Does Human Capital Risk Explain The Value Premium Puzzle?*

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Abstract

Using a general equilibrium model with endogenous growth, I show that risk to human capital leads to a “Value” premium in equity returns. In particular, firms with relatively more firm-specific human capital or more positive covariance between asset growth and returns on human capital are less valuable (and hence have greater Book-to-Market Equity) and yield greater expected equity returns since human capital is more tied to the fate of said firms. Thus, I reproduce some of the results of Fama and French (1996) and show that in the model their HmL factor is a proxy for human capital risk as measured by macroeconomic and financial variables such as the covariance between human capital growth, or labor income growth, with the growth rate of firm assets. The model implies relatively lower investment-to-asset ratio and lower average asset growth for Value firms as observed in data and as argued in Zhang (2005). Furthermore, the model yields counter-cyclical Value premium and relative Book-to-Market Equity, greater long-run risk exposure for Value firms, and failure of the CAPM. Hence, it replicates several results from the related literature.

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7 Appendix

7.1 Limiting cases of SDU

\[ f \{C, V\} = \frac{\beta}{\rho} \left( \frac{C^\rho}{((1 - \gamma)V)^{1/\gamma} - (1 - \gamma)V} \right) \]

7.1.1 \( \gamma = (1 - \rho) = 1 \)

If \( \gamma = (1 - \rho) \) we have

\[ f \{C, V\} = \frac{\beta}{\rho} (C^\rho - \rho V) \]

As seen in Duffie and Skiadas (1994) we obtain the usual time-separable utility when \( \gamma = (1 - \rho) \).

Therefore with \( \gamma = (1 - \rho) \to 1 \) we obtain

\[ \lim_{\rho \to 0} f \{C, V\} = \beta \ln(C) - \beta V \]

Now, recall that \( dV = -f dt + V \sigma_v dZ \). With \( \gamma = 1 \) we have \( \sigma_v = 0 \)

\[ \frac{dV}{dt} = -\beta \ln(C) + \beta V \]

\[ \mathcal{D} \left( V e^{-\beta t} \right) = -e^{-\beta t} \beta \ln(C) \]

\( V \{t\} = -\int_t^t e^{-\beta(s-t)} \beta \ln(C_s) \, ds \)

The transversality condition is \( \lim_{T \to \infty} E_t (V_T) = 0 \), therefore

\[ E_t (V_T - V_t) = -E_t \left[ \int_t^T e^{-\beta(s-t)} \beta \ln(C_s) \, ds \right] \]

Taking the limit as \( T \to \infty \)

\[ V_t = E_t \left[ \int_t^\infty e^{-\beta(s-t)} \beta \ln(C_s) \, ds \right] \]
7.1.2 \( \gamma \neq (1 - \rho) \) and \( \rho \to 0 \)

\[
\lim_{\rho \to 0} f\{C, V\} = \beta(1 - \gamma)V \lim_{\rho \to 0} \rho \left( \frac{C^\rho - ((1 - \gamma)V)^{\frac{\rho}{1 - \gamma}}}{((1 - \gamma)V)^{\frac{\rho}{1 - \gamma}}} \right)
\]

As in Section 2.4 of Duffie and Epstein (1992a)

\[
\lim_{\rho \to 0} f\{C, V\} = \beta(1 - \gamma)V \left( \ln(C) - \frac{1}{1 - \gamma} \ln((1 - \gamma)V) \right)
\]

7.2 Difficulties with non-linear production

I have also considered using a Cobb-Douglas production function. With linear production the model yields natural boundary conditions necessary for solving the PDE we obtain from the Hamiltonian-Jacobi-Bellman equation. We obtain these boundary conditions by solving the model with a single capital type (human capital, Type A physical capital, or Type B physical capital). With any CES production function, \( Y^A = A \left( \alpha H^{\frac{\eta - 1}{\eta}} + (1 - \alpha) \left( K^A \right)^{\frac{\eta - 1}{\eta}} \right)^{\frac{\eta}{\eta - 1}} \), with elasticity of substitution \( \eta \leq 1 \) as found in empirical estimates (and of which Cobb-Douglas production is a special case) and \( \alpha \in (0, 1) \) the boundary value \( F\{0, 0\} \) is not well defined. In particular, the boundary case with solely human capital would imply

\[
\lim_{K^A \to 0} Y = \lim_{K^A \to 0} Y^A = \lim_{K^A \to 0} \left( \alpha \left( H^A \right)^{\frac{\eta - 1}{\eta}} + (1 - \alpha) \left( K^A \right)^{\frac{\eta - 1}{\eta}} \right)^{\frac{\eta}{\eta - 1}} = 0
\]

hence the HJB is not well-defined and \( F\{0, 0\} \) is indeterminate. Thus to solve the model with Cobb-Douglas production (or CES with \( \eta \leq 1 \)) one would need to impose an ad hoc boundary condition. I have thus far found that imposing such a boundary condition and solving the model with Cobb-Douglas production is not possible. Furthermore, even with an ad hoc boundary condition the problem would still be ill-defined at the bound \( x^A = x^B = 0 \). The state variables can visit this bound. Indeed the state variables can be absorbed at the boundaries because from Equation 7.5 we have

\[
\mu_{x,A,t}\{0, x^B_t\} = \mu_{x,A,t}\{1, x^B_t\} = 0 \quad \& \quad \sigma_{x,A,t}\{0, x^B_t\} = \sigma_{x,A,t}\{1, x^B_t\} = 0
\]

\[
\mu_{x,B,t}\{x^A_t, 0\} = \mu_{x,B,t}\{x^A_t, 1\} = 0 \quad \& \quad \sigma_{x,B,t}\{x^A_t, 0\} = \sigma_{x,B,t}\{x^A_t, 1\} = 0
\]

Jones and Manuelli (2004) show how to solve the model with Cobb-Douglas production but without adjustment cost. They also discuss the asset pricing implications of human capital in
such a context. Without adjustment costs solving the model no longer requires solving a PDE; we need only solve a system of non-linear equations for the now constant shares of capital $x^A$ and $x^B$. Without the adjustment cost the model loses much of the interesting dynamics since the Marginal Q’s are equal to 1 (so BE/ME and expected equity returns are the same for the two types of securities, see Section 7.11 of the Appendix). Furthermore, without adjustment costs the model requires very specific values for the parameters to ensure that all types of capital co-exist in equilibrium. Palacios (2013) solves a similar model with human capital, one type of physical capital and CES production. However, with one type of physical capital there is only one state variable and hence his model is much simpler than mine. Furthermore, the author assumes that the stochastic component of output comes solely from physical capital, $dY = dK = \mathcal{A} \left( \alpha H^{\frac{\eta - 1}{\eta}} + (1 - \alpha) (K)^{\frac{\eta - 1}{\eta}} \right)^{\frac{\eta}{\eta - 1}} d\sigma K d\mathcal{Z}$ rather than $Y = \mathcal{A} \left( \alpha H^{\frac{\eta - 1}{\eta}} + (1 - \alpha) (K)^{\frac{\eta - 1}{\eta}} \right)^{\frac{\eta}{\eta - 1}} \Rightarrow dY = \frac{\partial Y}{\partial H} dH + \frac{\partial Y}{\partial K} dK + \frac{1}{2} \left( \frac{\partial^2 Y}{\partial H^2} dH^2 + \frac{\partial^2 Y}{\partial K^2} dK^2 + 2 \frac{\partial^2 Y}{\partial H \partial K} dH dK \right)$. This ensures that the local volatility is constant and avoids some additional complexity which is present in my model. For tractability, the author also makes the simplifying assumptions that the production of human capital requires only human capital as input.

A discrete-time version of the model would not be easier to solve. Indeed, as argued in Kaltenbrunner and Lochstoer (2008) and Gourio (2013) to make quantitative statements we would need to solve the model globally rather than solving some first or second-order local approximations. The more common approaches to solving discrete-time models globally include function iterations and value function approximation with high order polynomials (as described in Judd; 1998 and Miranda and Fackler; 2002). In a discrete-time model, regardless of the solution method used we would need to carefully compute the expectation of the continuation value. Since the two state variables are endogenous, stochastic and not independently distributed, it is not trivial to compute this expectation. We avoid this issue with a continuous-time model. Obtaining a global solution to the discrete-time model would also yield numerical difficulties at the bounds of the state variables if we use a CES production function with an elasticity of substitution $\eta \leq 1$.

### 7.3 Total factor productivity

The following derivation is based on the results of Brunnermeier and Sannikov (2011). Let $h_t, k^i_t$ denote the levels of human and physical capital of Type $i$

\[ dk^i_t = k^i_t \left( \Gamma \ln \left( 1 + \frac{I^i_t}{\theta \delta k^i_t} \right) - \delta \right) dt \quad \text{with a similar process for } h_t \]
The TFP shocks follow

\[ da_t^i = a_t^i (\bar{a} dt + \sigma dZ_t^i) \]

In endogenous growth models, \( \bar{a} = 0 \), thus

the TFP is Martingale and any growth will be endogenous.

The production process follows

\[ Y_t = A (a_t^A h_t + a_t^A k_t^A + a_t^B k_t^B) = A (H_t + K_t^A + K_t^B) \]

The price of human and physical capital are now defined as \( a_t^A p_t \) and \( a_t^i q_t^i \). Introducing the TFP shocks in the law of motion of capital traces back to Cox, Ingersoll and Ross (1985) and Jones and Manuelli (2004). As argued in Brunnermeier and Sannikov (2011), for scale invariance the adjustment function has to be a function of effective capital.

"The Brownian shocks \( dZ_t \) reflect the fact that one learns over time how “effective” the capital stock is. That is, the shocks \( dZ_t \) capture changes in expectations about the future productivity of capital, and \( [K_t] \) reflects the “efficiency units” of capital, measured in expected future output rather than in simple units of physical capital (number of machines). For example, when a company reports current earnings it not only reveals information about current but also future expected cash flow. In this sense our model is also linked to the literature on news to business cycles, see e.g. Jaimovich and Rebelo (2009). [...] To preserve the tractable scale invariance property one has to modify the adjustment cost function \((\Phi(t_t/a_t))\). The fact that adjustment costs are higher for high \( a_t \) can be justified by the fact that high TFP economies are more specialized." – Brunnermeier and Sannikov (2011)

Notice that the value function is unchanged and \( a_t \) is not a state variable since \( \{Y,C\} \) scale with \( (a_t^A h_t + a_t^A k_t^A + a_t^B k_t^B) \)

\[ 0 = \max_{I^A,I^B,I^H} f \{C,V\} + V_H E \left( d \{a^Ah\} \right) \\
+ V_A E \left( d \{a^Ak^A\} \right) + V_B E \left( d \{a^Bk^B\} \right) \\
+ \frac{1}{2} \left( V_{AA} \left( d \{a^Ak^A\} \right)^2 + V_{BB} \left( d \{a^Bk^B\} \right)^2 + 2V_{AH} d \{a^Ah\} d \{a^Ak^A\} \right) \]

we then plug in the conjecture for the value function

\[ (V \{a^Ah, a^Ak^A, a^Bk^B, x^A, x^B\} = \left( \frac{\left(a^Ah + a^Ak^A + a^Bk^B\right)}{1-\gamma} F \{x^A, x^B\} \right)^{1-\gamma} \text{ with } x^i = \frac{a^ik^i}{a^Ah + a^Ak^A + a^Bk^B} \]
simplify
\[ 0 = \frac{\beta}{\rho} \left( \frac{c}{F} \right)^\rho + \phi \{ x^A, x^B \} + \ln \left( \frac{e^{H + \theta}}{\theta} \right) \left( x^A + x^B - 1 \right) \left( \frac{\Gamma x^A F_A}{F} + \frac{\Gamma x^B F_B}{F} - \Gamma \right) \]
\[ + \ln \left( \frac{e^{B + \theta}}{\theta} \right) x^B \left( - \frac{\Gamma x^A F_A}{F} - \frac{(x^B - 1) F_B}{F} + \Gamma \right) \]
\[ + \ln \left( \frac{e^{A + \theta}}{\theta} \right) x^A \left( - \frac{\Gamma (x^A - 1) F_A}{F} - \frac{\Gamma x^B F_B}{F} + \Gamma \right) \]

This is exactly what the current formulation of the model yields. To reduce the complexity of the notation and to maintain the notation from the recent literature on endogenous growth, I will use \( K^i = a^i k^i \) and \( H = a^A h \).

Using Ito’s Lemma we then have
\[ \frac{d \left( a^i k_t \right)}{a^i_k} = \frac{dK^i_t}{K^i_t} = \left( \Gamma ln \left( 1 + \frac{I^i_t}{\theta K^i_t} \right) - \delta \right) dt + \sigma dZ^i_t \]
Similarly
\[ \frac{d \left( a^i h_t \right)}{a^i_h} = \frac{dH_t}{H_t} = \left( \Gamma ln \left( 1 + \frac{I^H_t}{\theta H_t} \right) - \delta \right) dt + \sigma dZ^A_t \]

7.4 Processes for aggregate capital

\[ dK^i_t = d \left( \int_{j} K^i_{j,t} dj \right) \]
\[ = \int_{j} (dK^i_{j,t}) dj \]
\[ = \int_{j} K^i_{j,t} \left( \Gamma ln \left( 1 + \frac{I^i_{j,t}}{\theta K^i_{j,t}} \right) dt - \delta dt + \sigma dZ^i_t \right) dj \]
\[ \Rightarrow dK^i_t = \int_{j} K^i_{j,t} \left( \Gamma ln \left( 1 + \frac{I^i_{j,t}}{\theta K^i_{j,t}} \right) dt \right) dj - K^i_t \delta dt + K^i_t \sigma dZ^i_t \]

We can conjecture and verify from the decentralized problem that
\[ \frac{I^i_{j,t}}{K^i_{j,t}} \]

is independent of \( j \) since all firms in each sector are identical.

