,t} \right) dt \quad \text{s.t.:}$$

$$dK^B_{j,t} / K^B_{j,t} = \Gamma \ln \left( 1 + \frac{I^B_{j,t}}{\theta K^B_{j,t}} \right) dt - \delta dt + \sigma dZ^B_t$$
Proposition 1 (Cont.)

3) The aggregate resource constraint and market clearing conditions are

\[ C_t + I_t^A + I_t^B + I_t^H = Y_t \]

where \( Y_t = A \left( H_t + K_t^A + K_t^B \right) \)

\[ \varpi_t^A \tilde{W}_t + \varpi_t^B \tilde{W}_t = S_t^A + S_t^B \]

market for risky securities clears

\[ 1 - \varpi_t^A - \varpi_t^B = 0 \]

zero net bond holdings

The last 2 conditions imply

\[ \tilde{W}_t = S_t^A + S_t^B \]

\[ \Rightarrow \tilde{W}_t = q_t^A K_t + q_t^B K_t^B \]

where \( \varpi_t = \int_j \varpi_{j,t} dj \), \( C_t = \int_j C_{j,t} dj \), \( I_t^i = \int_j I_{j,t}^i dj \), \( I_t^H = \int_j I_{j,t}^H dj \),

\[ \tilde{W}_t = \int_j \tilde{W}_{j,t} dj \]
\[ C_t + I_t^A + I_t^B + I_t^H = Y_t \]
where \( Y_t = A \left( H_t + K_t^A + K_t^B \right) \)

\[ \tilde{\omega}^A_t \tilde{W}_t + \tilde{\omega}^B_t \tilde{W}_t = S_t^A + S_t^B \]
market for risky securities clears

\[ 1 - \tilde{\omega}_t^A - \tilde{\omega}_t^B = 0 \]
zero net bond holdings

The last 2 conditions imply

\[ \tilde{W}_t = S_t^A + S_t^B \]
\[ \Rightarrow \tilde{W}_t = q_t^A K_t + q_t^B K_t^B \]

Using the static constraint

\[ \tilde{W}_t = E_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} \left( C_s + I_s^H - \omega_s H_s \right) ds \]

Using \( S_t^H = p_t H_t = E_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} \left( AH_s - I_s^H \right) ds \) (given by FOCs and Prop. 2)

\[ E_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} \left( C_s + I_s^H - \omega_s H_s \right) ds = q_t^A K_t + q_t^B K_t^B \]
\[ \Rightarrow E_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} C_s ds = p_t H_t + q_t^A K_t + q_t^B K_t^B \] as expected
Appendix

Proposition 2

The agent’s and firm problems can be re-written as

\[
\max_{\{C_{jt}, I_{jt}^H, H_{jt}\}_{t=0}^\infty} E_0 \int_0^\infty f(C_{jt}, V_{jt}) \, dt \quad \text{s.t.:}
\]

\[
\tilde{W}_0 + p_0 H_0 = E_0 \int_0^\infty \Lambda_t (C_{jt} + I_{jt}^H - \omega_t H_{jt}) \, dt
\]

\[
+ E_0 \int_0^\infty \Lambda_t p_t H_{jt} \left( \frac{\bar{\psi}_t}{p_t} - \Gamma \ln \left( 1 + \frac{I_{jt}^H}{\theta H_{jt}} \right) + \delta \right) \, dt
\]

\[
\max_{K_{jt}^A, H_{jt}, I_{jt}^A} \int_0^\infty \Lambda_t \left( A (K_{jt}^A + H_{jt}) - \omega_t H_{jt} - I_{jt}^A \right) \, dt \quad \text{s.t.:}
\]

\[
K_0^A q_0^A = E_0 \int_0^\infty \Lambda_t q_t^A K_{jt}^A \left( \frac{\psi_t^A}{q_t^A} - \Gamma \ln \left( 1 + \frac{I_{jt}^A}{\theta K_{jt}^A} \right) + \delta \right) \, dt
\]

