Frege versus Cantor and Dedekind: On the Concept of Number

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There can be no doubt about the value of Frege's contributions to the philosophy of mathematics. First, he invented quantification theory and this was the first step toward making precise the notion of a purely logical deduction. Secondly, he was the first to publish a logical analysis of the ancestral R* of a relation R, which yields a definition of R* in second-order logic.[1] Only a narrow and arid conception of philosophy would exclude these two achievements. Thirdly and very importantly, the discussion in §§58-60 of the *Grundlagen* defends a conception of mathematical existence, to be found in Cantor (1883) and later in the writings of Dedekind and Hilbert, by basing it upon considerations about meaning which have *general* application, outside mathematics.[2]

Michael Dummett, in his book [Dummett (1991)][3] on Frege's philosophy of mathematics, is rather stronger in his evaluation. He writes "For all his mistakes and omissions, he was the greatest philosopher of mathematics yet to have written" (P. 321). I think that one has to have a rather circumscribed view of what constitutes philosophy to subscribe to such a statement - or indeed to any

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[1]Frege (1879). Dedekind (1887) similarly analyzed the ancestral F* in the case of a one-to-one function F from a set into a proper subset. In the preface to the first edition, Dedekind stated that, in the years 1872-78, he had written a first draft, containing all the essential ideas of his monograph.

[2]However, it was only in the hands of Wittgenstein, in *Philosophical Investigations*, that this critique of meaning was fully and convincingly elaborated.

[3]All references to Dummett will be to this work, unless otherwise specified.

ranking in philosophy of mathematics. If I had to choose, I would perhaps rank Plato first, on grounds of priority, since he was first, as far as we know, to conceive of the idea of a priori science, that is science based on primitive truths from which we reason purely deductively. But, if Plato seems too remote, then Frege still has some strong competitors even in the nineteenth century, for example Bolzano, Riemann, Weierstrass and, especially, Cantor and Dedekind. Contributing to Dummett's assessment is, I think, a tendency to make a sharp distinction between what is philosophical and what is technical and outside the domain of philosophy, a sharper distinction between philosophy and science than is historically justified or reasonable. Thus we read that Frege had answers (although not always the right ones) "to all the philosophical problems concerning the branches of mathematics with which he dealt. He had an account to offer of the applications of arithmetic; of the status of its objects; of the kind of necessity attaching to arithmetic truths; and of how to reconcile their a priori character with our attainment of new knowledge about arithmetic." (p. 292) The question of existence of mathematical objects, their 'status', certainly needed clarification; but, otherwise, are these the most important philosophical problems associated with the branches of mathematics with which he dealt? Surely the most important philosophical problem of Frege's time and ours, and one certainly connected with the investigation of the concept of number, is the clarification of the infinite, initiated by Bolzano and Cantor and seriously misunderstood by Frege. Likewise, the important distinction between cardinal and ordinal numbers, introduced by Cantor, and (especially in connection with the question of mathematical existence) the characterization of the system of finite numbers to within isomorphism as a simply infinite system, introduced by Dedekind, are of central importance in the philosophy of mathematics. Also, the issue of constructive versus non-constructive reasoning in mathematics, which Frege nowhere discussed, was very much alive by 1884, when he published his *Grundlagen.* Finally, although Frege took up the problem of the analysis of the continuum, his treatment of it appeared about thirty years after the work of Weierstrass, Cantor, Dedekind, Heine and Meray (the latter four in 1872) and, besides, was incomplete. What it lacked was, essentially, just what the earlier works supplied, a construction (at least up to isomorphism) of the complete ordered additive group of real numbers. Whether Frege had, as he thought, something to add to that construction in the definition of the real numbers is a question on which I shall briefly comment in §VII, where I discuss the analogous question of Frege's versus Dedekind's treatment of finite cardinal numbers. The issue here concerns the matter of applications. It is true that Frege offered an account of application of the natural numbers and the real numbers and that this account structured his treatment of the real numbers and possibly, as Dummett suggests, his treatment of the natural numbers. But there is some question as to whether his account of application should enhance his stature as a philosopher.

However, more important to me in this paper than the question of Frege's own importance in philosophy is the tendency in the literature on philosophy to contrast the superior clarity of thought and powers of conceptual analysis that Frege brought to bear on the foundations of arithmetic, especially in the *Grundlagen*, with the conceptual confusion of his predecessors and contemporaries on this topic. Thus, in Dummett (1991), p.292: "In Frege's writings, by contrast [to those of Brouwer and Hilbert], everything is lucid and explicit: when there are mistakes, they are set out clearly for all to recognize." Aside from the contrast with Brouwer, I don't believe that this evaluation survives close examination. Frege's discussions of other writers are often characterized less by clarity than by misinterpretation and lack of charity, and, on many matters, both of criticism of other scholars and of substance, his analysis is defective. Dummett agrees with part of this assessment in so far as Volume II of the *Grundgesetze* (1903) is concerned. He writes