Therefore
\[ \frac{I^i_{j,t}}{K^i_{j,t}} = \frac{I^i_t}{K^i_t} \]
\[ \Rightarrow \frac{dK^i_t}{K^i_t} = \Gamma ln \left( 1 + \frac{I^i_t}{\theta K^i_t} \right) dt - \delta dt + \sigma dZ^i_t \quad \text{for} \ i \in \{ A, B \} \]
Similarly
\[
\frac{dH_t}{H_t} = \Gamma \ln \left( 1 + \frac{I^H_t}{\partial H_t} \right) dt - \delta dt + \sigma_h dZ^A_t
\]

Despite the linear production processes, the adjustment cost implies that the planner will not simply invest all resources in one type of capital. For example, let us compare investing only in human capital or physical capital of Type A to investing in both. Given an initial positive stock of capital \( \{H_t, K^A_t\} \), the expected growth rate of the stock if the planner invests in both is

\[
E_t \left( \frac{dH_t + dK^A_t}{H_t + K^A_t} \right) = \frac{H_t}{H_t + K^A_t} \Gamma \ln \left( 1 + \frac{I^H_t}{\partial H_t} \right) dt + \frac{K^A_t}{H_t + K^A_t} \Gamma \ln \left( 1 + \frac{I^A_t}{\partial K^A_t} \right) dt - \delta dt
\]

(7.1)

If instead the planner were to invest entirely in \( K^A_t \) we would have

\[
E_t \left( \frac{dH_t + dK^A_t}{H_t + K^A_t} \right) = \frac{K^A_t}{H_t + K^A_t} \Gamma \ln \left( 1 + \frac{I^A_t + I^H_t}{\partial K^A_t} \right) dt - \delta dt
\]

(7.2)

Now, with \( \sigma_h = \sigma \), we have \( H = K^A \) and equations (7.1) and (7.2) become

\[
E_t \left( \frac{dH_t + dK^A_t}{H_t + K^A_t} \right) = \Gamma \ln \left( 1 + \frac{I^A_t}{\partial K^A_t} \right) dt - \delta dt
\]

\[
E_t \left( \frac{dH_t + dK^A_t}{H_t + K^A_t} \right) = \frac{1}{2} \Gamma \ln \left( 1 + 2 \frac{I^A_t}{\partial K^A_t} \right) dt - \delta dt
\]

Due to concavity

\[
E_t \left( \frac{dH_t + dK^A_t}{H_t + K^A_t} \right) > E_t \left( \frac{dH_t + dK^A_t}{H_t + K^A_t} \right)
\]

Thus even if the agents were risk neutral, it would be preferable to invest in both types of capital despite having linear production. The simulation results will confirm that on average it is optimal to invest in all three types of capital. All the qualitative results would be unchanged with if the production function for were a Cobb-Douglas function in \( H \) and \( K^A \). However, using a Cobb-Douglas (or other non-linear) production function makes the model significantly more difficult to solve.
If there were no adjustment cost the equations above would imply

\[ E_t \left( \frac{dH_t + dK^A_t}{H_t + K^A_t} \right) = \frac{I^H_t}{H_t + K^A_t} dt + \frac{I^A_t}{H_t + K^A_t} dt - \delta dt = E_t \left( \frac{dH_t + dK^A_t}{H_t + K^A_t} \right) \]

Thus, aggregate production would be independent of the allocation of resources. The planner would be indifferent between putting all resources in any type of capital if \( \sigma_h = \sigma \) and would otherwise strictly prefer the less risky technology.

### 7.5 State variables

The state variables are \( x^i = \frac{K^i_t}{H_t + K^A_t + K^B_t} \in [0, 1] \). With linear production, the boundaries \( \{0, 1\} \) are absorbing.

\[
\begin{align*}
    dx^i_t &= \mu_{x,i,t} dt + \sigma_{x,i,t} dZ_t \\
    \mu_{x,A,t} &= x^A \left( \Gamma (x^A + x^B - 1) \ln \left( \frac{\Gamma P \{x^A, x^B\}}{\gamma} \right) + (\Gamma - \Gamma x^A) \ln \left( \frac{\Gamma q \{x^A, x^B\}}{\gamma} \right) \right) \\
    &\quad - \Gamma x^B \ln \left( \frac{\Gamma q \{x^A, x^B\}}{\gamma} \right) + \sigma_h^2 (x^A + x^B - 1)^2 \\
    &\quad - \sigma (2x^A - 1) \sigma_h (x^A + x^B - 1) + \sigma^2 \left( (x^A - 1)x^A + (x^B)^2 \right) \\
    \sigma_{x,A,t} &= \{ x^A \sigma_h (x^A + x^B - 1) + \sigma (1 - x^A) \} \quad , \quad -\sigma x^A x^B \\
    \mu_{x,B,t} &= x^B \left( \Gamma (x^A + x^B - 1) \ln \left( \frac{\Gamma P \{x^A, x^B\}}{\gamma} \right) - \Gamma x^A \ln \left( \frac{\Gamma q \{x^A, x^B\}}{\gamma} \right) \right) \\
    &\quad - \Gamma (x^B - 1) \ln \left( \frac{\Gamma q \{x^A, x^B\}}{\gamma} \right) - 2 \sigma x^A \sigma_h (x^A + x^B - 1) \\
    &\quad + \sigma_h^2 (x^A + x^B - 1)^2 + \sigma^2 \left( (x^A)^2 + (x^B - 1) x^B \right) \\
    \sigma_{x,B,t} &= \{ x^B (\sigma_h (x^A + x^B - 1) - \sigma x^A) \} \quad , \quad -\sigma x^B (x^B - 1) \}
\end{align*}
\]

### 7.6 The HJB

\[
V \{H, K^A, K^B, x^A, x^B\} = \frac{(H + K^A + K^B) F \{x^A, x^B\}^{1-\gamma}}{1-\gamma}
\]
\begin{align*}
0 &= \max_{I^A, I^B, I^H} \{ C, V \} + V_H E (dH) \\
&\quad + V_A E (dK^A) + V_B E (dK^B) \\
&\quad + \frac{1}{2} \left( V_{AA} (dK^A)^2 + V_{BB} (dK^B)^2 + V_{HH} (dH)^2 + 2V_{AH} dH dK^A \right) \\
\end{align*}

where \( E(dH), E(dK^A), \) and \( E(dK^B) \) denote the drifts of \( H, K^A, \) and \( K^B. \)

Plugging in the conjecture for \( V \) and simplifying a bit yields the desired result. The function \( \phi \) is

\[
\phi \{ x^A, x^B \} = F_B \left( \frac{\gamma x^A x^B F_A (2\sigma^2 x^A x^B + \sigma_h (\sigma - \sigma_h) (x^A + x^B - 1)^2 - \sigma (x^A + x^B - 1) (x^A (\sigma - \sigma_h) + x^B (\sigma + \sigma_h)))}{2F^2} \\
+ \frac{\gamma x^B (\sigma^2 (x^A)^2 + (x^B - 1) x^B) - 2\sigma_h x^A (x^A + x^B - 1) + \sigma^2_h (x^A + x^B - 1)^2}{2F^2} \\
\right) \\
+ \left( \frac{\gamma (x^A)^2 F_A \left( \sigma^2 \left( (x^A)^2 + (x^B - 1)^2 \right) - 2\sigma_h (x^A - 1) (x^A + x^B - 1) + \sigma^2_h (x^A + x^B - 1)^2 \right)}{2F^2} \\
+ \frac{\gamma x^A F_A \left( \sigma^2 \left( (x^A - 1) x^A + (x^B)^2 \right) - 2\sigma_h (2x^A - 1) (x^A + x^B - 1) + \sigma^2_h (x^A + x^B - 1)^2 \right)}{2F^2} \\
\right) \\
+ \left( \frac{\gamma (x^B)^2 F_B \left( \sigma^2 \left( (x^A)^2 + (x^B - 1)^2 \right) - 2\sigma_h x^A (x^A + x^B - 1) + \sigma^2_h (x^A + x^B - 1)^2 \right)}{2F^2} \\
+ \frac{\gamma x^B F_B \left( \sigma^2 \left( (x^A)^2 + (x^B - 1)^2 \right) - 2\sigma_h x^B (x^A + x^B - 1) + \sigma^2_h (x^A + x^B - 1)^2 \right)}{2F^2} \\
\right) \\
+ \frac{\rho (x^A)^2 F_{AB} \left( \sigma^2 \left( (x^A)^2 + (x^B - 1)^2 \right) - 2\sigma_h x^A (x^A + x^B - 1) + \sigma^2_h (x^A + x^B - 1)^2 \right)}{2F^2} \\
+ \frac{\rho (x^B)^2 F_{AB} \left( \sigma^2 \left( (x^A)^2 + (x^B - 1)^2 \right) - 2\sigma_h x^B (x^A + x^B - 1) + \sigma^2_h (x^A + x^B - 1)^2 \right)}{2F^2} \\
\right)
\]

We find the boundary conditions by plugging \( \{ x^A = 1, x^B = 0 \}, \{ x^A = 0, x^B = 1 \}, \) and \( \{ x^A = 0, x^B = 0 \} \) in equations (2.6)-(2.9) and solving the resulting non-linear equation for the constant \( \tilde{F} = \hat{F}. \)

### 7.7 Martingale method

I will suppress the subscript \( j \) for clarity. Let \( \Lambda_t \) denote the state price density (SPD). Let

\[
\tilde{\psi}_t = p_t \left( -E_t \left[ \frac{d(\Lambda_t p_t)}{\Lambda_t p_t} \right] / dt + \sigma_h \tilde{\Lambda}_t \right) \\
\psi_t^i = q_t^i \left( -E_t \left[ \frac{d(\Lambda_t q_t^i)}{\Lambda_t q_t^i} \right] / dt + \sigma_i \tilde{\Lambda}_t \right)
\]

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\[ \tilde{\sigma}_{\Lambda,t} = \{ \tilde{\sigma}_{\Lambda,t}^{(A)}, \tilde{\sigma}_{\Lambda,t}^{(B)} \} = \text{Diffusion} \left[ \frac{d(\Lambda_t p_t)}{\Lambda_t p_t} \right] \]

\[ \tilde{\sigma}_{\Lambda,i,t} = \{ \tilde{\sigma}_{\Lambda,i,t}^{(A)}, \tilde{\sigma}_{\Lambda,i,t}^{(B)} \} = \text{Diffusion} \left[ \frac{d(\Lambda_t q_{it})}{\Lambda_t q_{it}} \right] \]

Using the Martingale Method, I will derive some static constraints

\[ \tilde{\Lambda}_t = \Lambda_t p_t \]

\[ \frac{d\tilde{\Lambda}_t}{\Lambda_t} \equiv -\tilde{r}_t dt - \tilde{\sigma}_{\Lambda,t} dZ_t \]

\[ d\left( \tilde{\Lambda}_t H_t \right) = H_t d\tilde{\Lambda}_t + \tilde{\Lambda}_t dH_t + d\tilde{\Lambda}_t dH_t \]

\[ \Rightarrow \tilde{\Lambda}_T H_T - \tilde{\Lambda}_t H_t = \int_t^T H_r d\tilde{\Lambda}_r + \int_t^T \tilde{\Lambda}_r dH_r + \int_t^T d\tilde{\Lambda}_r dH_r \]

The transversality condition is \( \lim_{T \to \infty} E_t \left( \tilde{\Lambda}_T H_T \right) = 0 \)

\[ \Rightarrow \Lambda_s H_s p_s = E_s \int_s^\infty \Lambda_t p_t H_t \left( \tilde{\psi}_t^i - \Gamma ln \left( 1 + \frac{\tilde{H}_t}{\theta} \right) + \delta \right) dt \quad (7.3) \]

Notice that

\[ \tilde{r}_t = -\text{Drift} \left[ \frac{d(\Lambda_t q_{it})}{\Lambda_t q_{it}} \right] \]

We can define

\[ dp \equiv \tilde{\varphi}_t dt + p_t \tilde{\varphi}_t dZ_t \]

with

\[ \tilde{\varphi}_t = -\tilde{\psi}_t + p_t \tilde{\varphi}_t + p_t \tilde{\varphi}_t \sigma_{\Lambda,t} + p_t \sigma_h \left( \sigma_{\Lambda,t}^{(A)} - \tilde{\varphi}_t^{(A)} \right) \quad (7.4) \]

Using the same steps we can derive

\[ \Lambda_s K_s^i q_s^i = E_s \int_s^\infty \Lambda_t q_{it}^i K_{it} \left( \tilde{\psi}_{it}^i - \Gamma ln \left( 1 + \frac{\tilde{H}_t}{\theta} \right) + \delta \right) dt \quad (7.5) \]

\[ dq_t^i \equiv \varphi_t^i dt + q_t^i \varphi_t^i dZ_t \]

\[ \varphi_t^i = -\psi_t^i + q_t^i r_t + q_t^i \varphi_t^i \sigma_{\Lambda,t} + q_t^i 1_t \left( \sigma_{\Lambda,t} - \varphi_t^i \right) \sigma \]
where \( \mathbf{1}_A = \{1, 0\} \), \( \mathbf{1}_B = \{0, 1\} \) and

\[
\Lambda_s \tilde{W}_s = E_s \int_s^\infty \Lambda_t \left( C_t + I_t^H - \omega_t H_t \right) dt
\]

We can write the agent and firm problems as

\[
\max_{\{C_{j,t}, I_{j,t}^H, H_{j,t}\}_{t=0}^\infty} E_0 \int_0^\infty f \left( C_{j,t}, V_{j,t} \right) dt \quad \text{s.t.:}
\]

\[
\hat{W}_0 + p_0 H_0 = E_0 \int_0^\infty \Lambda_t \left( C_{j,t} + I_{j,t}^H - \omega_t H_{j,t} \right) dt
\]

\[
+ E_0 \int_0^\infty \Lambda_t p_t H_{j,t} \left( \frac{\psi_t}{p_t} - \Gamma ln \left( 1 + \frac{I_{j,t}^H}{\theta H_{j,t}} \right) + \delta \right) dt
\]