We can similarly re-write Firm B’s optimization...
Applying Ito’s Lemma to $\Lambda_t H_t p_t, \Lambda_t K_t q_t$, and $\Lambda_t \tilde{W}_t$ integrating and using a transversality condition yields (respectively)

1)  

$$\tilde{W}_0 = E_0 \int_0^\infty \frac{\Lambda_t}{\Lambda_0} \left( C_t + l_t^H - \omega_t H_t \right) dt$$

2)  

$$H_0 p_0 = E_0 \int_0^\infty \frac{\Lambda_t}{\Lambda_0} p_t h_t \left( \frac{q_t}{p_t} - \Gamma \ln \left( 1 + \frac{l_t}{\partial H_t} \right) + \delta \right) dt$$

$$\Rightarrow \quad \underbrace{H_0 p_0}_{\text{Current Value of Capital}} + E_0 \int_0^\infty \frac{\Lambda_t}{\Lambda_0} p_t H_t E_t \left( \frac{dH_t}{H_t} \right) dt = E_0 \int_0^\infty \frac{\Lambda_t}{\Lambda_0} H_t \tilde{\psi}_t dt$$

$$\underbrace{\text{PV Expected Growth in capital}}_{\text{PV Total Surplus}}$$

3)  

$$K_0^i q_0^i = E_0 \int_0^\infty \frac{\Lambda_t}{\Lambda_0} q_t^i K_t^i \left( \frac{\psi_t^i}{q_t^i} - \Gamma \ln \left( 1 + \frac{l_t^i}{\partial K_t^i} \right) + \delta \right) dt$$

$$\Rightarrow \quad \underbrace{K_0^i q_0^i}_{\text{Current Value of Capital}} + E_0 \int_0^\infty \frac{\Lambda_t}{\Lambda_0} q_t^i K_t^i E_t \left( \frac{dK_t^i}{K_t^i} \right) dt = E_0 \int_0^\infty \frac{\Lambda_t}{\Lambda_0} K_t^i \psi_t^i dt$$

$$\underbrace{\text{PV Expected Growth in capital}}_{\text{PV Total Surplus}}$$
Appendix

\[
\bar{\psi}_t \equiv p_t \left( -E_t \left[ \frac{d(\Lambda_t \rho_t)}{\Lambda_t \rho_t} \right] + \sigma_h \tilde{\sigma}^{(A)}_{\Lambda,t} \right)
\]

\[
\psi^i_t \equiv q^i_t \left( -E_t \left[ \frac{d(\Lambda_t q^i_t)}{\Lambda_t q^i_t} \right] + \sigma^{(i)} \tilde{\sigma}^{(i)}_{\Lambda,i,t} \right)
\]

\[
\tilde{\sigma}_{\Lambda,t} = \{ \tilde{\sigma}^{(A)}_{\Lambda,t}, \tilde{\sigma}^{(B)}_{\Lambda,t} \} = \text{Diffusion} \left[ \frac{d(\Lambda_t \rho_t)}{\Lambda_t \rho_t} \right]
\]

\[
\tilde{\sigma}_{\Lambda,i,t} = \{ \tilde{\sigma}^{(A)}_{\Lambda,i,t}, \tilde{\sigma}^{(B)}_{\Lambda,i,t} \} = \text{Diffusion} \left[ \frac{d(\Lambda_t q^i_t)}{\Lambda_t q^i_t} \right]
\]

In equilibrium we have

**FOC for Cand H:** \( \bar{\psi}_t = \omega_t - \left( -p_t \left( \Gamma \ln \left( \frac{\rho_t \Gamma}{\theta} \right) - \delta \right) + p_t \Gamma - \theta \right) \)

- marginal benefit from human capital
- marginal cost to human capital

**FOC for K^i:** \( \psi^i_t = A - \left( -q^i_t \left( \Gamma \ln \left( \frac{q^i_t \Gamma}{\theta} \right) - \delta \right) + q^i_t \Gamma - \theta \right) \)