> The critical sections of *Grundlagen* follow one another in a logical sequence; each is devoted to a question concerning arithmetic and the natural numbers, and other writers are cited only when either some view they express or the refutation of their errors contributes positively to answering the question. In Part III.1 of *Grundgesetze*, the sections follow no logical sequence. Each after the first ... is devoted to a particular rival mathematician or group of mathematicians .... From their content, the reader cannot but think that Frege is anxious to direct at his competitors any criticism to which they lay themselves open, regardless of whether it advances his argument or not. He acknowledges no merit in the work of those he criticizes; nor, with the exception only of Newton and Gauss, is anyone quoted with approbation. The Frege who wrote Volume II of *Grundgesetze* was a very different man from the Frege who had written *Grundlagen*: an embittered man whose concern to give a convincing exposition of his theory of the foundations of analysis was repeatedly overpowered by his desire for revenge on those who had ignored or failed to understand his work. (pp. 242-43)

Concerning the relative coherence of the two works, Dummett is surely right. But I think that, in Frege's treatment of other scholars, we can very well recognize the later Frege in the earlier one.

Establishing this purely negative fact about Frege would be, by itself, very small potatoes. But unfortunately, his assessment of his contemporaries in *Grundlagen* and elsewhere lives on in much of the philosophical literature, where respected mathematicians, such as Heine, Lipschitz, Schröder and Thomae, are regarded as utterly muddled about the concept of number and great philosophers, such as Cantor and Dedekind, are treated as philosophical naifs, however creative, whose work provides, at best, fodder for philosophical chewing. Not only have we inherited from Frege a poor regard for his contemporaries, but, taking the critical parts of his *Grundlagen* as a model, we in the Anglo-American tradition of analytic philosophy have inherited a poor vision of what philosophy is.

I

The conception of sets and of ordinal and cardinal numbers for which Cantor is perhaps best known first appeared in print in 1888 and represents a significant and, to my mind, unfortunate change in his position. He first introduced the concept of two arbitrary sets, finite or infinite, having the same power in Cantor (1878). In Cantor (1874) he had already in effect shown that there are at least two infinite powers (although he had not yet defined the general notion of equipollence). Prior to 1883, all of the sets that he had been considering were subsets of finite-dimensional Euclidean spaces, all of which he had shown to have the power of the continuum. New sets, the number classes, with successively higher powers, were introduced in Cantor (1883). So here, for the first time, he obtained sets which might have powers greater than that of the continuum. In this connection, it should be noted that, although he defined the concept of a well-ordered set and noted that the ordinal numbers corresponded to the order types of well-ordered sets,[4] the ordinal numbers themselves were defined *autonomously* and not *as* the order types of well-ordered sets.[5] Indeed, in general, the only well-ordered set of order type a available to him was the set of predecessors of a. In discussing what had been gained by his construction of the ordinals, the application to well-ordered sets is mentioned only *second*, after the founding of

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[4]Cantor 1932, p.168.

[5]Frege obviously appreciated this point. In (1884) §86 he wrote "I find special reason to wellcome in Cantor's investigations an extension of the frontiers of science, because they have led to the construction of a purely arithmetical route to higher transfinite numbers (powers)."

the theory of powers. For Cantor, at this time, the construction of the number classes was essential to the theory of powers. In speaking of their significance, he writes

> Our aforementioned number classes of determinately infinite real whole numbers [i.e. the ordinals] now show themselves to be the natural uniform representatives of the lawful sequence of ascending powers of well-defined sets. [Cantor 1932), p. 167]

Just prior to this he wrote that "Every well-defined set has a determinate power", so his view at that time was that every infinite well-defined set is equipollent to a number class.[6] In particular, he notes in (1883) that neither the totality of all ordinals nor the totality of all cardinals has a power. It follows then that neither is a well-defined set.

It was in "Mitteilungen zur Lehre vom Transfiniten" (1887-88) and, later, in "Beiträge zur Begründung der transfitiniten Mengenlehre" (1895-97) that Cantor introduced the much-criticized abstractionist conception of the cardinals and ordinals. To quote from the "Beiträge":

> *By the "power" or "cardinal number" of M we mean the general concept, which arises with the help of our active faculty of thought from the set M, in that we abstract from the nature of the particular elements of M and from the order in which they are presented. ...* Since every single element m [of M], if we abstract from its nature, becomes a 'unit', the cardinal number ... [of M] is a definite aggregate composed of units, and this number has existence in our mind as an