Following Duffie and Skiadas (1994), the first order condition for consumption is

\[
G_{fC} = \Lambda_t
\]

\[
G = \exp \left( \int_0^t f_V ds \right)
\]

The first order condition for human capital implies

\[
0 = -p_t \Gamma + \theta - \psi_t + \omega_t + p_t \Gamma \ln \left( \frac{\Gamma}{\theta} \right)
\]

\[
\Rightarrow \psi_t = \frac{\omega_t}{\text{marginal benefit from human capital}} \left( -p_t \left( \Gamma \ln \left( \frac{p_t}{\theta} \right) - \delta \right) + p_t \Gamma - \theta \right)
\]

The first order condition for investment in human capital implies

\[
\frac{I_{j,t}^H}{H_{j,t}} = p_t \Gamma - \theta
\]

Thus we have

\[
\frac{I_{j,t}^H}{H_{j,t}} = \frac{I_{k,t}^H}{H_{k,t}} \quad \forall \{j, k\} \in J
\]

\[
\Rightarrow \frac{I_{j,t}^H}{H_{j,t}} = \frac{I_t^H}{H_t} = \iota_t^H
\]
The firms problems are

$$\max_{K_{j,t},H_{j,t},\epsilon_{j,t}} \int_0^\infty \Lambda_t \left( \mathcal{A}\left(K_{j,t}^A + H_{j,t}\right) - \omega_t H_{j,t} - I_{j,t}^A \right) dt \quad \text{s.t.}:
\begin{align*}
K_{0,t}^A = E_0 \int_0^\infty \Lambda_t q_t^A K_{j,t}^A \left( \frac{\psi_{t}^A}{q_{t}} - \Gamma \ln \left( 1 + \frac{I_{j,t}^A}{\theta K_{j,t}^A} \right) + \delta \right) dt
\end{align*}$$

$$\max_{K_{j,t},\epsilon_{j,t}} \int_0^\infty \Lambda_t \left( \mathcal{A}K_{j,t}^A - I_{j,t}^A \right) dt \quad \text{s.t.}:
\begin{align*}
K_{0,t}^B = E_0 \int_0^\infty \Lambda_t q_t^B K_{j,t}^B \left( \frac{\psi_{t}^B}{q_{t}} - \Gamma \ln \left( 1 + \frac{I_{j,t}^B}{\theta K_{j,t}^B} \right) + \delta \right) dt
\end{align*}$$

The first order conditions for physical capital\(^51\) imply

$$\psi_{t}^i = \mathcal{A} - \left( -q_t^i \left( \Gamma \ln \left( \frac{\Gamma}{q_t^i} \right) - \delta \right) + q_t^i \Gamma - \theta \right)$$

The first order conditions for investment in physical capital imply

$$\frac{I_{j,t}^i}{K_{j,t}^i} = q_t^i \Gamma - \theta$$

$$\Rightarrow \frac{I_{j,t}^i}{K_{j,t}^i} = \frac{I_t^i}{K_t^i} = i_t^i$$

For firms of Type A the first order condition with respect to human capital implies

$$\omega_t = \mathcal{A}$$

By using the FOC’s from the planner’s problem we then have

$$p = \frac{1}{\beta} \left( \mu^H + \theta \right) = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} \left( F - x_A F_A - x_B F_B \right)$$

$$q^A = \frac{1}{\beta} \left( \mu^A + \theta \right) = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} \left( F - (x_A - 1) F_A - x_B F_B \right)$$

$$q^B = \frac{1}{\beta} \left( \mu^B + \theta \right) = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} \left( F - x_A F_A - (x_B - 1) F_B \right)$$

\(^51\)We can think of \(p\) and \(q^i\) as already including the Lagrange multipliers.
where \( \dot{t}^H = \dot{I}^H / H \) and \( \dot{t}^i = \dot{I}^i / K^i \). Plugging in the first order conditions from the decentralized problem in the static constraints (7.3) and (7.5) yields

\[
\Lambda_s s^H_s = \Lambda_s H_s p_s = E_s \int_s^\infty \Lambda_t p_t H_t (A - e_t^H) \, dt = E_s \int_s^\infty \Lambda_t H_t p_t \left( \frac{A - p_t \Gamma - \theta}{p_t} \right) \, dt = E_s \int_s^\infty \Lambda_t D_t^H \, dt
\]

Dividend Yield: \( D^H / S^H \)

\[
\Lambda_s s^i_s = \Lambda_s K^i_s q^i_s = E_s \int_s^\infty \Lambda_t q^i_t K^i_t (A - e_t^i) \, dt = E_s \int_s^\infty \Lambda_t K^i_t q^i_t \left( \frac{A - q^i_t \Gamma - \theta}{q^i_t} \right) \, dt = E_s \int_s^\infty \Lambda_t D_t^i \, dt
\]

Dividend Yield: \( D^i / S^i \)

We can combine and re-write these last equations as

\[
E_s \int_s^\infty \frac{\Lambda_t}{\Lambda_s} C_t \, dt = p_s H_s + q^A_s K_s + q^B_s K^B_s
\]

\[
\Lambda_s s^H_s + \int_0^s \Lambda_t D_t^H \, dt = E_s \int_0^\infty \Lambda_t D_t^H \, dt \quad (7.6)
\]

\[
\Lambda_s s^i_s + \int_0^s \Lambda_t D_t^i \, dt = E_s \int_0^\infty \Lambda_t D_t^i \, dt \quad (7.7)
\]

The resource constraint and market clearing conditions are

\[
C_t + I_t^A + I_t^B + I_t^H = Y_t \quad \text{where } Y_t = A \left( H_t + K_t^A + K_t^B \right)
\]

\[
\omega_t^A \tilde{W}_t + \omega_t^B \tilde{W}_t = S_t^A + S_t^B \quad \text{market for risky securities clears}
\]

\[
1 - \omega_t^A - \omega_t^B = 0 \quad \text{zero net bond holdings}
\]

The last two conditions imply

\[
\tilde{W}_t = S_t^A + S_t^B = q^A_t K_t + q^B_t K^B_t = E_t \int_t^\infty \Lambda_s \left( C_s + I_s^H - \omega_s H_s \right) \, ds
\]

as expected. I will summarize some of the above results in the following proposition.

**Proposition 3.** We can re-write the agent and firms problems as follow
• The agent solves
\[
\max_{\{C_{j,t}, H_{j,t}, I_{j,t}\}} \int_0^\infty E_0 f \{C_{j,t}, V_{j,t}\} \, dt \quad \text{s.t.:}
\]
\[
\tilde{W}_0 + p_0 H_0 = E_0 \int_0^\infty \Lambda_t (C_{j,t} + H_{j,t} - \omega_t H_{j,t}) \, dt
\]
\[
+ E_0 \int_0^\infty \Lambda_t p_t H_{j,t} \left( \frac{\psi_{t}}{p_t} - \Gamma \ln \left( 1 + \frac{I^H_{j,t}}{\theta H_{j,t}} \right) + \delta \right) \, dt
\]

• Firms A and B (respectively) solve
\[
\max_{K^A_{j,t}, H_{j,t}, I^A_{j,t}} \int_0^\infty \Lambda_t \left( A \left( K^A_{j,t} + H_{j,t} - \omega_t H_{j,t} - I^A_{j,t} \right) \right) \, dt \quad \text{s.t.:}
\]
\[
K^A_0 q^A_0 = E_0 \int_0^\infty \Lambda_t q^A_t K^A_{j,t} \left( \frac{\psi^A_{t}}{q^A_{t}} - \Gamma \ln \left( 1 + \frac{I^A_{j,t}}{\theta K^A_{j,t}} \right) + \delta \right) \, dt
\]
\[
\max_{K^B_{j,t}, I^B_{j,t}} \int_0^\infty \Lambda_t \left( A K^B_{j,t} - I^A_{j,t} \right) \, dt \quad \text{s.t.:}
\]
\[
K^B_0 q^B_0 = E_0 \int_0^\infty \Lambda_t q^B_t K^B_{j,t} \left( \frac{\psi^B_{t}}{q^B_{t}} - \Gamma \ln \left( 1 + \frac{I^B_{j,t}}{\theta K^B_{j,t}} \right) + \delta \right) \, dt
\]

Proof. We can obtain these results by applying the Martingale Method from Karatzas and Shreve (1991).

7.8 Asset pricing formulas

Since markets are complete, there exists a unique state price density (SPD), $\Lambda_t$. I will conjecture and verify the following process for the SPD
\[
\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \sigma_{\Lambda,t} dZ_t
\]

From Duffie and Skiadas (1994) we have
\[
\Lambda_t = G f_C
\]
where
\[
G = \exp \left( \int_0^t f_V \, ds \right)
Applying Ito’s Lemma to $\Lambda_t$ and collecting the drift and diffusion terms yields

$$
    r_t = (1 - \rho)\mu_{c,t} - \frac{1}{2}(\rho - 2)(\rho - 1)\sigma_{c,t}^2 - \frac{(\rho - 1)(\gamma + \rho - 1)\sigma_{c,t}\sigma_{v,t}}{\gamma - 1} - \frac{\rho(\gamma + \rho - 1)\sigma_{v,t}^2}{2(\gamma - 1)^2} + \beta \\

    \sigma_{\Lambda,t} = -(\rho - 1)\sigma_{c,t} - \left(\frac{\rho}{\gamma - 1} + 1\right)\sigma_{v,t}
$$

Thus the conjecture is verified. $\mu_{c,t}$ and $\sigma_{c,t}$ are the conditional mean and volatility of consumption growth

$$
    \frac{dC_t}{C_t} = \mu_{c,t}dt + \sigma_{c,t}dZ_t
$$

Plugging the first order conditions for investment in the aggregate resource constraint yields

$$
    C_t = A\left(\mu_i + K_i^A + K_i^B\right) - H_t\left(\Gamma_p\{x_i^A, x_i^B\} - \theta\right) - K_i^A\left(\Gamma_q^A\{x_i^A, x_i^B\} - \theta\right) - K_i^B\left(\Gamma_q^B\{x_i^A, x_i^B\} - \theta\right)
$$

Applying Ito’s Lemma to $C_t$ yields

$$
    \mu_{c,t} = \frac{1}{C_t} \left( \frac{\partial C_t}{\partial H} \times H_t \left( \Gamma \ln \left(1 + \frac{H_t}{\delta}\right) - \delta \right) + \frac{1}{2} \frac{\partial^2 C_t}{\partial H^2} \times H_t \frac{\sigma_h^2}{C_t} + \frac{\partial^2 C_t}{\partial H \partial K^A_t} \times H_t K^A_t \sigma_h \right)

    + \frac{\partial C_t}{\partial K^A_t} \times K^A_t \left( \Gamma \ln \left(1 + \frac{K^A_t}{\delta}\right) - \delta \right) + \frac{1}{2} \frac{\partial^2 C_t}{\partial K^A_t^2} \times K^A_t \sigma^2

    + \frac{\partial C_t}{\partial K^B_t} \times K^B_t \left( \Gamma \ln \left(1 + \frac{K^B_t}{\delta}\right) - \delta \right) + \frac{1}{2} \frac{\partial^2 C_t}{\partial K^B_t^2} \times K^B_t \sigma^2

    + \frac{1}{C_t} \left( \frac{\partial C_t}{\partial H} \times H_t \sigma_h + \frac{\partial C_t}{\partial K^A_t} \times K^A_t \sigma + \frac{\partial C_t}{\partial K^B_t} \times K^B_t \sigma \right)
$$

The value function can be written as

$$
    V\{H + K^A + K^B, x^A, x^B\} = \frac{1}{1 - \gamma} \left( (H + K^A + K^B) F\{x^A, x^B\} \right)^{1 - \gamma}

    \sigma_{v,t} = \left\{ \begin{array}{l}
    \frac{(\gamma - 1)\sigma_x^A(F_A(x_i^A - 1) + F_Bx_i^B - F)}{F} \\
    + \frac{(\gamma - 1)\sigma_x^B(F_Ax_i^A + F_Bx_i^B - 1)}{F} \\
    + \frac{(\gamma - 1)\sigma_x^B(F_Ax_i^A + F_B(x_i^B - 1) - F)}{F}
    \end{array} \right\}
$$

with

$$
    dV_t = -f\{C_t, V_t\} dt + V_t \sigma_{v,t} dZ_t
$$
I define the risky security prices as

$$dS_t = \left( \mu_t \, \text{diag}(S_t) - D_t \right) dt + \left( \begin{array}{c} S_t^A \varsigma_t^A' \\ S_t^B \varsigma_t^B' \end{array} \right) dZ_t$$

where $S_t^i = K_t^i q_t^i$. Using the static constraints from Section 7.7 of the Appendix we can write

$$S_t^i = E_t \int_t^\infty \frac{\Lambda_t}{\Lambda_t} D_t^i d\tau$$

$$D_t^i = K_t^i q_t^i \left( \frac{A - q_t^i \Gamma - \theta}{\Gamma} \right)$$

Similarly for human capital we have

$$S_t^H = H_t p_t = E_t \int_t^\infty \frac{\Lambda_t}{\Lambda_t} D_t^H d\tau$$

$$D_t^H = H_t p_t \left( \frac{A - p_t \Gamma - \theta}{p_t} \right)$$

$$dS_t^H = \left( \mu_t^H S_t^H - D_t \right) dt + S_t^H \varsigma_t^H' dZ_t$$

The static constraints in Section 7.7 of the Appendix imply that the gains processes adjusted by the SPD is Martingales, therefore

$$E_t \left( \Lambda_T S_T^i + \int_0^T \Lambda_t D_t^i dt \right) = \Lambda_t S_t^i + \int_0^t \Lambda_t D_t dt$$

$$\Rightarrow 0 = E_t \left( d \left( \Lambda_T S_t^i + \int_0^t \Lambda_t D_t dt \right) \right)$$

$$\Rightarrow \mu_t^i = r_t + \sigma_{A,t} \cdot \varsigma_t^i + \left( \frac{D_t^i}{S_t^i} - \frac{D_t^i}{S_t^i} \right) = r_t + \sigma_{A,t} \cdot \varsigma_t^i$$

where $\varsigma_t^A = \frac{1}{q_t^i} \frac{\partial q_t^A}{\partial x_t^i} \sigma_{x,A,t} + \left( \frac{1}{q_t^i} \frac{\partial q_t^A}{\partial x_t^i} \sigma_{x,B,t} + \{\sigma,0\}' \right)$ and $\varsigma_t^B = \frac{1}{q_t^i} \frac{\partial q_t^B}{\partial x_t^i} \sigma_{x,A,t} + \left( \frac{1}{q_t^i} \frac{\partial q_t^B}{\partial x_t^i} \sigma_{x,B,t} + \{0,\sigma\}' \right)$. The returns on the risky securities are

$$dR_t^i = \frac{D_t^i}{S_t^i} dt + \frac{dS_t^i}{S_t^i} = \mu_t^i dt + \varsigma_t^i dZ_t = \left( r_t + \sigma_{A,t} \cdot \varsigma_t^i \right) dt + \varsigma_t^i dZ_t$$