- marginal product of physical capital
- marginal cost to physical capital
Proposition 3.1: The state price density, the consumption growth and the value function follow

\[
\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \sigma_{\Lambda,t} dZ_t
\]
\[
\frac{dC_t}{C_t} = \mu_{c,t} dt + \sigma_{c,t} dZ_t
\]
\[
dV_t = -f(C_t, V_t) dt + V_t \sigma_{v,t} dZ_t
\]

where

\[
r_t = (1 - \rho)\mu_{c,t} - \frac{1}{2}(\rho - 2)(\rho - 1)\sigma_{c,t}^2 - \frac{(\rho - 1)(\gamma + \rho - 1)\sigma_{c,t} \sigma_{v,t}}{\gamma - 1} - \frac{\rho(\gamma + \rho - 1)\sigma_{v,t}^2}{2(\gamma - 1)^2} + \beta
\]
\[
\sigma_{\Lambda,t} = -(\rho - 1)\sigma_{c,t} - \left(\frac{\rho}{\gamma - 1} + 1\right)\sigma_{v,t}
\]
\[
\sigma_{v,t} = \left\{ \begin{array}{c}
\frac{(\gamma - 1)\sigma x_t^A \left( F_A (x_t^A - 1) + F_B x_t^B - F \right) + (\gamma - 1)\sigma_h (x_t^A + x_t^B - 1) \left( -F_A x_t^A - F_B x_t^B + F \right)}{F} \\
\frac{(\gamma - 1)\sigma x_t^B \left( F_A x_t^A + F_B (x_t^B - 1) - F \right)}{F}
\end{array} \right\}
\]
\[ C_t = \mathcal{A} \left( H_t + K_t^A + K_t^B \right) - H_t \left( \Gamma p_t \left\{ x_t^A, x_t^B \right\} - \theta \right) - K_t^A \left( \Gamma q_t^A \left\{ x_t^A, x_t^B \right\} - \theta \right) \\
- K_t^B \left( \Gamma q_t^B \left\{ x_t^A, x_t^B \right\} - \theta \right) \]

Using Ito's Lemma

\[ \mu_{c,t} = \frac{1}{C_t} \left\{ \frac{\partial C_t}{\partial H_t} \times H_t \left( \Gamma \ln \left( 1 + \frac{l_t^H}{\theta H_t} \right) - \delta \right) + \frac{1}{2} \frac{\partial^2 C_t}{\partial H_t^2} \times H_t \sigma_h^2 + \frac{\partial^2 C_t}{\partial H_t \partial K_t^A} \times H_t K_t^A \sigma \sigma_h + \frac{\partial^2 C_t}{\partial K_t^A} \times K_t^A \sigma^2 \right\} \]

\[ \sigma_{c,t} = \frac{1}{C_t} \left\{ \frac{\partial C_t}{\partial H_t} \times H_t \sigma_h + \frac{\partial C_t}{\partial K_t^A} \times K_t^A \sigma \right\} \]

\[ \frac{\partial C_t}{\partial K_t^B} \times K_t^B \sigma \]
\[ dx^i_t = \mu_{x,i,t} dt + \sigma'_{x,i,t} dZ_t \]

\[
\mu_{x,A,t} = x^A \left( \sigma_h^2 \left( x^A + x^B - 1 \right)^2 - \sigma \left( 2x^A - 1 \right) \sigma_h \left( x^A + x^B - 1 \right) \right. \\
\left. + \sigma^2 \left( (x^A - 1) x^A + (x^B)^2 \right) \right)
\]

\[
\sigma_{x,A,t} = \left\{ x^A \left( \sigma_h \left( x^A + x^B - 1 \right) + \sigma \left( 1 - x^A \right) \right) , -\sigma x^A x^B \right\}
\]

\[
\mu_{x,B,t} = x^B \left( -2\sigma x^A \sigma_h \left( x^A + x^B - 1 \right) + \sigma_h^2 \left( x^A + x^B - 1 \right)^2 \right. \\
\left. + \sigma^2 \left( (x^A)^2 + (x^B - 1) x^B \right) \right)
\]