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[6] At the beginning of §3 of (1883), Cantor explicitly states as a 'law of thought' that every set can be well-ordered. His assertion that the powers form an absolute infinity seems to imply that the construction of the number classes is to be continued beyond the finite number classes. He isn't explicit about how one proceeds to construct the ath-number class for limit ordinal a, but presumably, if its power is to be the next highest after those of all the b number classes for $b<a$, we should take it to be the union of the number classes of smaller index. But *when* should we introduce the ath number class for limit a? If we require that a be already obtained in some earlier number class, then the only ordinals Cantor's scheme yields are those less than the least fixed point $a=A_a$. But if, as seems perfectly consistent with Cantor's ideology, we require only that the cofinality of a be obtained in some earlier number class, then the ordinals that would be obtained are precisely those less than the least weakly inaccessible cardinal (i.e. the least *regular* fixed point of A). After his proof in Cantor (1891-92) that the power of a set is strictly less than that of its power set, possibly higher powers are obtained. The analogous hierarchy of powers leads to the least strongly inaccessible cardinal. Not until Zermelo (1930) does it seem that anyone pursued this 'constructive' approach to set theory, to obtain ordinals beyond the least strongly inaccessible cardinal.

> intellectual image or projection of the given aggregate M. [Cantor (1932), p. 282-83.]

In the analogous way, he introduced the *order type* of a linearly ordered set M:

> By this we understand *the general concept which arises from M when we abstract only from the nature of the elements of M, retaining the order of precedence among them.* ... Thus, the order type ... is *itself an ordered set* whose elements are pure units ... . (p. 297)

In particular, ordinal numbers are identified in the "Mitteilungen" and the "Beiträge" with the order types of well-ordered sets. It would be interesting to conjecture about the reasons for the change from the point of view of (1883) to that of (1887-88); but I shall not go into that here, other than to register my regret.[7] Husserl (1890), who notes the change with approval, defends the later definition and, as does Cantor, argues that the essential principle that, if two sets are equipollent, then they have the same cardinal, is derivable from it. (Two sets are *equipollent* or, as Cantor expressed it, *equivalent* if they are in one-to-one correspondence.) But the argument is not entirely clear: Why should abstraction from two equipollent sets lead to the *same* set of 'pure units'? And the conception of the cardinal number as a set plays no other role in Cantor's theory. We shall discuss this further in §VIII.

II

In §71 of "Was sind und was sollen die Zahlen?" (1888), Dedekind defines the notion of a 'simply infinite system' M with respect to a one-to-one function f:M@M. Namely, there is an element e of M which is not in the range of f and M is the least set containing e and closed under f. f is said to *order* M and e is called the *base element* of M (with respect to f). He then goes on to introduce the simply infinite system of natural numbers:

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[7]The assimilation of the theory of ordinals to the more general theory of order types may be part of the explanation. The discovery that a set is strictly less in power than its power set does not seem to be part of the explanation, since the "Mitteilungen" preceeds that discovery.

> 73. Definition. If, in considering a simply infinite system [M] ordered by the mapping f, we completely disregard the particular nature of the elements, retaining only their distinguishability and considering only those relationships in which they are placed to one another by the ordering map f, then these elements are called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base element [e] is called the *base element* of the *number series* [M]. In consideration of this freeing of the elements from every other content (abstraction) one can with justice call the numbers a free creation of the human intellect (menschlichen Geistes).

When M is a simply infinite system with respect to the map f and base element e, let us call the triple **M**=«M,f,e» a *simply infinite set*. We may denote the system of numbers by **N**=«N,&,1». Dedekind goes on to cite his proof in §134 that all simply infinite sets are isomorphic, in order to show that arithmetic depends only on the axioms of the (second order) theory of simply infinite sets and not on the choice of any particular such system.[8] The step from **M** to **N** is an instance of Cantor's abstraction in the case of ordered sets.

III

The response in the literature on philosophy of mathematics to Cantor's and Dedekind's abstractionist treatment of numbers has generally been negative.

An early direct attack on Cantor is contained in Frege (1891), a review of Cantor (1890) (which includes the "Mitteilungen".)[9] In what must count as one of the more impertinent passages in the history of philosophy, he writes

> If Mr. Cantor had not only reviewed my *Grundlagen der Arithmetik* but also read it thoughtfully, he would have avoided many mistakes. I believe that I have done there already a long time ago what he is here trying in vain to do. Mr. Cantor repeats (p. 13) a definition he had given in his review of my book as his own intellectual property. It seemed to me at the time that it differed from mine, not in its essentials, but only in its wording ... I now see that the truths I enunciated in my book were not, after all, like coins dropped in the

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[8]The axioms in question are

:x(e≠f(x))
:xy(f(x)=f(y)@x=y)
:Z[eMZ◊:x(xMZ@f(x)MZ)@:x(xMZ)].

[9]A more virulent attack, which Frege chose not to publish, is contained in a partial draft of that review [Frege (1979), p. 68-71].