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Equivalently, using Ito’s Lemma we can write

\[ dR_i = \frac{D_i}{S_i} dt + \frac{dS_i}{S_i} dt + \left( \frac{A - q_i \Gamma + \theta}{q_i} \right) dt + \left( \frac{dq_i}{q_i} \right) + \frac{dK_i}{K_i} + \frac{dK_i}{K_i} \times \frac{dq_i}{q_i} \]

dividend yield = \( \frac{D_i}{S_i} \)

capital gains = \( \frac{dS_i}{S_i} \)

The total wealth is

\[ S_t = p_t H_t + q_i^A K^A_t + q_i^B K^B_t \]

The marketable wealth is

\[ S_t^m = q_i^A K^A_t + q_i^B K^B_t \]

It will be convenient to define

\[ \omega_i = \frac{S_i}{S_t^w} \]

Let us derive the return on the total wealth portfolio. The above results imply that the gains process for the total wealth portfolio is Martingale when adjusted by the SPD

\[ 0 = E_t \left( d\left( \Lambda_t S_t^w + \int_0^t \Lambda_t \left( D^H_t + D^A_t + D^B_t \right) dt \right) \right) \]

\[ 0 = \frac{D^H_t}{S_t^H} (1 - \omega_A - \omega_B) + \frac{D^A_t}{S_t^A} \omega_A + \frac{D^B_t}{S_t^B} \omega_B - r_t \]

\[ + \mu^w - \frac{D^H_t}{S_t^H} (1 - \omega_A - \omega_B) - \frac{D^A_t}{S_t^A} \omega_A - \frac{D^B_t}{S_t^B} \omega_B \]

\[ - \sigma_{A,t} \cdot \left( \zeta^H (1 - \omega_A - \omega_B) + \zeta^A \omega_A + \zeta^B \omega_B \right) \]

\[ \Rightarrow \mu^w = r_t + \sigma_{A,t} \cdot \zeta^w \]

where

\[ \mu^w = \mu^H (1 - \omega_A - \omega_B) + \mu^A \omega_A + \mu^B \omega_B \]

and

\[ \zeta^w = \zeta^H (1 - \omega_A - \omega_B) + \zeta^A \omega_A + \zeta^B \omega_B \]

Let \( D_t^w \) denote the dividend on the total wealth portfolio

\[ \frac{D_t^w}{S_t^w} = \frac{D^H_t}{S_t^H} (1 - \omega_A - \omega_B) + \frac{D^A_t}{S_t^A} \omega_A + \frac{D^B_t}{S_t^B} \omega_B = C_t \]
Thus the return on the total wealth portfolio is

\[ dR_t^w = \frac{D_t^w}{S_t^w} dt + \frac{dS_t^w}{S_t^w} dt = (r_t + \sigma A \cdot \omega^w) dt + \omega^w dZ_t = \mu_t^w + \omega^w dZ_t \]

Similarly the return on the market portfolio is

\[ dR_t^m = \frac{D_t^m}{S_t^m} dt + \frac{dS_t^m}{S_t^m} dt = (r_t + \sigma A \cdot \omega^m) dt + \omega^m dZ_t = \mu_t^m + \omega^m dZ_t \]

where

\[ \omega^m = \omega^A \frac{S^A}{S^m} + \omega^B \frac{S^B}{S^m} = \frac{S^w}{S^m} \left( \omega^A_A + \omega^B_B \right) \]

\[ \mu_t^m = \mu^A \frac{S^A}{S^m} + \mu^B \frac{S^B}{S^m} = \frac{S^w}{S^m} \left( \mu^A_A + \mu^B_B \right) \]

We can write the return on the market portfolio as a value-weighted sum of the two risky securities return

\[ dR_t^m = dR_t^A \frac{S^A}{S^m} + dR_t^B \frac{S^B}{S^m} \]

\[ dR_t^m = \frac{S^w}{S^m} (dR_t^A \omega^A_A + dR_t^B \omega^B_B) \]

Notice that when \( \rho = 0 \)

\[ \frac{D_t^w}{S_t^w} = \frac{D_t^H + D^A + D^B}{p_t H_t + q_t^A K_t^A + q_t^B K_t^B} = \frac{A (H + K^A + K^B) - (\epsilon^H - \epsilon^A - \epsilon^B)}{p_t H_t + q_t^A K_t^A + q_t^B K_t^B} \]

\[ \Rightarrow \frac{D_t^w}{S_t^w} = \frac{C}{p_t H_t + q_t^A K_t^A + q_t^B K_t^B} = \beta \]

Back to the more general case we have, the beta with the wealth portfolio

\[ \beta_{i,w} = \frac{\text{cov}(dR_t^i, dR_t^w)}{\omega^w} = \frac{\omega^m_{i} \omega^w_{i}}{\omega^w \omega^w} \]

Beta with the market portfolio

\[ \beta_{i,m} = \frac{\text{cov}(dR_t^i, dR_t^m)}{\omega^m_{i} \omega^m_{i}} = \frac{\omega^m_{i} \omega^m_{i}}{\omega^m \omega^m} \]
The expected excess return of stock $A$ relative to stock $B$ is

\[ E_t\left(dR^A_t\right) - E_t\left(dR^B_t\right) = \sigma_{\Lambda,t} \cdot (\xi^A_t - \xi^B_t) \, dt \]

\[ = Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq^A_t}{q^A_t}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq^B_t}{q^B_t}\right) + \sigma_{\Lambda,t} \cdot \{1, -1\}' \, \sigma dt \]

After simulating the model, I find that the instantaneous covariance between the SPD and the price of physical capital of Type $A$ is on average smaller than that between the SPD and the price of physical capital of Type $B$. In particular, I find

\[ E\left(\sigma_{\Lambda,t} \cdot \{1, -1\}' \, \sigma\right) \, dt > E\left(Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq^A_t}{q^A_t}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq^B_t}{q^B_t}\right)\right) > 0 \]

Thus most of the action comes from the fact that the price of risk associated with $dZ^A$ is on average larger than that associated with $dZ^B$. This is because negative shocks to firms of Type $A$ are costlier to the agents since they coincide with negative shocks to the human capital.

Let us define the $Q$ measure as the measure under which the gains processes are Martingale. Let $\xi_t$ denote the corresponding Radon-Nikodym derivative (with $\xi_0 = 1$). $\Lambda_t = \xi_t e^{-\int_0^t r_s \, ds}$. Thus, $\xi_0 = 1 \Rightarrow \Lambda_0 = 1$.

### 7.9 Market Completeness

With $Corr_t\left(\frac{dH_t}{H_t}, \frac{dK^A_t}{K^A_t}\right) / dt = Corr_t\left(dZ^A, dZ^H\right) / dt = q = 1$ markets are dynamically complete as long as the two risky securities (claims on the profits of firms of types $A$ and $B$) are traded. Thus with $q = 1$ the agents can implicitly issue claims on the value of their human capital. Now, let $1 > q > 0$. In that case, markets are dynamically complete only if we explicitly introduce a third risky security by allowing agents to issue claims on the value of their human capital. With complete markets, we can use the solution to the planner’s problem. The asset pricing formulas would be slightly different and the Value premium would be lower (compared to when $q = 1$)

\[ E_t\left(dR^A_t\right) - E_t\left(dR^B_t\right) = Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq^A_t}{q^A_t}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq^B_t}{q^B_t}\right) \]

\[ + \left\{ \sigma_{\Lambda,t}^{(1)} + \sigma_{\Lambda,t}^{(3)} \, q, \sigma_{\Lambda,t}^{(2)} \right\} \cdot \{1, -1\}' \, \sigma dt \]

Notice that with $q = 1$ we have $Cov_t\left(\frac{dH_t}{H_t}, \frac{dK^A_t}{K^A_t}\right) = \sigma_h \, \sigma dt$ and with $1 > q > 0$ it is $Cov_t\left(\frac{dH_t}{H_t}, \frac{dK^A_t}{K^A_t}\right) = \tilde{\sigma}_h \, \sigma dt$. Thus, we can always set $\sigma_h = \tilde{\sigma}_h \, q$ so this covariance is the same in both models. Furthermore, the expected excess return of Security $A$ with $q = 1$ is $Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq^A_t}{q^A_t}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq^B_t}{q^B_t}\right) +$
\(\sigma_{\Lambda,t} \cdot \{1, -1\}^t \sigma dt\). Thus we again recover a simple mapping between the two models. Hence, I conjecture that the quantitative results should not be very different. Furthermore, I should be able to calibrate the parameters of the model with \(1 > \varrho > 0\) (and complete markets) so that the key moments are exactly the same as those in the model with \(\varrho = 1\). The time paths will nonetheless not be identical because of non-linearities, state dependence, and because the introduction of the additional shock \(dZ^H\) changes the time paths of the state variables.

With \(1 > \varrho > 0\), if we do not allow agents to issue claims on the value of their human capital, markets are incomplete. As a result the price of risk is greater since the agents can no longer smooth consumption as much as before due to the restriction on security holdings. In particular, \(\sigma_{\Lambda,t}^{(3)}\) (and hence \(\{\sigma_{\Lambda,t}^{(1)} + \sigma_{\Lambda,t}^{(3)}\}, \sigma_{\Lambda,t}^{(2)}\} \cdot \{1, -1\}^t \sigma\) is greater (relatively to the case with complete markets and \(1 > \varrho > 0\)). Thus, market incompleteness can undo the effect\(^{52}\) of a decrease in \(\varrho\). When markets are dynamically complete, the Martingale Method is simple. If we introduced markets incompleteness, we would need to directly solve for the competitive equilibrium. With incomplete markets the Martingale Method would be more complex as it would require solving a min-max problem. We could instead use dynamic programming to solve for the competitive equilibrium.

### 7.10 Allowing for firm liabilities

Let us assume that at time \(t\) the agents hold an amount \(d_t^i S_t^i\) of debt from firms of Type \(i\). As before, \(\varpi_t^i\) is the fraction of financial wealth the agent invests in the equity of Type \(i\). The securities market clearing conditions would be

\[
\begin{align*}
\varpi_t^A \tilde{W}_t + \varpi_t^B \tilde{W}_t &= ((1 - d_t^A) S_t^A + (1 - d_t^B) S_t^B) \quad \text{equity market clears} \\
(1 - \varpi_t^A - \varpi_t^B) \tilde{W}_t &= d_t^A (\tilde{W}_t - S_t^B) + d_t^B (\tilde{W}_t - S_t^A) \quad \text{debt market clears}
\end{align*}
\]

where financial wealth is

\[
\tilde{W}_s = S_t^A + S_t^B
\]

By Walras Law one of these equations is redundant. Indeed, we can re-write these equations as

\[
\begin{align*}
(\varpi_t^A + \varpi_t^B - 1) \tilde{W}_t &= -d_t^A S_t^A - d_t^B S_t^B \quad \text{equity market clears} \\
(1 - \varpi_t^A - \varpi_t^B) \tilde{W}_t &= d_t^A S_t^A + d_t^B S_t^B \quad \text{debt market clears}
\end{align*}
\]

\(^{52}\)Of course market incompleteness would also affect all other endogenous variables of the model. Nonetheless, I expect that the mechanism highlighted above will be the dominant one.
Notice that when $d_t^A = d_t^B = 0$ we recover the outcome of the baseline model. When we allow for firm debt the return on equity becomes

$$d\tilde{R}_t^i = dR_t^i - r_t d_t^i dt \quad \text{for } i \in \{A, B\}$$

$$\Rightarrow d\tilde{R}_t^i = \left( r_t (1 - d_t^i) + \sigma_{\Lambda,t} \cdot \zeta_t^i \right) dt + \zeta_t^i \cdot dZ_t$$

The model’s BE/ME ratio is unchanged because of the Modigliani-Miller Theorem (the firm value is unaffected by how it is financed)

$$BE/ME = \frac{Assets - Liabilities}{Equity} = \frac{K_t^i - d_t^i S_t^i / q_t^i}{S_t^i - d_t^i S_t^i} = \frac{1 - d_t^i}{(1 - d_t^i)q_t^i} = \frac{1}{q_t^i}$$

Let us assume that $d_t^i$ is constant and that firm debt pays the risk-free rate (as in He and Krishna-murthy; 2012). We can obtain the parameters $\{d^A, d^B\}$ by using data on the Value (firms of Type A) and Growth firms (of Type B)

$$Leverage = \frac{Liabilities}{Assets} = \frac{d_t S_t^i / q_t^i}{S_t^i / q_t^i} = d_t$$

Using data from Compustat and the Fama and French breakpoint for Value and Growth, I find the ratios of total liabilities to total assets

$$d^A = 0.62$$

$$d^B = 0.58$$

The Type A security excess return becomes

$$d\tilde{R}_t^A - d\tilde{R}_t^B = \left( r_t (d_t^B - d_t^A) + \sigma_{\Lambda,t} \cdot (\zeta_t^A - \zeta_t^B) \right) dt + (\zeta_t^A - \zeta_t^B) \cdot dZ_t$$

With an average risk-free rate of 1% we would have

$$E \left( d\tilde{R}_t^A - d\tilde{R}_t^B \right) = E \left( r_t (d_t^B - d_t^A) \right) dt + E \left( dR_t^A - dR_t^B \right) \approx -0.0005$$
7.11 Cox, Ingersoll and Ross (1985a)

To better understand the implications of the adjustment costs, I will now consider the case without adjustment costs as in Cox, Ingersoll and Ross (1985).

\[
\frac{dH_{j,t}}{H_{j,t}} = \frac{I_{j,t}}{H_{j,t}} dt - \delta dt + \sigma_t dZ^A_t
\]

\[
\frac{dK_{j,t}^i}{K_{j,t}^i} = \frac{I_{j,t}^i}{K_{j,t}^i} dt - \delta dt + \sigma_t dZ^i_t \quad \text{for} \ i \in \{A, B\}
\]

From the Martingale Method in Section 7.7 we have

\[
\max_{\{C_{j,t}, I_{j,t}, H_{j,t}\}_{t=0}^\infty} E_0 \int_0^\infty f \{C_{j,t}, V_{j,t}\} dt \quad \text{s.t.:}
\]

\[
\tilde{W}_0 + p_0 H_0 = E_0 \int_0^\infty \Lambda_t \left( C_{j,t} + I_{j,t}^H - \omega_t H_{j,t} \right) dt
\]

\[
+ E_0 \int_0^\infty \Lambda_t p_t H_{j,t} \left( \frac{\tilde{v}_t}{p_t} - \frac{I_{j,t}^H}{H_{j,t}} + \delta \right) dt
\]

The FOC for investment in human capital implies

\[
p_t = 1
\]

Similarly from the firms problems we have

\[
q^i_t = 1 \quad \text{for} \ i \in \{A, B\}
\]

This is the same result reported in Section 5.3 of Eberly and Wang (2011) and Appendix A.4.3 of Kozak (2012). Therefore, without adjustment cost the BE/ME is the same for both types of securities.