\[
\sigma_{x,B,t} = \left\{ x^B \left( \sigma_h \left( x^A + x^B - 1 \right) - \sigma x^A \right) , -\sigma x^B \left( x^B - 1 \right) \right\}
\]
\[ dR^m = dR^A \frac{q^A K^A}{q^A K^A + q^B K^B} + dR^B \frac{q^B K^B}{q^A K^A + q^B K^B} \]

\[ dR^w = \left( dR^H \frac{pH}{pH + q^A K^A + q^B K^B} + dR^A \frac{q^A K^A}{pH + q^A K^A + q^B K^B} \right) \\
\quad + dR^B \frac{q^B K^B}{pH + q^A K^A + q^B K^B} \]
Applying Ito’s Lemma to $\Lambda_t S_t = \Lambda_t K_t q_t$, integrating and using a transversality condition yields

$$K^i q_s = E_s \int_s^\infty \frac{\Lambda_t}{\Lambda_s} q_t K_t \left( \frac{q_t}{K_t} - \Gamma \log \left( 1 + \frac{e_t}{\theta K_t} \right) + \delta \right) dt$$

$$\Rightarrow \quad K^i q_s = E_s \int_s^\infty \frac{\Lambda_t}{\Lambda_s} q_t K_t E_t \left( \frac{dK_t}{K_t} \right) dt = E_s \int_s^\infty \frac{\Lambda_t}{\Lambda_s} K_t \psi_t dt$$

where we define $\{\psi_t, \psi_t\}$. Then plugging in the FOCs yields the desired result, $\Lambda_t S_t = E_t \int_t^\infty \Lambda_\tau D_\tau d\tau$.

We obtain the second result by applying Ito’s lemma to $\Lambda_t S_t$ and then dividing by $S_t$

$$\frac{dS_t^i}{S_t^i} = d \left( \frac{q_t^i K_t^i}{q_t^i K_t^i} \right) = \frac{dq_t^i}{q_t^i} + \frac{dK_t^i}{K_t^i} \times \frac{dq_t^i}{q_t^i}$$

$$dR_t^i = \frac{D_t^i}{S_t^i} + \frac{dS_t^i}{S_t^i}$$
Agents hold $d_t^i S_t^i$ of debt from firms of Type $i$. The model’s BE/ME ratio is unchanged because of the Modigliani-Miller Theorem (firm value is unaffected by how it is financed)

$$BE/ME = \frac{\text{Assets} - \text{Liabilities}}{\text{Equity}} = \frac{K_t^i - d_t^i S_t^i / q_t^i}{S_t^i - d_t^i S_t^i} = \frac{K_t^i - d_t^i K_t^i}{S_t^i - d_t^i S_t^i} = \frac{1 - d_t^i}{(1 - d_t^i) q_t^i} = \frac{1}{q_t^i}$$

Assume (as in He and Krishnamurthy; 2012) that debt pays the risk-free rate and $d_t^i$ is constant. When we allow for firm debt the return on equity becomes

$$d \tilde{R}_t^i = \left( r_t (1 - d^i) + \sigma_{\Lambda,t} \cdot \zeta_t^i \right) dt + \zeta_t^i \cdot dZ_t \quad \text{for } i \in \{A, B\}$$

We can obtain the parameters $\{d^A, d^B\}$ by using data on the Value (firms of Type A) and Growth firms (of Type B)

$$d^A = 0.54$$
$$d^B = 0.49$$

With an average risk-free rate of 1% we would have

$$E \left( d \tilde{R}_t^A - d \tilde{R}_t^B \right) = E \left( r_t (d^B - d^A) \right) dt + E \left( dR_t^A - dR_t^B \right) \approx -0.0005$$

Alternatively we could introduce risk-less capital (Kozak; 2012) or use the approach from Brunnermeier and Sannikov (2011)
The Book-to-Market values are \[ \frac{K_t^i}{K_t^i q_t^i} = \frac{1}{q_t^i} \] and the dividends are
\[ D_t^i = K_t^i A - I_t^i = K_t^i (A - q_t^i \Gamma + \theta) \]