> street which anybody could make his own simply by bending down. For Mr. Cantor goes on to give some other definitions (pp. 23 and 56) which show that he is still firmly ensconced in an antiquated position. He is asking for impossible abstractions and it is unclear to him what is to be understood by a 'set', even though he has an inkling of the correct answer, which comes out faintly when he says (p. 67 n.): 'A set is already completely delimited by the fact that everything that belongs to it is determined in itself and well distinguished from everything that does not belong to it.' This delimitation is, of course, achieved by characteristic marks and is nothing other than the definition of a concept. On this point compare my proposition (*Grundlagen*, §46): '... the content of a statement of number is an assertion about a concept'. (1984, p. 179)

And shortly after:

> ... we once again encounter those unfortunate ones which are different even though there is nothing to distinguish them from one another. The author evidently did not have the slightest inkling of the presence of this difficulty, which I dealt with at length in §§34 to 54 of my *Grundlagen*.

The page references are to Cantor (1890). But the 'other definitions (pp. 23 and 56)' are the "Mitteilungen" versions of the above quoted definitions of power and order type, in Cantor (1932), p. 387 and p. 422, respectively. The definition given 'as his own intellectual property' in Cantor's review (1885) of Frege (1884) and which he 'repeats' is clearly the definition of the cardinal number or power of an aggregate (inbegriff) or set as 'that general concept under which all and only those sets fall which are equivalent to the given set.' [Cantor (1932), p. 380]. Frege's own definition is that the number $N_xF(x)$ of F's, where F is a concept, is the extension of the second level concept 'is equipollent to F'. Frege had thought that it differs only in wording from his own, because he thought that it is inessential whether one speaks of a general concept here or of its extension and because he thought, incorrectly (see §XII below), that Cantor's notion of a set could be understood to mean 'extension of a concept' in his sense.[10]

In (1883), §1, Cantor writes "Every well-defined set M has a power, such that two sets have the same power when they [are equipollent]" The passage from this, in response to the question "What *is* the cardinal of M?", to the definition of the cardinal of M as the general concept under which fall precisely those sets equipollent to M, surely owes nothing to Frege. In defining the cardinal number

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[10]See the footnote at the end of §68 of Frege (1884) and his reply in Frege (1984), p. 120, to Cantor's review [Cantor (1932), pp. 440-41]

of M as a 'general concept', it seems clear that Cantor did not have in mind Frege's technical notion of a concept; rather, he was following the traditional view according to which, for example, the numeral '10' is a common name, under which falls all ten-element sets. (For example, see Aristotle's *Physics*, 224a3-16.) Moreover, as we shall see, Cantor had already pointed out, implicitly in his (1883) and explicitly in his review of Frege's (1884), that it is a mistake to take the notion of a set (i.e. of that which has a cardinal number) to simply mean the extension of a concept. Finally, the very definition of equinumerosity in terms of equipollence, applied to sets *in general*, is due to no one but Cantor - not to Hume, Kossak or Schröder, whom Frege cites but who were concerned entirely with finite sets. See §X below. As for Frege, who did intend the definition to apply also to the infinite, he was not only anticipated by Cantor by seven years, but even after that time and, presumably, after having read Cantor's (1883) (to which he refers in his (1884)), he was unaware of the difficulties involved in treating the infinite. See §XII below.

But Frege is right that Cantor crucially modifies his definition of the power of a set M when he goes on to define it, not merely as a concept under which fall all sets equipollent to M, but also as an equipollent set of pure units: the concept becomes a paradigm instance of itself. Frege's own conception of abstraction (although he disapproves of the term) is, as we shall see, in agreement with the view that abstracting from the particular nature of the elements of M would yield the concept under which fall all sets equipollent to M. His target was the idea that abstraction leads to the paradigm set of pure units. We have already noted a difficulty with this idea, at least with respect to the work to which Cantor wanted to put it. But Frege's own arguments against Cantor's conception, that he cites from §§34-54 of his (1884), are invalid. See §VIII below.

A very influential attack on Dedekind's theory occurs in Russell's *Principles of Mathematics* (1903):

> Moreover, it is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colours from sounds. (p. 249)

Russell's point is often expressed by saying that it is impossible that objects should 'have only structural properties'. The confusion that lies behind this objection is discussed in §VII. His criticism is echoed in Dummett (1991) In Chapter 5, comparing Frege's and

Dedekind's treatment of the foundations of arithmetic, Dummett writes

> One of the mental operations most frequently credited with creative powers was that of abstracting from particular features of some object or system of objects, that is, ceasing to take any account of them. It was virtually an orthodoxy, subscribed to by many philosophers and mathematicians, including Husserl and Cantor, that the mind could, by this means, create an object or system of objects lacking the features abstracted from, but not possessing any others in their place. It was to this operation that Dedekind appealed in order to explain what the natural numbers are. His procedure differed from the usual one. Husserl ... supposed that each individual cardinal number was created by a special act of abstraction: starting with any arbitrary set having that number of elements, we abstract from all properties possessed by the individual members of the set, thus transforming them into featureless units; the set comprising these units was then the relevant cardinal number. Cantor's variation on this account was a trifle more complex: we start with an ordered set, and abstract from all the features of the individual members, but not from their ordering, and thus obtain their [sic] order type; next, we abstract from the ordering relation, and obtain the cardinal number as an unordered set of featureless units, as before. Frege devoted a lengthy section of *Grundlagen* , §§29-44, to a detailed and conclusive critique of this misbegotten theory; it was a bitter disappointment to him that it had not the slightest effect. (p.50)