In particular, the FOC’s from planner’s problem become

\[
1 = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} \left( F - x^A F_A - x^B F_B \right)
\]

\[
1 = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} \left( F - (x^A - 1) F_A - x^B F_B \right)
\]

\[
1 = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} \left( F - x^A F_A - (x^B - 1) F_B \right)
\]
They imply
\[ c = F \left( \frac{\beta}{F} \right)^{\frac{1}{1-\rho}} \]

\[ F_A = F_B = 0 \]

This is exactly as shown in Appendix A.4.3 of Kozak (2012). The HJB becomes
\[ 0 = \frac{\beta}{\rho} \left( \frac{c}{F} \right)^\rho + \phi \{ x^A, x^B \} + \iota^H (1 - x^A - x^B) + \iota^B x^B + \iota^A x^A \]

where
\[ \iota^H (1 - x^A - x^B) + \iota^B x^B + \iota^A x^A = A - c \]

\[ c = F \left( \frac{\beta}{F} \right)^{\frac{1}{1-\rho}} \]

\[ \phi \{ x^A, x^B \} = \frac{-2\beta - \rho \left( 2\delta + \gamma \left( \sigma^2 \left( x^A \right)^2 + (x^B)^2 \right) - 2\sigma \sigma_h x^A (x^A + x^B - 1) \right)}{2\rho} \]

We can thus find \( F \)
\[ F = \left( \frac{\rho(A + \phi \{ x^A, x^B \})^{\frac{1}{\rho-1}}}{\rho - 1} \right)^{-\frac{1-\rho}{\rho}} \]

As in Eberly and Wang (2011) and Kozak (2012), \( x^A \) and \( x^B \) are no longer state variables but (constant) choice variables which maximize the value function (at date zero). \( \{ x^A, x^B \} \) solve
\[ 0 = \max_{x^A, x^B} \left( \frac{\rho(A + \phi \{ x^A, x^B \})^{\frac{1}{\rho-1}}}{\rho - 1} \right)^{-\frac{1-\rho}{\rho}} \]

The solution is
\[ x^B = \bar{x}^B = 1/2 \]
\[ x^A = \bar{x}^A \in [0, 1/2] \]
To summarize, we have
\[ F = \hat{F} = \left( \frac{\rho(A + \phi(A, B))}{\rho - 1} \right)^{-\frac{1-\rho}{\rho}} \]
and the optimal consumption to capital ratio and total investment to capital ratio
\[ c = \bar{c} = \hat{F} \left( \frac{\beta}{F} \right)^{\frac{1}{1-\rho}} \]
\[ x^A t^A + x^B t^B + (1 - x^A - x^B) t^H = \bar{\psi} = A - \bar{c} \]

The asset pricing implications are as follows
\[ \bar{\psi}_t = r_t + \sigma_h \sigma^{(A)}_{\lambda, t} \]
\[ \bar{\psi}^i_t = r_t + \sigma^{(i)}_{\lambda, t} \]

The first order condition for human capital implies
\[ 0 = -\bar{\psi}_t + \omega_t \]
\[ \Rightarrow \bar{\psi}_t = A \]

Similarly, the first order condition for physical capital implies
\[ \bar{\psi}^i_t = A \]

Thus we must have
\[ A = r_t + \sigma_h \sigma^{(A)}_{\lambda, t} \]
\[ A = r_t + \sigma^{(A)}_{\lambda, t} \]
\[ A = r_t + \sigma^{(B)}_{\lambda, t} \]

which implies
\[ \sigma_h = \sigma \]
\[ \sigma^{(A)}_{\Lambda,t} = \sigma^{(B)}_{\Lambda,t} \]

\[ \mathcal{A} = r_t + \sigma_{\Lambda,t}^{(A)} \]

The diffusions are

\[ \sigma_{v,t} = \begin{cases} - (\gamma - 1) \sigma x^A + (\gamma - 1) \sigma h (x^A + x^B - 1) , & - (\gamma - 1) \sigma x^B \end{cases} \]

\[ \sigma_{c,t} = \begin{cases} (1 - x^A - x^B) \sigma h + x^A \sigma , & x^B \sigma \end{cases} \quad \text{(since } c \text{ is constant)} \]

\[ \sigma_{\Lambda,t} = - (\rho - 1) \sigma_{c,t} - \left( \frac{\rho}{\gamma - 1} + 1 \right) \sigma_{v,t} \]

\[ \sigma^{(B)}_{\Lambda,t} = \gamma \sigma x^B \]

\[ \sigma^{(A)}_{\Lambda,t} = \gamma \sigma (1 - x^B) \]

Thus,

\[ r_t = \mathcal{A} - \frac{1}{2} \gamma \sigma^2 \]

\[ \sigma^{(A)}_{\Lambda,t} = \sigma^{(B)}_{\Lambda,t} = \frac{1}{2} \gamma \sigma \]

The volatility of returns are

\[ \varsigma_t^A = \{ \sigma, 0 \} \]

\[ \varsigma_t^B = \{ 0, \sigma \} \]

and

\[ dR_t^i = \left( r_t + \frac{1}{2} \gamma \sigma^2 \right) dt + \varsigma_t^i dZ_t \]

\[ \Rightarrow dR_t^i = Adt + \varsigma_t^i dZ_t \]

Thus the expected excess return \( (E_t(dR_t^A) - E_t(dR_t^B)) \) is zero.
7.12 Alternative calibration

I search for a combination of \( \{A, c^*, F^*, \gamma, \theta, \rho, \sigma\} \) over the corresponding seven-dimensional space to satisfy the following seven conditions: an expected output growth of 2% in a one-capital economy, a risk-free rate of 0.90%, a volatility of output of 4%, a consumption-to-output ratio of 90% in a one-capital economy, the resource constraint in a one-capital economy, the first-order condition for investment in a one-capital economy, and the HJB in a one-capital economy.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name/Calculation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>Marginal Product of Capital</td>
<td>20.63%</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Subjective Discount Factor</td>
<td>7.00%</td>
</tr>
<tr>
<td>( c^* )</td>
<td>Consumption ( \frac{\text{Capital}}{\text{Capital}} ) in one-capital economy</td>
<td>18.63%</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Depreciation</td>
<td>0.00%</td>
</tr>
<tr>
<td>( F^* )</td>
<td>Normalized Value Function, ( F ), in one-capital economy</td>
<td>0.0791</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Risk Aversion Parameter</td>
<td>59.35</td>
</tr>
<tr>
<td>( \Gamma = \theta )</td>
<td>Adjustment Cost Parameters</td>
<td>2.73%</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Implied IES Parameter = 2.0025</td>
<td>0.5006</td>
</tr>
<tr>
<td>( \sigma = \sigma_h )</td>
<td>Standard deviation of Capital Growth</td>
<td>5.50%</td>
</tr>
</tbody>
</table>

Table 17: Parameters of the model

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data/Targets (%)</th>
<th>Model (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Consumption Growth</td>
<td>2.00</td>
<td>2.02</td>
</tr>
<tr>
<td>Mean Output Growth*</td>
<td>2.00</td>
<td>2.02</td>
</tr>
<tr>
<td>Mean Risk-free Return*</td>
<td>0.90</td>
<td>0.90</td>
</tr>
<tr>
<td>Standard Deviation of Risk-free Return</td>
<td>2.00</td>
<td>0.14</td>
</tr>
<tr>
<td>Mean Return of Value Stocks</td>
<td>10.32</td>
<td>10.97</td>
</tr>
<tr>
<td>Standard Deviation of Value Stocks</td>
<td>16.73</td>
<td>4.66</td>
</tr>
<tr>
<td>Sharpe Ratio of Value Stocks</td>
<td>56.31</td>
<td>216</td>
</tr>
<tr>
<td>Mean Return of Growth Stocks</td>
<td>6.24</td>
<td>9.34</td>
</tr>
<tr>
<td>Standard Deviation of Growth Stocks</td>
<td>16.62</td>
<td>4.15</td>
</tr>
<tr>
<td>Sharpe Ratio of Growth Stocks</td>
<td>32.13</td>
<td>204</td>
</tr>
<tr>
<td>Mean Value premium</td>
<td>4.08</td>
<td>([1.64, 1.65])</td>
</tr>
<tr>
<td>Sharpe Ratio of Value premium</td>
<td>38.50</td>
<td>41.42</td>
</tr>
<tr>
<td>Mean Market Return</td>
<td>7.16</td>
<td>10.05</td>
</tr>
<tr>
<td>Standard Deviation of Market Return</td>
<td>15.45</td>
<td>3.92</td>
</tr>
</tbody>
</table>

Table 18: The parameters are set to to match some annual moments. The moments with * are some of the moments that I target in the calibration. To obtain the model’s moments, I simulate 25 years of observations 10,000 times with a monthly frequency. I then take the sample mean (or standard deviation) over time and then its average across simulations. All values are in annual units.

This alternative calibration yields a Value premium that is very close to that from the main calibration. The Sharpe ratios are much larger because the return volatilities are small. All other results of the model are qualitatively identical and quantitatively similar to those from the main
calibration.

The volatility of equity returns is much lower than what is observed in data. We can nonetheless reconcile the two by adjusting for leverage. Indeed, the model does not include debt financing. In data, firms are levered and the observed volatility of equity is the volatility of levered equity, which is much higher than that of equity with zero leverage.

7.13 More on the conditional CAPM

In the table below I reproduce the regressions from Table 4 while introducing some error in $\beta_t^{i,m}$, $\beta_t^{i,w}$, $\varsigma_t^m$, and $\varsigma_t^w$. In particular, for each observation of $\{\varsigma_t^A, \varsigma_t^B, \varsigma_t^m, \varsigma_t^w\}$ construct $\{\hat{\varsigma}_t^A, \hat{\varsigma}_t^B, \hat{\varsigma}_t^m, \hat{\varsigma}_t^w\}$ where $\hat{\varsigma}_t^A = \varsigma_t^A + \epsilon_t^A$, $\hat{\varsigma}_t^B = \varsigma_t^B + \epsilon_t^B$, $\hat{\varsigma}_t^m = \varsigma_t^m + \epsilon_t^m$, $\hat{\varsigma}_t^w = \varsigma_t^w + \epsilon_t^w$ and the $\epsilon$'s are Normal random variables with mean zero and standard deviation equal to that of the corresponding $\varsigma$. Lastly, I define $\hat{\beta}_t^{i,w} = \frac{\hat{\varsigma}_t^w}{\hat{\varsigma}_t^m}$ and $\hat{\beta}_t^{i,m} = \frac{\hat{\varsigma}_t^i}{\hat{\varsigma}_t^m}$.