The return on the market portfolio is value weighted average of both stock returns
\[ dR_t^m = \mu_m dt + \zeta^m \cdot dZ_t \]

Beta with the market
\[ \beta^i = \frac{\text{cov}(dR_t^i, dR_t^m)}{\zeta^m \cdot \zeta^m} = \frac{\zeta_t^{i'} \zeta^m}{\zeta^m \cdot \zeta^m} \]

Volatility of returns
\[ \zeta_t^i = \frac{1}{q_t^i} \frac{\partial q_t^i}{\partial x_t^A} \sigma_{x,A,t} + \frac{1}{q_t^i} \frac{\partial q_t^i}{\partial x_t^B} \sigma_{x,B,t} + \sigma 1_i \]

where \( 1_A = \{1, 0\} \)’ \( 1_B = \{0, 1\} \)'
Set $\beta = \bar{\beta}, \delta = 0, \Gamma = \theta,$ and $\sigma = \sigma_h$.

Search for 7 unknowns $\{\theta, A, c^*, F^*, \gamma, \rho, \sigma\}$ to solve 7 equations:

- In a 1-capital economy: 1) output growth of 2%, 2) $\frac{C}{Y}$ of 90%, 3) resource constraint, 4) FOC for investment, 5) HJB
- 6) risk-free rate of 0.90%, 7) volatility of market portfolio of 16%

*Note: there is no gov’t exp. so $C/Y$ is higher than in data*

### Calibration

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name/Calculation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Marginal Product of Capital</td>
<td>20.63%</td>
</tr>
<tr>
<td>$c^*$</td>
<td>$\frac{\text{Consumption}}{\text{Capital}}$ in one-capital economy</td>
<td>18.63%</td>
</tr>
<tr>
<td>$F^*$</td>
<td>Normalized Value Function, $F$, in one-capital economy</td>
<td>0.0791</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Risk Aversion Parameter</td>
<td>3.97</td>
</tr>
<tr>
<td>$\Gamma = \theta$</td>
<td>Adjustment Cost Parameters</td>
<td>2.73%</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Implied IES Parameter $= 2.0025$</td>
<td>0.5006</td>
</tr>
<tr>
<td>$\sigma = \sigma_h$</td>
<td>Standard deviation of Capital Growth</td>
<td>21.27%</td>
</tr>
</tbody>
</table>
Possible model extension to improve the fit

Sharpe ratio for the Value premium \( \left( \frac{E_t(dR^A_t) - E_t(dR^B_t)}{\sqrt{\sigma_t^A \cdot \sigma_t^A - 2\sigma_t^A \cdot \sigma_t^B + \sigma_t^B \cdot \sigma_t^B}} \right) \) is fairly small because \( \sigma_t^A \cdot \sigma_t^B \) is fairly small.

Cannot match both \( \sigma_y \) and \( \zeta^m \) because there is no exogenous variation to increase the volatility of \( q \) independently of \( \sigma_y \). So \( \sigma \) determines both \( \sigma_y \) and \( \zeta^m \).

Solution: introduce common variation in the two types of firms which is locally orthogonal to (or has the same local covariance with) both \( dZ^A_t \) and \( dZ^B_t \).

This can be done with \( \gamma \) stochastic (and mean reverting for stationarity), or stochastic \( \theta \).
Impulse Responses

Let $\chi_t$ denote a given endogenous variable of the model. Following Koop, Pesaran and Potter (1996) and Kozak (2012), I construct two non-linear impulse responses for $\chi_t$ using

$$IRF_{\chi} \left\{ x_t^A, x_t^B \right\} = E \left( \chi_t \left| x_0^A = \bar{x}^A, x_0^B = \bar{x}^B, dZ_0 = 1_i \right. \right)$$

$$1_i \in \left\{ \{1, 0\}', \{0, 1\}', \{0.5, 1\}' \right\}$$

Thus, starting at the mean values for the state variables $\{\bar{x}^A, \bar{x}^B\}$, I introduce a one-standard-deviation positive shock through $dZ^A$ or $dZ^B$ or both at date zero and simulate twenty-five years of observations one-hundred-thousand times. I then compute the average across simulations. I define the impulse response relative to a baseline with no shock at date zero.
Appendix