In questioning that "the mind could, by this means, create an object or system of objects lacking the features abstracted from, but not possessing any others in their place", Dummett seems to be merging Russell's criticism with another: Is it abstract objects to which we should object or is it their creation by the mind? The same double-barreled objection arises immediately after when, discussing Russell's reaction to (and misunderstanding of) Dedekind, Dummett writes that Dedekind "believed that the magical operation of abstraction can provide us with specific objects having only structural properties: Russell did not understand that belief because, very rightly, he had no faith in abstraction thus understood." (p. 52) Dummett is taking 'abstraction' here to be a psychological term.

> Dedekind's philosophy of mathematics was that mathematical objects are 'free creations of the human mind', as he says in the Preface. The idea, widely shared by his contemporaries, was that abstract objects are actually created by operations of our minds. This would seem to lead to a solipsistic conception of mathematics; but it is implicit in this conception that each subject is entitled to feel assured that what he creates by means of his own mental operations

> will coincide, at least in its properties, with what others have created by means of analogous operations. For Frege, such an assurance would be without foundation: for him, the contents of our minds are wholly subjective; since there is no means of comparing them, I cannot know whether my idea is the same as yours. (p. 49)

So Dummett's believes that Cantor's and Dedekind's operation of abstraction is psychologistic.

Frege himself does not criticize Dedekind's treatment of the number concept on grounds of psychologism when he discusses it in the introduction to the *Grundgesetze I* (1893). And, in his review of Cantor (1890), he makes the criticism rather mildly when he writes: "Besides, the verb 'abstract' is a psychological expression and, as such, ought to be avoided in mathematics." (p. 181) But this is by no means his principal attack on Cantor's conception of cardinal and ordinal numbers. *His* reference to his refutation of what Dummett calls the 'misbegotten theory' of Cantor is not to §§29-44 in his (1884), but to §§34-54, in which psychologism is not the issue.

IV

Actually, Dedekind did not say in the Preface that mathematical objects are free creations of the human mind. He did say this of the natural numbers and there is little doubt that he would have said it also of the real numbers; but it is too hasty to reduce his 'philosophy of mathematics' to a psychologistic reading of this metaphor. Indeed, this tendency to attack forms of expression rather than attempting to appreciate what is actually being said is one of the more unfortunate habits that analytic philosophy inherited from Frege. If one reads §73, quoted above, the metaphor of 'free creation' is justified by the fact that we arrive at the system of numbers by abstraction, by freeing the elements from every other content. Therefore, it is reasonable to conclude that Dedekind's conception is psychologistic only if that is the only way to understand the abstraction that is involved. And we shall see that it is not.

The difficulty with abstraction as a psychological operation would be that what is abstracted is mental, that what I abstract is mine and what you abstract is yours. (See Frege (1884), §§26-27.) We are no more communicating when I say "0<1", meaning that my 0 is (my) less than my 1, and you say, "No, 1<0", meaning that your 1 is (your) less than your 0, than when I say "I am shorter than

Jones" and you say, "No, I am taller than Jones". Of course, this does not mean that you and I cannot argue objectively about your, my or some third party's mental states as an empirical question. But when we discuss the nature of arithmetic truth, that is not what is going on: on whatever grounds I might be seduced into thinking that I am expressing something about *my* mental states when I assert that 0<1, you would, on the same grounds, be seduced into thinking that you are expressing something about *your* mental states when you assert that 1<0. Frege's point is that the objectivity of mathematics demands that we both resist this seduction.

So Frege's argument against psychologism in the context of abstraction is not that the source of judgement about the abstracted objects is not in some sense to be found in the common human psyche, but rather that the objects abstracted should not be found in the individual psyche.[11] For example, he himself ultimately traces the source of logical principles to our logical disposition. [Frege (1979), p. 269.] So the objectivity of logic rests, for him, upon the fact that we are disposed to agree in logical judgement.[12] The same point is illustrated by his defense of Kant's conception of geometry as objective in (1884), §26.

> Space, according to Kant, belongs to appearance. For other rational beings it might take some form quite different from that in which we know it. Indeed, we cannot even know whether it appears the same to one man as to another; for we cannot, in order to compare them, lay one man's intuition of space beside another's. Nevertheless, there is something objective in space all the same; everyone recognizes the same geometrical axioms, even if only by his behavior, and must do so if he is to find his way about in the world. What is objective in it is what is subject to laws, what can be conceived and judged, what is expressible in words. What is purely intuitable is not communicable.