<table>
<thead>
<tr>
<th>Conditional CAPM Regressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Wealth Portfolio: $E_t (dR_t^A) - E_t (dR_t^B) =$</td>
</tr>
<tr>
<td>(2) Market Portfolio: $E_t (dR_t^A) - E_t (dR_t^B) =$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma = (1 - \rho) = 1$</th>
<th>$\gamma \neq (1 - \rho) = 1$</th>
<th>$\gamma &gt; (1 - \rho) \neq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0$ (%)</td>
<td>$0.34$</td>
<td>$0.98$</td>
</tr>
<tr>
<td>t-stat</td>
<td>$6.3$</td>
<td>$7.7$</td>
</tr>
<tr>
<td>Slope</td>
<td>$-0.71$</td>
<td>$-0.78$</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>$0.66$</td>
<td>$1.5$</td>
</tr>
<tr>
<td>t-stat</td>
<td>$24.$</td>
<td>$18.$</td>
</tr>
<tr>
<td>Slope</td>
<td>$0.03$</td>
<td>$0.01$</td>
</tr>
</tbody>
</table>

Table 19: Each panel shows the regression results for three different specifications for preferences. To obtain the regression coefficients I simulate the model 10,000 times and run a regression for each simulation. The coefficients displayed are the average coefficients across simulations. I then calculate the standard errors as the standard deviation of the statistics across simulations. The t-stats are the ratio of the average statistic and their standard error. In the columns labelled (1), I include the results from the CAPM regressions of excess returns against excess betas on the wealth portfolio. In the columns labelled (2), I include the results from the CAPM regressions of excess returns against excess betas on the market portfolio. The column labelled Diff shows the difference between the statistics from regressions (1) and (2) as well as the corresponding t-stats.
7.14 Deterministic model

I now turn off all the shocks in the model by setting $\sigma = \sigma_h = 0$. The processes for the state variables become

$$dx^i_t = \mu_{x,i,t}dt \quad \text{for } i \in \{A, B\}$$

$$\mu_{x,A,t} = x^A_t \left( \Gamma (x^A_t + x^B_t - 1) ln \left( \frac{\Gamma p_t}{\theta} \right) - (1 - \Gamma x^A_t) ln \left( \frac{\Gamma q^A_t}{\theta} \right) - \Gamma x^B_t ln \left( \frac{\Gamma q^B_t}{\theta} \right) \right)$$

$$\mu_{x,B,t} = x^B_t \left( \Gamma (x^A_t + x^B_t - 1) ln \left( \frac{\Gamma p_t}{\theta} \right) - \Gamma x^A_t ln \left( \frac{\Gamma q^A_t}{\theta} \right) + (1 - \Gamma x^B_t) ln \left( \frac{\Gamma q^B_t}{\theta} \right) \right)$$

At the deterministic steady state we have

$$0 = \Gamma (x^A_t + x^B_t - 1) ln \left( \frac{\Gamma p_t}{\theta} \right) + (1 - \Gamma x^A_t) ln \left( \frac{\Gamma q^A_t}{\theta} \right) - \Gamma x^B_t ln \left( \frac{\Gamma q^B_t}{\theta} \right)$$

which implies that

$$p = q^A = q^B$$

$$dR^A = dR^B = r dt = (1 - \rho) \mu_c + \beta dt$$

$$\mu_c = \Gamma ln \left( \frac{p}{\theta} \right) - \delta$$

In steady state, $x^A$ and $x^B$ are indeterminate because $H$, $K^A$ and $K^B$ are risk-less and have the same productivity, $A$. Using Corollary 2 and the HJB (2.10), we then have that $p$ and $F$ jointly solve

$$p = \frac{1}{\beta} \left( \frac{A - \Gamma p + \theta}{F} \right)^{1-\rho} F$$

$$0 = \frac{\beta}{\rho} \left( \frac{A - \Gamma p + \theta}{F} \right)^{\rho} - \frac{\beta}{\rho} + \Gamma ln \left( \frac{p}{\theta} \right) - \delta$$

Lastly, in steady state we also have $\ell^A = \ell^B = \ell^H = \Gamma p - \theta$.

7.15 Joint distribution of the state variables
Figure 11: I pool all observations (across time and simulations) from the model for \{x^A, x^B\} to produce the joint distribution.

The red dot shows the mean of the state variables. Because \(x^A \in [0,1]\) and \(x^B \in [0,1]\), the state space is triangular (and is delineated with the thin black line). However because of the strong symmetry of the model, there is a strong relationship between \(x^A_t\) and \(x^B_t\) and the realizations of \(\{x^A_t, x^B_t\}\) are in a tight band. In particular, the thick black curve is actually a scatter plot from 10,000 simulation of the model. The dark shading shows where the observations are relatively more concentrated. I also plot the 45-degree line and the percentage of observations in the two halves of the state space. Furthermore, from the 10,000 simulations of the model I find that the boundaries are never visited: \(x^A_t \in [0.0038, 0.4976]\) and \(x^B_t \in [0.0034, 0.9924]\).

7.16 Long run risk

Following Borovicka et al. (2011) and Hansen (2011), I construct risk-price elasticities for securities \(A\) and \(B\). Let \(x\) denote the vector of state variables. We have

\[
\begin{align*}
\frac{dx_t}{dt} &= \mu_{x,t}dt + \sigma_{x,t}dZ_t \\
\mu_{x,t} &= \begin{pmatrix} \mu_{x,A,t} \\ \mu_{x,B,t} \end{pmatrix} \quad \text{and} \quad \sigma_{x,t} = \begin{pmatrix} \sigma'_{x,A,t} \\ \sigma'_{x,B,t} \end{pmatrix}
\end{align*}
\]
I introduce the perturbation with \( \zeta_t \{ \epsilon \} \) where

\[
\ln (\zeta_t \{ \epsilon \}) = \int_0^t \frac{1}{2} \epsilon^2 \alpha \cdot \alpha ds + \int_0^t \epsilon \alpha \cdot dZ_s
\]

and \( \alpha = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\} \). 

The risk-price elasticities are

\[
\pi^i \{ x, t \} = \frac{1}{t} \left. \frac{d}{d\epsilon} \ln \left( E (S^i_t \zeta_t \{ \epsilon \} \mid x_0 = x) \right) \right|_{\epsilon = 0} \quad \text{risk-exposure elasticity}
\]

\[
- \frac{1}{t} \left. \frac{d}{d\epsilon} \ln \left( E (\Lambda_t S^i_t \zeta_t \{ \epsilon \} \mid x_0 = x) \right) \right|_{\epsilon = 0} \quad \text{risk-value elasticity}
\]

for \( i \in \{ A, B \} \).

The first step is to produce the dynamic valuation decomposition for \( S^i_t \) and \( \Lambda_t S^i_t \). Since the approach is the same in both cases, I will produce the decomposition for a generic multiplicative functional \( M_t \) and later specialize to \( M_t = S^i_t \) and \( M_t = \Lambda_t S^i_t \).

\[
dl = \mu_{M,t} dt + \sigma_{M,t} \cdot dZ_t
\]

\[
\Rightarrow M_t = M_0 \exp \left( \int_0^t (\mu_{M,s}) ds + \int_0^t \sigma_{M,s} \cdot dZ_s \right)
\]

I look for an eigenfunction \( e \{ x \} = \exp \{ g \{ x \} \} \) and an eigenvalue \( \nu \) which solve

\[
E \left( \frac{M_t e \{ x_t \} \mid x_0 = x}{M \{ x \}} \right) = \exp (\nu t) e \{ x \}
\]

Since this equation holds for any \( t \), it can be localized as

\[
\lim_{t \downarrow 0} \frac{E \left( M_t e \{ x_t \} \mid x_0 = x \right) - \exp (\nu t) e \{ x \} M \{ x \}}{t} = 0
\]

\[
\Rightarrow \frac{\partial e}{\partial x} (\mu_x + \sigma_x \sigma_M) + \frac{1}{2} \text{trace} \left( \left( \frac{\partial}{\partial x} \left( \frac{\partial e}{\partial x'} \right) \right) \sigma_x \sigma_x \right) + e \left( \mu_M + \frac{1}{2} \sigma_M \cdot \sigma_M \right) = \nu e
\]

\[
\Rightarrow \frac{\partial g}{\partial x} (\mu_x + \sigma_x \sigma_M) + \frac{1}{2} \frac{\partial g}{\partial x'} \sigma_x \sigma_x \frac{\partial g}{\partial x} + \frac{1}{2} \text{trace} \left( \left( \frac{\partial}{\partial x} \left( \frac{\partial g}{\partial x'} \right) \right) \sigma_x \sigma_x \right) + \left( \mu_M + \frac{1}{2} \sigma_M \cdot \sigma_M \right) = \nu
\]

\( \nu \) is also the long run growth (or decay) rate, \( \nu = \lim_{t \to \infty} \frac{1}{t} \ln \left( E \left( M_t \mid x_0 = x \right) \right) \). To obtain \( \nu \) we can collect the constant terms from the above eigenfunction equation (7.8) by evaluating this equation at the \( x^A = x^B = 0 \) boundary. As expected, I find that \( \nu \) is larger for \( M_t = S_t^A \) (Value stocks) than \( M_t = S_t^B \) since Value stocks have greater returns on average. In particular,

\[
\nu^A = \beta - \frac{A - \Gamma q^A \{0,0\}}{q^A \{0,0\}} + \theta + \frac{1}{2} \gamma \sigma_h (2\sigma + (-2 + \rho)\sigma_h) + (1 - \rho) \left( \ln \left( \frac{\Gamma p^A \{0,0\}}{\theta} \right) - \delta \right)
\]

\[
\nu^B = \beta - \frac{A - \Gamma q^B \{0,0\}}{q^B \{0,0\}} + \theta + \frac{1}{2} \gamma (-2 + \rho)\sigma_h^2 + (1 - \rho) \left( \ln \left( \frac{\Gamma p^A \{0,0\}}{\theta} \right) - \delta \right)
\]

where \( q^B \{0,0\} > q^A \{0,0\} > p^A \{0,0\} \). The difference in the long-run growth of stock prices between Value and Growth is

\[
\nu^A - \nu^B = \frac{(q^A \{0,0\} - q^B \{0,0\}) (A + \theta)}{q^A \{0,0\} q^B \{0,0\}} + \gamma \sigma \sigma_h > 0
\]

Similarly, for \( M_t = \Lambda_t S_t^\dagger \), I find

\[
\hat{\nu}^A = -\frac{A - \Gamma q^A \{0,0\}}{q^A \{0,0\}}
\]

\[
\hat{\nu}^B = -\frac{A - \Gamma q^B \{0,0\}}{q^B \{0,0\}}
\]

Thus, following Section 6.4 of Borovicka et al. (2011), the limiting Value premium is

\[
\lim_{t \to \infty} \left\{ \frac{1}{t} \left[ \ln \left( E \left( S_t^A \mid x_0 = x \right) \right) - \ln \left( E \left( \Lambda_t S_t^A \mid x_0 = x \right) \right) \right] \right\} = (\nu^A - \hat{\nu}^A) - (\nu^B - \hat{\nu}^B) = \gamma \sigma \sigma_h
\]

The \( \sigma \sigma_h \) term comes from the relative covariance of human capital growth with the asset growth of Type A firms; that is \( \text{Cov} \left( \frac{dH}{\delta t}, \frac{dK^A}{K^A} \right) = \text{Cov} \left( \frac{dH}{\delta t}, \frac{dK^H}{K^H} \right) = \sigma \sigma_h dt \). This is by design. Nonetheless, it is interesting that we are able to characterize the limiting Value premium by the product of the risk aversion governing parameter and the relative covariance of human capital growth with the asset growth of Value firms.

Continuing with the decomposition, I can now write

\[
M_t = \exp(\nu t) \tilde{M}_t \frac{\hat{\epsilon} \{x_t\}}{\hat{\epsilon} \{x\}}
\]

where \( \hat{\epsilon} \{x\} = 1/e \{x\} \) and \( \tilde{M}_t \) is Martingale. Notice that at the boundaries we have

\[
M_t = \exp(\nu t) \tilde{M}_t \frac{\hat{\epsilon} \{x_t\}}{\hat{\epsilon} \{x\}} = \exp(\nu t) \tilde{M}_t
\]

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Therefore, we can set
\[ e\{1,0\} = e\{1,0\} = e\{0,0\} = 1 \]
\[ \Rightarrow g\{1,0\} = g\{0,1\} = g\{0,0\} = 0 \]
The above boundary conditions are necessary for solving the partial differential equation in (7.8).
\[ \ln(\tilde{M}_t) = \ln(M_t) + \ln(e \{x_t\}) - \ln(e \{x\}) - \nu t \]
\[ d\ln(\tilde{M}_t) = \frac{1}{2} \tilde{\sigma}_{M,t} \cdot \tilde{\sigma}_{M,t} dt + \tilde{\sigma}_{M,t} \cdot dZ_t \]
\[ \Rightarrow \tilde{\sigma}_{M,t} = \sigma_{M,t} + \sigma_x \frac{\partial}{\partial x} g \{x_t\} \]
I now define an alternative measure such that for any function \( n \{x_t\} \) we have
\[ E(\tilde{M}_t \exp(n \{x_t\}) | x_0 = x) = \tilde{E} \left( \exp(n \{x_t\}) | x_0 = x \right) \]
\( d\tilde{Z}_t \) is a standard Brownian motion increment under the alternative measure and
\[ dZ_t = \Theta_t dt + d\tilde{Z}_t \]
\[ dx_t = \mu_{x,t} dt + \sigma_{x,t} \left( \Theta_t dt + d\tilde{Z}_t \right) \]
\[ dn = \left( \frac{\partial}{\partial x} n \{x\} \right) dx + O(dx^2) = \mu_n dt + \left( \frac{\partial}{\partial x} n \{x\} \right) \sigma_x dZ \]
Therefore, \( E(\tilde{M}_t \exp(n \{x_t\}) | x_0 = x) = \tilde{E} \left( \exp(n \{x_t\}) | x_0 = x \right) \) implies \( \Theta_t = \tilde{\sigma}_{M,t} \).
Now, we have
\[ \frac{1}{t} \frac{d}{d\epsilon} \ln \left\{ E(\tilde{M}_t \zeta_t \{\epsilon\} | x_0 = x) \right\} \bigg|_{\epsilon = 0} = \frac{1}{t} \frac{d}{d\epsilon} \ln \left\{ \exp(\nu t) \tilde{E} \left( \frac{\hat{e} \{x_t\} \zeta_t \{\epsilon\} | x_0 = x} {\hat{e} \{x\}} \right) \right\} \bigg|_{\epsilon = 0} \]
\[ \frac{1}{t} \frac{d}{d\epsilon} \ln \left\{ E(\tilde{M}_t \zeta_t \{\epsilon\} | x_0 = x) \right\} \bigg|_{\epsilon = 0} = \frac{1}{t} \frac{\tilde{E} \left( \hat{e} \{x_t\} \left( \int_0^t \alpha \cdot d\tilde{Z}_s \right) | x_0 = x \right)} {\tilde{E} \{ \hat{e} \{x_t\} | x_0 = x \}} + \frac{1}{t} \frac{\tilde{E} \left( \hat{e} \{x_t\} \left( \int_0^t \alpha \cdot \tilde{\sigma}_{M,s} ds \right) | x_0 = x \right)} {\tilde{E} \{ \hat{e} \{x_t\} | x_0 = x \}} \]
Using the innovation representation

\[ \hat{e}\{x_t\} = \int_0^t \chi\{x_s, t - s\} d\hat{Z}_s + \tilde{E}(\hat{e}\{x_t\} \mid x_0 = x) \]

\[ \chi\{x, t - s\} = \sigma_x \frac{\partial}{\partial x} \tilde{E}(\hat{e}\{x_t\} \mid x_s = x) \]

\[ \chi\{x, t - s\} = \tilde{E}(\hat{e}\{x_t\} \mid x_s = x) \frac{\sigma_x \frac{\partial}{\partial x} \ln(\tilde{E}(\hat{e}\{x_t\} \mid x_s = x))}{\phi_{\{x, t - s\}}} \]

\[ \Rightarrow \frac{1}{t} \frac{d}{de} \ln \{E(M_t \zeta_t \mid e \mid x_0 = x)\} \bigg|_{e=0} = \frac{1}{t} \left( \frac{\tilde{E}(\hat{e}\{x_t\} \int_0^t \varepsilon\{x_s, t - s\} ds \mid x_0 = x)}{\tilde{E}(\hat{e}\{x_t\} \mid x_0 = x)} \right) \]

\[ \varepsilon\{x, t\} = \alpha \cdot \left( \phi\{x, t\} + \sigma_M\{x\} + \sigma_x\{x\} \frac{\partial}{\partial x} g\{x\} \right) \]