Properties of the Equilibrium

$q_t^i$ depends on states and the firm type. $p_t$ depends on states

In equilibrium, $H$ and $K^i$ follow

\[
dH_t = H_t \left( \Gamma \ln \left( p_t \frac{\Gamma}{\theta} \right) - \delta \right) dt + H_t \sigma_h dZ_t^A
\]

\[
dK_t^i = K_t^i \left( \Gamma \ln \left( q_t^i \frac{\Gamma}{\theta} \right) - \delta \right) dt + K_t^i \sigma dZ_t^i
\]
Deterministic Model $\sigma = \sigma_h = 0$

\[
dx_t^i = \mu_{x,i,t} dt \quad \text{for } i \in \{A, B\}
\]

\[
\mu_{x,A,t} = x_t^A \left( \Gamma \left( x_t^A + x_t^B - 1 \right) \ln \left( \frac{\Gamma p_t}{\theta} \right) + (\Gamma - \Gamma x_t^A) \ln \left( \frac{\Gamma q_t^A}{\theta} \right) - \Gamma x^B \ln \left( \frac{\Gamma q_t^B}{\theta} \right) \right)
\]

\[
\mu_{x,B,t} = x_t^B \left( \Gamma \left( x_t^A + x_t^B - 1 \right) \ln \left( \frac{\Gamma p_t}{\theta} \right) - \Gamma x^A \ln \left( \frac{\Gamma q_t^A}{\theta} \right) + (\Gamma - \Gamma x^B) \ln \left( \frac{\Gamma q_t^B}{\theta} \right) \right)
\]

At the deterministic steady state we have

\[
\ell^A = \ell^B = \ell^H
\]

\[
p = q^A = q^B
\]

\[
dR^A = dR^B = rdt = \left( \Gamma \ln \left( \frac{\Gamma}{p} \right) - \delta + \beta \right) dt
\]

The steady state $x^A$ and $x^B$ are indeterminate because $H$, $K^A$ and $K^B$ are risk-less and have the same productivity, $\mathcal{A}$. 
The instantaneous (local) risk-price elasticity is independent of $i$

$$\pi^i \{x, 0\} = \pi \{x, 0\} = \alpha \cdot \sigma \{x\} \quad i \in \{A, B\}$$

Because of symmetry and endogenous reallocation, $\pi \{x, 0\} = \bar{\pi}$

The long horizon risk-price elasticities are

$$\pi^i \{x, \infty\} = \alpha \cdot \left(\sigma \{x\} + \sigma_x \frac{\partial}{\partial x} g^i \{x\} - \sigma_x \frac{\partial}{\partial x} g^i \{x\} \right) \quad i \in \{A, B\}$$

where $g^i \{x\}$ and $g^i \{x\}$ solve the PDE below for $M_t = S^i_t$ and $M_t = \Lambda_t S^i_t$ respectively

$$\frac{\partial g}{\partial x} (\mu_x + \sigma_x \sigma_M) + \frac{1}{2} \frac{\partial g}{\partial x} \sigma_x \sigma_x \frac{\partial g}{\partial x} + \frac{1}{2} trace \left\{ \left( \frac{\partial}{\partial x'} \left( \frac{\partial g}{\partial x} \right) \right) \sigma_x \sigma_x \right\}$$

$$+ \left( \mu_M + \frac{1}{2} \sigma_M \cdot \sigma_M \right) = \nu$$

and $\nu = \lim_{t \to \infty} \frac{1}{t} \ln \left( E \left( M_t \big| x_0 = x \right) \right)$
Following Section 6.4 of Borovicka et al. (2011), the limiting Value premium is