The abstractionism of neither Cantor nor Dedekind is subject to the criticism that it is psychologistic: For neither of them are numbers psychological objects nor are the laws of number to be understood in any way as subjective.

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[11]There is of course another side to psychologism, which Frege discusses and opposes in (1884), p. x and §60. There the issue is the confusion of the logical with the psychological, e.g. of the senses of words or sentences with our (pschological ideas). But it would seem that Dummett's charge against Dedekind is not this, but that he is takeing the *reference* of number words to be ideas.

[12]It should be noted that Frege does not observe, as Wittgenstein later did in the *Investigations*, that objectivity requires not only agreement in judgement concerning the laws of logic, but also agreement drawing consequences, i.e. in moving from premises to conclusion (computing according to a rule).

V

Concerning the notion of abstraction, Frege writes

> For suppose that we do, as Thomae demands, "abstract from the peculiarities of the individual members of a set of things", or "disregard, in considering separate things, those characteristics which serve to distinguish them". In that event we are not left, as Lipschitz maintains, with "the concept of the number of the things considered"; what we get is rather a general concept under which the things considered fall. The things themselves do not in the process lose any of their special characteristics. For example, if I, in considering a white cat and a black cat, disregard the properties which serve to distinguish them, then I get presumably the concept "cat". [(1884), §34]

There is a serious misunderstanding of both Thomae and Lipschitz in this passage, which we shall take up later. But the point I want to make here is that Frege is not really correct about abstraction resulting always in concepts, at least not if he is referring to the traditional meaning of the term 'abstraction'. For example, although it is true that Aristotle regards attributes such as 'white' (the concept 'x is white', for Frege) to be obtained by abstraction, he also regards geometric objects such as lines and surfaces to be obtained by abstraction from sensible things (*Metaphysics* 1061a29).

But of course there is a sense in which no object is really created by this latter kind of abstraction. The geometric magnitude, e.g. the line segment or the plane figure, was for Aristotle just the sensible substance; but in geometry we regard it, not *qua* sensible object, but only with respect to those properties it has in virtue of its extension. On this view, truths about a geometric object are simply truths about the sensible object, but restricted to the language of geometry. Analogous to this is the so-called 'forgetful functor' by means of which we pass, say, from a ring «M,+,≈» to the corresponding group «M,+». But these cases are somewhat different from the case of the abstraction of the power $|A|$ from the set A. For if A is equipollent to the set B, then $|A|=|B|$. But this does not imply that A=B. Hence, we cannot regard $|A|$ as really being just A, taken in abstraction, unless 'taking in abstraction' has the power to identify distinct sets. Or rather, since Frege is certainly right that the process cannot literally identify distinct things, the process of abstraction in this case must be understood to create new objects.

Nevertheless, there is something common to Aristotle's conception of geometry, right or wrong, and the abstraction of cardinal numbers. Namely, sentences about the abstract objects have a canonical, truth-preserving, translation into sentences about the objects from which they are abstracted. In the case of cardinals, the objects from which the powers are abstracted are sets. The relation ? of equipollence is clearly an equivalence relation among sets and it respects the relation

$$X\tilde{}Y$$

of X being equipollent to a subset of Y as well as the operations of *sum* or *disjoint union*

$$S_{iMI}X_i$$

(i.e. the set $\{(i,a)|\ iMI\ \&\ aMX_i\}$) and *cartesian product*

$$P_{iMI}X_i$$

of a family «X| iMI» of sets. In view of this fact, the order relation ≤ among the cardinals and the arithmetical operations of addition and multiplication on cardinals may be defined by

$$|X| \leq |Y| \ldots X\tilde{}Y$$
$$S_i|X_i| = |S_iX_i|$$
$$P_i|X_i| = |P_iX_i|.$$

It follows that any proposition about the arithmetic and ordering of cardinal numbers translates into a proposition about sets, providing only that '=' and '≤' are translated as '?' and '˜', respectively, and 'S' and 'P' are interpreted as the corresponding operations on families of sets. The one difficulty with this translation is that we are passing from the cardinal |X| to the set X, and it could happen that X is itself a set containing cardinals or whose transitive closure contains cardinals;[13] and so the translation does not entirely eliminate reference to cardinals. But, assuming the well-ordering principle, we can always take the representative X of the cardinal |X|

[13]The transitive closure of a set X is the least transitive set which includes X. A set X is transitive iff YM X and bMY imply that bM X.

to be a pure set, i.e. one, such as the corresponding initial von Neumann ordinal, whose transitive closure contains only sets.

Of course I am not accurately presenting Cantor's abstractionist conception of the cardinals here, since for him they are sets of pure units. But I will leave this aside for the moment, noting only that it is entirely inessential to the reduction.