Notice that

\[ \phi\{x, t\} = \sigma_x \frac{\partial}{\partial x} \ln(\tilde{E}(\exp(-g\{x_t\}) \mid x_0 = x)) \]

\[ \phi\{x, 0\} = -\sigma_x \frac{\partial}{\partial x} g\{x\} \]

I will assume that process for the state vector, \( x \), is stochastically stable under the alternative measure. Thus \( \lim_{t \to \infty} \tilde{E}(\exp(-g\{x_t\}) \mid x_s = x) = \tilde{E}(\exp(-g\{x_t\})) \) and

\[ \lim_{t \to \infty} \phi\{x, t\} = 0 \]

### 7.16.1 Limiting behavior

Following the approach from Section 3.4 of Borovicka et al. (2011) I derive the limiting behavior. The long horizon risk-price elasticities are

\[ \pi^i\{x, \infty\} = \alpha \cdot \left( \sigma_A\{x\} + \sigma_x \frac{\partial}{\partial x} g^i\{x\} - \sigma_x \frac{\partial}{\partial x} g^i\{x\} \right) \quad i \in \{A, B\} \]

where \( g^i\{x\} \) and \( g^i\{x\} \) solve equation (7.8) for \( M_t = S_t^i \) and \( M_t = \Lambda_t S_t^i \) respectively. When we condition on the mean value of the state vector (\( \bar{x} \)) we find \( \pi^A\{\bar{x}, \infty\} > \pi^B\{\bar{x}, \infty\} \). Thus, in the model the Type A firms are indeed more exposed to long-run risk than the Type B firms.
Figure 12: Figure (a) shows the relative long-horizon risk-price elasticity as a function of the state variables. Figure (b) shows the distribution of the relative risk-price elasticities from simulations of the model.

From Figure 12 we can see that \( \pi^A \{ \bar{x}, \infty \} > \pi^B \{ \bar{x}, \infty \} \) (where \( \bar{x} \) is the mean of the state variables, that is \( \bar{x} = E(x) \) and \( E(\pi^A \{ x, \infty \}) > E(\pi^B \{ x, \infty \}) \). Thus the Type A firms are indeed more exposed to long-run risk.

### 7.16.2 Transitional dynamics

Let

\[
\begin{align*}
    f \{ x, t - s \} &= \tilde{E} \left( \exp \left( -g \{ x_t \} \right) \bigg| x_s = x \right) \\
    \Rightarrow f \{ x, 0 \} &= \exp \left( -g \{ x \} \right)
\end{align*}
\]

Stochastic stability implies

\[
\begin{align*}
    \lim_{t-s \to \infty} \phi \{ x, t - s \} &= \lim_{t-s \to \infty} \sigma_x \frac{\partial}{\partial x} \log \left( \tilde{E} \left( \exp \left( -g \{ x_t \} \right) \bigg| x_s = x \right) \right) \\
    &= \lim_{t-s \to \infty} \sigma_x \frac{\partial}{\partial x} \log \left( f \{ x, t - s \} \right) \\
    \Rightarrow \lim_{t-s \to \infty} \frac{\partial}{\partial x} \log \left( f \{ x, t - s \} \right) &= \{0,0\}'
\end{align*}
\]
A conditional expectation is Martingale, therefore

\[ \Rightarrow 0 = \frac{\partial}{\partial s} f + \left( \frac{\partial}{\partial x} f \right) (\mu_x + \sigma_x \sigma_M + \sigma_x \sigma_x g_x) + \frac{1}{2} \text{trace} \left\{ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} f \right) \sigma_x \sigma_x \right\} \]

Let \( u = t - s \), now we can find \( f \{ x, u \} \) by solving

\[ 0 = -\frac{\partial}{\partial u} f + \left( \frac{\partial}{\partial x} f \right) (\mu_x + \sigma_x \sigma_M + \sigma_x \sigma_x g_x) + \frac{1}{2} \text{trace} \left\{ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} f \right) \sigma_x \sigma_x \right\} \]

\[ f \{ x, 0 \} = \exp (-g \{ x \}) \]

\[ \lim_{u \to \infty} \frac{\partial}{\partial x} \ln (f \{ x, u \}) = \{0, 0\}' \]

The risk-price elasticities are

\[ \pi^i \{ x, t - s \} = \frac{1}{t} \left( \frac{\bar{E} \left( \tilde{\epsilon}_g^i \{ x_t \} \int_0^t \tilde{\epsilon}_g^i \{ x_s, t - s \} ds \middle| x_0 = x \right)}{\bar{E} (\tilde{\epsilon}_g^i \{ x_t \} \middle| x_0 = x)} \right) \]

\[ -\frac{1}{t} \left( \frac{\bar{E} \left( \tilde{\epsilon}_v^i \{ x_t \} \int_0^t \tilde{\epsilon}_v^i \{ x_s, t - s \} ds \middle| x_0 = x \right)}{\bar{E} (\tilde{\epsilon}_v^i \{ x_t \} \middle| x_0 = x)} \right) \]

for \( i \in \{ A, B \} \)

where

\[ \tilde{\epsilon}_g^i \{ x_t \} = 1/\exp (g_g^i \{ x_t \}) \]

\[ \tilde{\epsilon}_v^i \{ x_t \} = 1/\exp (g_v^i \{ x_t \}) \]

\[ \epsilon_g^i \{ x, t - s \} = \alpha \cdot \left( \phi_g^i \{ x, t - s \} + \zeta^i \{ x \} - \sigma \{ x \} + \sigma_x \{ x \} \frac{\partial}{\partial x} g_g^i \{ x \} \right) \]

\[ \epsilon_v^i \{ x, t - s \} = \alpha \cdot \left( \phi_v^i \{ x, t - s \} + \zeta^i \{ x \} - \sigma \{ x \} + \sigma_x \{ x \} \frac{\partial}{\partial x} g_v^i \{ x \} \right) \]

\[ \phi_g^i \{ x, t - s \} = \sigma_x \frac{\partial}{\partial x} f_g^i \{ x, t - s \} \]

\[ \phi_v^i \{ x, t - s \} = \sigma_x \frac{\partial}{\partial x} f_v^i \{ x, t - s \} \]
and $f^l_y(x, u)$ and $f^l_v(x, u)$ solve equation (7.9) for $M_t = S^l_t$ and $M_t = \Lambda_t S^l_t$ respectively.

### 7.16.3 Autocorrelations and covariances with consumption growth

![Graphs of covariances and autocorrelations with consumption growth](image)

**Figure 13:** These figures show covariances of security returns with consumption growth over $\tau$ years and autocorrelations with lags of $\tau$ years.

First, I write the process for consumption as

$$
dlnC_t = \mu_{c,t} dt - \frac{1}{2} \sigma_{c,t} \cdot \sigma_{c,t} dt + \sigma_{c,t} \cdot dZ_t
$$
Then, I calculate the covariances between equity returns and consumption growth as follows.

\[
\text{Cov}_t \left( \frac{C_{t+\tau}}{C_t} - 1, dR_t^i \right) = \text{Cov}_t \left( \exp \left( \int_t^{t+\tau} (\mu_{c,s} - \frac{1}{2} \sigma_{c,s} \cdot \sigma_{c,s}) ds + \int_t^{t+\tau} \sigma_{c,s} \cdot dZ_s \right), dR_t^i \right)
\]

\[
\text{Cov}_t \left( \frac{C_{t+\tau}}{C_t} - 1, dR_t^i \right) = E_t \left( \exp \left( \int_t^{t+\tau} (\mu_{c,s} - \frac{1}{2} \sigma_{c,s} \cdot \sigma_{c,s}) ds + \int_t^{t+\tau} \sigma_{c,s} \cdot dZ_s \right) dR_t^i \right) - E_t \left( \exp \left( \int_t^{t+\tau} (\mu_{c,s} - \frac{1}{2} \sigma_{c,s} \cdot \sigma_{c,s}) ds + \int_t^{t+\tau} \sigma_{c,s} \cdot dZ_s \right) \right) E_t (dR_t^i)
\]

As expected the returns on the Type A security covary much more with consumption growth over horizons \( \tau > 0 \) compared to the Type B security. This result is reminiscent of that from Hansen, Heaton and Li (2008) and shows that the Type A security is relatively more exposed to long horizon risk than the Type B security. However, in my model it is also true at shorter horizons. The contemporaneous correlations between consumption growth and equity returns are in Table 20. They are larger than what is observed in data because the model is not tailored to match these moments.

<table>
<thead>
<tr>
<th></th>
<th>( dR^A_t )</th>
<th>( dR^B_t )</th>
<th>( dR^m_t )</th>
<th>( dR^w_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Cor}_{t} \left( \frac{dC_t}{C_t}, dR_t \right) )</td>
<td>0.90</td>
<td>0.74</td>
<td>0.98</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 20: Contemporaneous correlations between consumption growth and equity returns

The autocorrelations are somewhat large for expected consumption growth (\( \mu_{c,t} \)), expected returns, as well as the levels and expected growth rate of the state variables (\( x_t^i \) and \( \mu_{x,i,t} \) for \( i \in \{A, B\} \) respectively). That is because the adjustment costs prevent instantaneous reallocation of capital. As a result the capital shares, \( \{x_t^A, x_t^B\} \), have large autocorrelations for \( \tau < 3 \). Since the expected consumption growth, the expected returns and the expected growth rate of the capital shares are functions of state variables, these drifts also display strong autocorrelations for \( \tau < 3 \).

### 7.17 Controlling for the number of employees

Since Growth firms tend to hire more employees, in the regressions below I control for the level (or growth rate) of employment.
7.18 Univariate betas of monthly equity returns with monthly returns on human Capital

<table>
<thead>
<tr>
<th>Value Portfolio</th>
<th>Aggregate Human Capital</th>
<th>P-val</th>
<th>~Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>β_{t,h}</td>
<td>0.91</td>
<td>0.07</td>
<td>600</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Growth Portfolio</th>
<th>Aggregate Human Capital</th>
<th>P-val</th>
<th>~Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>β_{t,h}</td>
<td>0.04</td>
<td>0.93</td>
<td>600</td>
</tr>
</tbody>
</table>

Firm level returns on human capital data is only available annually from Compustat. Thus, I am unable run the above regressions at a monthly frequency for portfolio or firm level human capital returns.

7.19 Annually Updated β_{h}
7.20 Extension of the model

The model yields a Sharpe ratio for the Value premium \( \frac{E_t(dR_t^A) - E_t(dR_t^B)}{\sqrt{\sigma_t^A + \sigma_t^B}} \) which is fairly small compared to what is observed in data. This is because the local covariance of the two risky securities \( (\sigma_t^A \cdot \sigma_t^B dt) \) is fairly small. I can correct this without affecting the average Value premium much by introducing some common variation in the two types of firms which is locally orthogonal to (or has the same local covariance with) both \( dZ_t^A \) and \( dZ_t^B \). I can do so by making the parameter which governs the risk aversion (\( \gamma_t \)) or the adjustment cost parameter (\( \Gamma_t = \theta_t \)) stochastic (and mean-reverting for stationarity). In the current model, the Tobin Q’s have a rather low volatility and most of the volatility of the capital gains comes from the volatility of capital. With \( \theta_t \) or \( \gamma_t \) stochastic, the Tobin Q’s would me more volatile and we could have highly volatile capital gains while keeping the volatility of capital low; which would be more factual. This model would be more difficult to solve because there would be on additional state variable, \( \theta_t \) or \( \gamma_t \). I go through some of the details of one of the aforementioned extensions of the model below.