\[
\lim_{t \to \infty} \left\{ \frac{1}{t} \left[ \ln \left( E \left( S_t^A \mid x_0 = x \right) \right) - \ln \left( E \left( \Lambda_t S_t^A \mid x_0 = x \right) \right) \right] - \frac{1}{t} \left[ \ln \left( E \left( S_t^B \mid x_0 = x \right) \right) - \ln \left( E \left( \Lambda_t S_t^B \mid x_0 = x \right) \right) \right] \right\} = \gamma \sigma \sigma_h
\]

where \( \sigma \sigma_h dt = \text{Cov} \left( \frac{dH}{H}, \frac{dK^A}{K^A} \right) - \text{Cov} \left( \frac{dH}{H}, \frac{dK^B}{K^B} \right) \)

We are able to characterize the limiting Value premium by the product of the risk aversion governing parameter and the relative covariance of human capital growth with the asset growth of Value firms.
Relative Long–Run Risk–Price Elasticity:

\[ \pi^A(x, \infty) - \pi^B(x, \infty) \]
Distribution For Relative Risk–Price Elasticity

\[ \pi^A\{x,\infty\} - \pi^B\{x,\infty\} \]

\[ E (\pi^A\{x, \infty\} - \pi^B\{x, \infty\}) = 0.06 \]
Covariances with consumption growth

\[ \text{Cov}_t(C_{t+\tau}/C_t-1, dR_t) \]
Autocorrelations with lag of $\tau$ years

$\mu_c$
$E_t(dR^A)$
$E_t(dR^B)$
$E_t(dR^m)$
$E_t(dR^w)$
Autocorrelations with lag of $\tau$ years

- $\mu_{x,A}$
- $x^A$
Autocorrelations with lag of $\tau$ years

![Graph showing autocorrelations with lag of $\tau$ years. The graph plots autocorrelation values against lag $\tau$ in years. The vertical axis ranges from $-0.2$ to $1.0$, and the horizontal axis ranges from $0$ to $20$. Two lines are plotted: $\mu_{x,B}$ and $x^B$.](image)
<table>
<thead>
<tr>
<th></th>
<th>$dR_t^A$</th>
<th>$dR_t^B$</th>
<th>$dR_t^m$</th>
<th>$dR_t^w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Corr_t \left( \frac{dC_t}{C_t}, dR_t \right)$</td>
<td>0.90</td>
<td>0.74</td>
<td>0.98</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Recall: $x_t^i = \frac{K_t^i}{H_t + K_A^t + K_B^t}$ for $i \in \{A, B\}$ and $dx_t^i = \mu_{x,i,t} dt + \sigma_{x,i,t} \cdot dZ_t$

\[
\mu_{x,A,t} \{0, x_t^B\} = \mu_{x,A,t} \{1, x_t^B\} = 0 \quad \& \quad \sigma_{x,A,t} \{0, x_t^B\} = \sigma_{x,A,t} \{1, x_t^B\} = 0
\]
\[
\mu_{x,B,t} \{x_t^A, 0\} = \mu_{x,B,t} \{x_t^A, 1\} = 0 \quad \& \quad \sigma_{x,B,t} \{x_t^A, 0\} = \sigma_{x,B,t} \{x_t^A, 1\} = 0
\]

This is fine with linear production (current model). But with CES production

\[
Y^A = \mathcal{A} \left( \alpha H \frac{n-1}{\eta} + (1 - \alpha) \left( K_A^t \right)^{\frac{n-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} \quad \text{with} \quad \eta \leq 1 \quad \text{and} \quad \alpha \in (0, 1) \quad \Rightarrow \quad \lim_{K_A^t \to 0} Y^A = 0
\]

One-capital economy with only $H \Rightarrow Y = Y^A \to 0$. Thus $F \{0, 0\}$ is indeterminate (because HJB not well-defined). Utility function may prevent $x^A \to 0$ but still need to impose ad-hoc boundary condition for (or instead of) $F \{0, 0\}$. Suggestions???
# Constructing Modified HmL Portfolios