In any case, what seems to me to be essential to this kind of abstraction is this: the propositions about the abstract objects translate into propositions about the things from which they are abstracted and, in particular, the truth of the former is founded upon the truth of the latter. So the abstraction in question has a strong claim to the title *logical abstraction*: the sense of a proposition about the abstract domain is given in terms of the sense of the corresponding proposition about the (relatively) concrete domain.[14]

Dedekind's treatment of the finite ordinals is also a case of logical abstraction, providing that we assume given some simply infinite set **M**=«M,f,e», such as the system of finite von Neumann ordinals. We introduce the simply infinite set **N**=«N,',1» of finite ordinals by stipulating that **M**÷**N**. As we noted, the isomorphism is unique. In terms of this isomorphism, any arithmetical proposition, i.e. proposition about **N**, translates into a proposition about **M**. Moreover, because all simply infinite sets are isomorphic, the truth value of the arithmetical proposition does not depend upon the particular simply infinite set **M**.[15]

Of course, this treatment depends upon having a simply infinite set **M** to begin with, from which to abstract **N**. Dedekind in fact showed that it suffices to have a so-called Dedekind infinite set «S,f», i.e. a set S with a one-to-one function F from S into a proper subset of S. As Frege had essentially done previously, Dedekind notes that, if e is an element of S which is not a value of F and M is the intersection of all subsets of S which contain e and are closed under f, then «M,f,e» is a simply infinite set.[16]

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[14]Dummett (1991), pp. 167-68, uses the term 'logical abstraction' for the construction of the abstract objects as equivalence classes. But it is not clear why we should call this construction 'logical'.

[15]For further discussion of Dedekind's conception, see Parsons (1990) and Tait (1986) and (1993).

[16]Dedekind in fact 'constructs' such a Dedekind infinite set, where S is the domain of all objects of thought and f(x) is the thought of x. He argues that his ego is in S but is not a thought; and so «S,f» is a Dedekind infinite set. Frege begins, in effect, with the Dedekind infinite set «S,f» in which S is the totality of all cardinals and f(m)=n means that n is the cardinal of some concept F and,

One may ask: What is the point of logical abstraction? That is, instead of abstracting the simply infinite set of numbers from an already given simply infinite set **M**, why did Dedekind not simply take the system of numbers to *be* this latter system? Similarly, noting that every ordinal or cardinal is the order type or cardinal of a unique von Neumann ordinal, why not take this pure set to *be* the ordinal or cardinal? Dedekind discussed this question, not in connection with his monograph on the natural numbers, but in a letter to Weber about his earlier monograph, Dedekind (1872), on the irrational numbers.[17] He explains why he takes an irrational number to be *represented* by the corresponding Dedekind cut rather than defining it to *be* that cut. His argument is that to identify the real numbers with cuts - or with the objects in any other representation of them - is to endow them with properties which have nothing to do with them *qua* numbers but only to do with a particular and arbitrary representation of them.[18] Dummett seems to entirely misunderstand Dedekind's point here when he writes of the latter's refusal to identify the real numbers with the cuts: "Dedekind's resort to construction was not a means of avoiding labor. It was due solely to his philosophical orientation, according to which mathematical entities are to be displayed as creations of the human mind." (p. 250).

An objection often leveled at the abstractions of Cantor and Dedekind is that the abstractions do no work - they play no role in proofs. Of course Cantor intended his conception of cardinals as sets of pure units to do work, namely to yield the equivalence of the equinumerosity of two sets with their equipollence; but this is not convincing. But it would seem that logical abstraction, as it is described here, does play a role, not in proofs, but in that it fixes grammar, the domain of meaningful propositions, concerning the objects in question, and so determines the appropriate subject matter of proofs. For example, proving the categoricity of the axioms of simply ordered sets fixes the sense of all propositions in

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for some b in the extension of F, m is the cardinal of 'Fx & $x \neq b$'. (See Frege (1884), §76.)

[17]*Werke*, vol. 3, pp. 489-90. This letter was brought to my attention by Howard Stein. Cf. Stein (1988)

[18]Dedekind's argument is close to Benacerraf's in his paper "What numbers could not be": The latter argues that there is no one representation of the numbers by sets and so nothing intrinsic to the notion of number itself which decides the answer to the question "Is 0M 2?", for example. Hence *no* representation of the numbers as sets should be regarded as *defining* the numbers.

the pure theory of numbers; but it would not do so if numbers were sets, since the sense of 0M1 is not fixed.

In Dedekind's final judgement of the matter, it is not clear that his foundation of arithmetic (as opposed to the foundation of the theory of real numbers) should be regarded as abstractionist. In the well-known letter to Keferstein in 1890,[19] in which he explained the argument of "Was sind und was sollen die Zahlen?", Dedekind casts a somewhat different light upon his foundation of arithmetic and, in particular, his construction of the simply infinite set from the Dedekind infinite set of all objects of thought. He writes

> ...the question arose: does such a system *exist* at all in the realm of our ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such proofs. (von Heijenhoort, p. 101.)