7.20.1 Stochastic Risk Aversion

Let \( \gamma = \gamma_t \) where

\[
    d\gamma_t = \theta_0 (\gamma_t - \gamma_t^*) \, dt + \theta_1 \gamma_t dZ_t^\gamma
\]
We can obtain the solution for $\gamma_t$ using the steps below.

$$
(d\gamma_t + \vartheta_0 \gamma_t dt - \vartheta_1 \gamma_t dZ_t^G) \exp \left\{ \left( \vartheta_0 + \frac{1}{2} \vartheta_1^2 \right) t - \int_0^t \vartheta_1 dZ_s^G \right\} = \vartheta_0 \gamma_t \exp \left\{ \left( \vartheta_0 + \frac{1}{2} \vartheta_1^2 \right) t - \int_0^t \vartheta_1 dZ_s^G \right\} dt
$$

$$
\Rightarrow d \left( \gamma_t \exp \left\{ \left( \vartheta_0 + \frac{1}{2} \vartheta_1^2 \right) t - \vartheta_1 Z_t^G \right\} \right) = \vartheta_0 \gamma_t \exp \left\{ \left( \vartheta_0 + \frac{1}{2} \vartheta_1^2 \right) t - \vartheta_1 Z_t^G \right\} dt
$$

$$
\Rightarrow \gamma_t = \frac{\vartheta_0 \gamma \int_0^t \exp \left\{ \left( \vartheta_0 + \frac{1}{2} \vartheta_1^2 \right) n - \vartheta_1 Z_n^G \right\} dn + \vartheta_0}{\exp \left\{ \left( \vartheta_0 + \frac{1}{2} \vartheta_1^2 \right) t - \vartheta_1 Z_t^G \right\}}
$$

Notice that with $\gamma_0 = \bar{\gamma}$, $\vartheta_0 > 0$, $\vartheta_1 > 0$ we have $\gamma_t \geq 0$. Furthermore,

$$
E_0 (\gamma_t) = \bar{\gamma} (1 - \exp \{-\vartheta_0 t\}) + \gamma_0 \exp \{-\vartheta_0 t\}
$$

$$
\lim_{t \to \infty} E_0 (\gamma_t) = \bar{\gamma}
$$

To maintain market completeness, I will set $dZ_t^G = dZ_t^A + dZ_t^B$. For homogeneity, I need to modify the value function slightly. In particular, I will use an aggregator, $\langle \tilde{f}, \tilde{A}_v \rangle$, where

$$
\tilde{f} \left\{ C_t, \tilde{V}_t \right\} = \frac{\beta}{\rho} \left( \frac{C_t}{V_t^{\theta - 1}} - \tilde{V}_t \right)
$$

$$
\tilde{A}_v \left\{ \tilde{V}_t \right\} = -\frac{\gamma_t}{\tilde{V}_t}
$$

the new value function is $\tilde{V}_t \equiv (1 - \gamma)V_t^{-\gamma}$. And $\tilde{f} \left\{ C_t, \tilde{V}_t \right\} \equiv \tilde{V}_t^{\gamma} f \left\{ C_t, V_t \right\}$. The process for the value function is

$$
d\tilde{V}_t = -\left( \tilde{f} \left\{ C_t, \tilde{V}_t \right\} + \frac{1}{2} \tilde{V}_t^2 \tilde{A}_v \left\{ \tilde{V}_t \right\} \sigma_v \cdot \sigma_v \right) dt + \tilde{V}_t \sigma_v dZ_t
$$

The variance multiplier, $A_v \left\{ V_t \right\}$, introduces a penalty for volatility in the value function.

In the case where $\gamma$ is constant, this new aggregator is ordinally equivalent to the original aggregator. Indeed, following Kozak (2012), let us define the change of variables $\chi \{ y \} = y^{1-\gamma}$. 95
This implies that $V_t = \chi \{ \tilde{V}_t \}$ and

$$\tilde{f} \left\{ C_t, \tilde{V}_t \right\} = \frac{f \{ C_t, V_t \}}{\chi' \{ \tilde{V}_t \}} \quad (7.10)$$

$$\tilde{A}_v \{ \tilde{V}_t \} = \frac{\chi'' \{ \tilde{V}_t \}}{\chi' \{ \tilde{V}_t \}}$$

Recall that the original aggregator was a normalized aggregator $< f, A_v >$ with

$$A_v \{ V_t \} = A_v \left\{ \chi \left\{ \tilde{V}_t \right\} \right\} = 0$$

Thus we can re-write $\tilde{A}_v \{ \tilde{V}_t \}$ as

$$\tilde{A}_v \{ \tilde{V}_t \} = \chi' \{ \tilde{V}_t \} A_v \left\{ \chi \{ \tilde{V}_t \} \right\} + \frac{\chi'' \{ \tilde{V}_t \}}{\chi' \{ \tilde{V}_t \}}$$  \quad (7.11)

Equations (7.10) and (7.11) imply that the aggregator $< \tilde{f}, \tilde{A}_v >$ and $f$ are ordinally equivalent (see Section 1.4 of Duffie and Epstein; 1992b). Thus the corresponding SPD’s are the same.

$$G_t = \exp \left( \int_0^t f_v \{ C_s, V_s \} \, ds \right)$$

$$\Lambda_t = G_t f_c \{ C_t, V_t \}$$

$$\frac{d \Lambda_t}{\Lambda_t} = -r_t dt - \sigma_{\Lambda, t} \cdot dZ_t$$

$$r_t = -f_v \{ C_t, V_t \} - E_t \left[ \frac{df_c \{ C_t, V_t \}}{f_c \{ C_t, V_t \}} \right]$$

$$\sigma_{\Lambda, t} = -\mathcal{L} \left[ \frac{df_c \{ C_t, V_t \}}{f_c \{ C_t, V_t \}} \right]$$

where $\mathcal{L} [\cdot]$ denotes the loading on $dZ_t$

$$f \{ C_t, V_t \} = \tilde{V}_t^{-\gamma} \tilde{f} \left\{ C_t, \tilde{V}_t \right\}$$

$$\tilde{V}_t = ((1 - \gamma)V_t)^{\frac{1}{1-\gamma}}$$

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\[ f_v = \tilde{f}_v - \gamma \tilde{f} \]
\[ f_c = \tilde{V}_t - \gamma \tilde{f}_c \]
\[ \frac{df_c}{f_c} = \frac{df_c}{f_c} - \gamma \frac{d\tilde{V}_t}{\tilde{V}_t} - \gamma \left( \frac{df_c}{f_c} \right) \left( \frac{d\tilde{V}_t}{\tilde{V}_t} \right) \]
\[ \Rightarrow r_t = -\left( \tilde{f}_v - \gamma \tilde{f} \right) - E_t \left[ \frac{df_c}{f_c} - \gamma \frac{d\tilde{V}_t}{\tilde{V}_t} - \gamma \left( \frac{df_c}{f_c} \right) \left( \frac{d\tilde{V}_t}{\tilde{V}_t} \right) \right] \]
\[ \Rightarrow \sigma_{\lambda,t} = -L \left[ \frac{df_c}{f_c} \left\{ C_t, \tilde{V}_t \right\} - \gamma \frac{d\tilde{V}_t}{\tilde{V}_t} \right] \]

Now, when the parameter which governs the risk aversion is stochastic \((\gamma = \gamma_t)\) we can guess and verify that the process for the SPD is

\[ \frac{d\lambda_t}{\lambda_t} = -r_t dt - \sigma_{\lambda,t} \cdot dZ_t \]
\[ r_t = -\left( \tilde{f}_v - \gamma_t \tilde{f} \right) - E_t \left[ \frac{df_c}{f_c} - \gamma_t \frac{d\tilde{V}_t}{\tilde{V}_t} - \gamma_t \left( \frac{df_c}{f_c} \right) \left( \frac{d\tilde{V}_t}{\tilde{V}_t} \right) \right] \]
\[ \sigma_{\lambda,t} = -L \left[ \frac{df_c}{f_c} \left\{ C_t, \tilde{V}_t \right\} - \gamma_t \frac{d\tilde{V}_t}{\tilde{V}_t} \right] \]

The following verification proof below is adapted from Kozak (2012). Recall that the agent’s is

\[ dW_{j,t} = \left( W_{j,t} r_t + \omega_{j,t} \cdot \tilde{W}_{j,t} (\mu_t - 1) - C_{j,t} - I_{j,t}^{H} + \omega_t H_{j,t} \right) dt + \tilde{W}_{j,t} \omega_{j,t} \delta_t dZ_t \]
\[ \frac{dH_{j,t}}{H_{j,t}} = \Gamma ln \left( 1 + \frac{I_{j,t}^{H}}{\theta H_{j,t}} \right) dt - \delta dt + \sigma dZ^\lambda_t \]
where \( \varsigma_t = \begin{pmatrix} \varsigma_t^A \\ \varsigma_t^B' \end{pmatrix} \) is a \( 2 \times 2 \) matrix. The corresponding HJB is

\[
0 = \max_{C_{j,t},I_{j,t},H_{j,t},\varpi_{j,t}} \left( \dot{f} + \frac{1}{2} \tilde{V}^2 \tilde{A}_v \sigma_{v,t} \cdot \sigma_{v,t} \right) dt
\]

\[+ \tilde{V}_X \cdot dX + \frac{1}{2} dX (V_{XX}) dX + \tilde{V}_w dW + \frac{1}{2} \tilde{V}_{ww} dW^2 + (\tilde{V}_{Xw} \cdot dX) dW \]

where

\[
X = \{x^A, x^B, \gamma\} \quad \text{and} \quad dX = \mu_X dt + \sigma_X \cdot dZ
\]

\[
(\tilde{V}_{Xw} \cdot dX) = \tilde{V}_{x^A} dx^A + \tilde{V}_{x^B} dx^B + \tilde{V}_{\gamma} d\gamma
\]

\[
\sigma_{v,t} = L \left[ \frac{d\tilde{V}}{V} \right] = \tilde{W}_{j,t} \frac{\tilde{V}_w}{V} \varpi_{j,t,s} \sigma_{v,t} + \frac{\tilde{V}_w \cdot \sigma_X}{V}
\]

The FOCs for \( C_{j,t} \) and \( \varpi_{j,t} \) are

\[
\dot{f}_c = \tilde{V}_w
\]

\[
\tilde{V}^2 \tilde{A}_v \tilde{W}_{j,t} \frac{\tilde{V}_w}{V} \varpi_{j,t,s} \sigma_{v,t} + \tilde{W}_{j,t} \tilde{V}_w (\mu_t - 1 r_t) + \tilde{W}_{j,t} \tilde{V}_{ww} \varpi_{j,t,s}^2 + \tilde{W}_{j,t} \left( \tilde{V}_{Xw} \cdot \sigma_X \right) \varsigma_t = 0
\]

This the implies

\[
\sigma_{\Lambda,t} = -L \left[ \frac{d\tilde{V}_w}{\tilde{V}_w} \right] + \gamma_t \sigma_{v,t}
\]

\[
\sigma_{\Lambda,t}' = -\tilde{W}_{j,t} \frac{\tilde{V}_{ww}}{\tilde{V}_w} \varpi_{j,t,s}^2 - \frac{\left( \tilde{V}_{Xw} \cdot \sigma_X \right)'}{\tilde{V}_w} + \gamma_t \sigma_{v,t}'
\]

\[
\tilde{W}_{j,t} \sigma_{\Lambda,t,s} = -\tilde{W}_{j,t}^2 \frac{\tilde{V}_{ww}}{\tilde{V}_w} \varpi_{j,t,s}^2 - \tilde{W}_{j,t} \frac{\left( \tilde{V}_{Xw} \cdot \sigma_X \right)'}{\tilde{V}_w} + \tilde{W}_{j,t} \gamma_t \sigma_{v,t,s}
\]

\[
\Rightarrow \sigma_{\Lambda,t,s} = (\mu_t - 1 r_t)
\]

(7.12)

Now, the gains processes adjusted by the SPD is Martingales, therefore

\[
E_t \left( \Lambda_T S_T + \int_0^T \Lambda_t D_t dt \right) = \Lambda_t S_t + \int_0^t \Lambda_t D_t dt
\]
\[ 0 = E_t \left( d \left( \Lambda_t S_t + \int_0^t \Lambda_t D_t dt \right) \right) \]

\[ \mu_t = 1_r_t + \sigma'_{\Lambda,t} S_t + \frac{D_t}{S_t} - \frac{D_t}{S_t} \]

\[ \Rightarrow \sigma'_{\Lambda,t} S_t = (\mu_t - 1_r_t) \quad (7.13) \]

Since equations (7.12) and (7.13) are the same, we have confirmed that the guess for \( \frac{d\Lambda_t}{\Lambda_t} \) when \( \gamma_t \) is stochastic is indeed correct.

### 7.20.2 Boundary conditions

Following Kozak (2012), we can guess and verify that the value function takes the form

\[ \tilde{V} \{ H + K^A + K^B, x^A, x^B, \gamma \} = (H + K^A + K^B) F \{ x^A, x^B, \gamma \} \]

Solving the model requires solving a system of five PDE’s (three FOC’s for investment, the HJB and the resource constraint) with three boundary conditions. I obtain the boundary conditions by solving the model with \( x^A = 1 \), \( x^B = 1 \), and \( 1 - x^A - x^B = 1 \). These boundary problems reduce to solving second order ODE’s for \( F \{ 1, 0, \gamma \} = F \{ 0, 1, \gamma \} = F \{ 0, 0, \gamma \} = F \{ \gamma \} \). I solve these second order ODE’s with the following conditions: \( F \{ \gamma = 0 \} = 1 \) (since with risk-neutral preferences the value function is linear in total capital) and \( F \{ \gamma = \infty \} = 0 \) (since the value function is zero for an infinitely risk averse agent). Solving this extension of the model requires a lot of computing power.

### 7.21 Solving PDEs

First, following Chap 6 of Judd (1998) I re-write the state variables as \( \{ \tilde{x}^A, \tilde{x}^B \} \), functions of Chebyshev nodes. I then approximate \( F \{ x^A, x^B \} \) with a complete Chebyshev polynomial, \( \tilde{F} \{ \tilde{x}^A, \tilde{x}^B \} \). I define the residual function, \( R \), as the PDE where I plug in the approximation \( \tilde{F} \) and the as well as the nodes \( \{ \tilde{x}^A, \tilde{x}^B \} \). Using the collocation approach, the vector of polynomial coefficients \( \alpha \), is chosen to solve \( R \{ \alpha \} = 0 \) on the grid \( \{ \tilde{x}^A, \tilde{x}^B \} \). I first choose the size of the \( n \times n \) grid. Then I start with low order polynomials and solve for \( \alpha \). I then use an interpolation method to obtain the solution for \( F \{ x^A, x^B \} \) over the continuous state space \( \{ x^A \in [0, 1], x^B \in [0, 1 - x^A] \} \). I plug this function, \( F \{ x^A, x^B \} \), back into the PDE and examine the size of the PDE errors over the continuous state space \( \{ x^A \in [0, 1], x^B \in [0, 1 - x^A] \} \). I steadily increase the degree of the polynomial and

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repeat the procedure until the PDE errors are minimized. I use the AMPL modeling language to write the problem and I solve it using SNOPT on the NEOS server.