<table>
<thead>
<tr>
<th>( \beta_h ) Quintiles</th>
<th>BE/ME Quintiles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>Low</td>
</tr>
<tr>
<td>2</td>
<td>Short</td>
</tr>
<tr>
<td>3</td>
<td>Short</td>
</tr>
<tr>
<td>4</td>
<td>Long</td>
</tr>
<tr>
<td>High</td>
<td>Long</td>
</tr>
</tbody>
</table>

The modified HmL portfolios are long the Value portfolio and short the Growth portfolio. The Value portfolio is a value-weighted sum of securities in the cells labelled “Long”. The Growth portfolio is a value-weighted sum of securities in the cells labelled “Short”. Modified Portfolio 1 contains securities from all cells labelled “Long” or “Short”. Modified Portfolio 2 contains only securities labelled in bold letters.
## Characteristics of Modified HmL Portfolios

<table>
<thead>
<tr>
<th>Portfolio Components</th>
<th>Long Value</th>
<th>Short Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average</td>
<td>%Small</td>
</tr>
<tr>
<td>Port. 1</td>
<td>515.76</td>
<td>84.46</td>
</tr>
<tr>
<td>Port. 2</td>
<td>267.27</td>
<td>84.18</td>
</tr>
<tr>
<td>Orig.</td>
<td>973.08</td>
<td>84.69</td>
</tr>
</tbody>
</table>

The column labelled “#Securities” respectively show the average number of securities every month in the long and short legs of the corresponding portfolio. “%Small” shows the percentage of these securities that are small stocks (below median market value). “%Big” shows the percentage of these securities that are large stocks (above median market value).
# Performance of Modified HmL Portfolios

| Portfolio Characteristics From Monthly Returns | | | |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Annualized (%)** | **E (R)** | **σ (R)** | **S.R.** | **CAPM Results** | **α** | **β** | **Correlations** | **Port. 1** | **Port. 2** | **Orig.** | **Mkt.** |
| Port. 1 | 5.59 | 13.30 | 42.03 | | | | 1.00 | 0.89 | 0.54 | −0.04 |
| | | | | | | | [3.00] | [−0.84] | | |
| Port. 2 | 5.70 | 13.37 | 42.63 | | | | 1.00 | 0.45 | −0.10 | |
| | | | | | | | [3.20] | [−2.29] | | |
| Orig. | 4.08 | 10.60 | 38.49 | | | | 1.00 | | −0.13 | |
| | | | | | | | [3.02] | [−3.23] | | |
| Mkt. | 7.16 | 15.45 | 40.52 | | | | | | | 1.00 | |

Calculations are done with monthly portfolio returns. Returns are annualized by multiplying the mean by 12 and the standard deviations by $\sqrt{12}$. 
### Appendix

**Portfolio Characteristics From Cumulative Annual Returns**

<table>
<thead>
<tr>
<th>(%)</th>
<th>CAPM Results</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E(R^e)$</td>
<td>$\sigma(R^e)$</td>
</tr>
<tr>
<td>Port. 1</td>
<td>5.83</td>
<td>12.85</td>
</tr>
<tr>
<td></td>
<td>[10.82]</td>
<td>[-4.00]</td>
</tr>
<tr>
<td>Port. 2</td>
<td>5.94</td>
<td>13.13</td>
</tr>
<tr>
<td></td>
<td>[12.86]</td>
<td>[-8.28]</td>
</tr>
<tr>
<td>Orig.</td>
<td>4.70</td>
<td>14.15</td>
</tr>
<tr>
<td></td>
<td>[10.27]</td>
<td>[-7.87]</td>
</tr>
<tr>
<td>Mkt.</td>
<td>7.56</td>
<td>16.74</td>
</tr>
</tbody>
</table>

Calculations are done with (twelve-month cumulative) annual portfolio returns.
12Mo Value–Growth Returns (%)