What is stressed here is not the abstractionist reduction of arithmetic to something else, but rather the question of the *internal* consistency of arithmetic itself. In this respect, Dedekind was the precursor of Hilbert's view that mathematical existence is established when one has proved completeness and consistency.[20] In view of this, there is all the more reason to question Dummett's psychologistic reading of Dedekind - in particular when he writes

> For Dedekind, however, the process of creation involved the operation of psychological abstraction, which needed a non-abstract system from which to begin; so it was for him a necessity, for the foundation of the mathematical theory, that there be such systems. That is why he included in his foundation for arithmetic a proof of the existence of a simply infinite system, which had, of necessity, to be a non-mathematical one. (p. 296)

VI

There is another objection to Dedekind's foundation of arithmetic raised by Dummett: the characterization of **N** as a simple infinity does not tell us whether it is the system of numbers

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[19]Dedekind(1932), p. 490.

[20]Of course, when second order logic is involved, completeness and consistency are not purely formal notions, because of the incompleteness of formal higher-order logic. But categoricity and existence of a model suffice to establish these properties, both in Dedekind's case and in Hilbert's case of Euclidean geometry.

beginning with zero or beginning with one (or beginning with some other number)- in other words, it does not tell us whether 1 is really zero or one. But why hasn't Dedekind eliminated that ambiguity by *telling* us that 1 is the number one? Another way to state Dummett's objection is this: Dedekind proves (§126) the general principle of definition by primitive recursion according to which we may define unique functions F and G on N such that

$$F(m,1) = m \qquad\qquad G(m,1) = m'$$
$$F(m,n') = F(m,n)' \qquad\qquad G(m,n') = G(m,n)'.$$

The ambiguity in question then concerns whether it is F or G that is to be called 'addition', and Dedekind himself opts for the latter. Dummett suggests that we could eliminate the ambiguity by identifying the system of numbers with, say, the structure **N+=**«N,&,1,G» rather than with **N**. But that is surely inadequate *unless we then specify that G is to denote addition*! After all, no matter what number we take "1" to denote, the function G is well-defined on N. But, if we must specify that G is addition in **N+**, then we might as well stick with the structure **N** and specify that 1 is the number one. Dummett introduces this topic in the context, not of Dedekind's theory, but of Benacerraf's 'neo-Dedekindian' thesis "that structure is all that matters, since we can specify a mathematical object only in terms of in the structure to which it belongs". He objects to eliminating the ambiguity by passing from **N** to **N+**, not because it doesn't work, but because he believes that it betrays Benacerraf's thesis. **N** is already characterizable to within isomorphism as a simple infinity and G is definable in the structure **N**. Therefore it is contrary to Benacerraf's thesis to consider the structure **N+** instead of **N** as giving the structure of the numbers. So Benacerraf's thesis is false. (P. 53) But, having noted that **N+** is simply a definitional expansion of **N**, how can Dummett believe that substituting the former for the latter would determine which number 1 denotes?

But putting this aside, it is clear that, for Dummett, it is the fact that Dedekind's definition of the numbers does not 'intrinsically' determine for the number 1&&, say, whether it is the cardinal of two-element sets or the cardinal of three-element sets that is the defect in Dedekind's treatment of number. He writes

> Frege and Dedekind were at odds over two interconnected questions: whether or not the use of natural numbers to give the cardinality of finite totalities is one of their distinguishing

> characteristics, which ought therefore to figure in their definition; and whether it is possible, not merely to characterize the abstract structure of the system of natural numbers, but to identify the natural numbers solely in terms of that structure. Unlike Frege's, Dedekind's natural numbers have no properties other than their positions in the ordering determined by their generating operation, and those derivable from them; the question is whether such a conception is coherent. (p. 51)[21]

In consequence of the above alleged ambiguity in the meaning of "1", he writes of Benacerraf's thesis

> The thesis is false, and the example Benacerraf chose to illustrate it is the very one that most clearly illustrates its falsity. The identity of a mathematical object may sometimes be fixed by its relation to what lies outside the structure to which it belongs; what is constitutive of the number 3 is not its position in any progression whatever, or even in some particular progression, nor yet the result of adding 3 to another number, or of multiplying it by 3, but something more fundamental than any of these: the fact that, if certain objects are counted 'One two, three', or, equally, 'Naught, one, two', then there are three of them. The point is so simple that it needs a sophisticated intellect to overlook it; and it shows Frege to have been right, as against Dedekind, to have made the use of the natural numbers as finite cardinals intrinsic to their characterization. (p. 53)

There are several difficulties with Dummett's assessment in this connection. One concerns the dominance in his argument *here* of the role of numbers as cardinals: Why should we single out one kind of application of the natural numbers as being of their essence?[22] We have already noted that Dedekind focused on a different one; namely their role as ordinals or counting numbers. Thus, Dummett notes (p. 51) as a point of criticism of Dedekind's account that whereas he defines the addition of natural numbers by the recursion equations for G, i.e. as ordinal addition, Frege defines it as cardinal addition. But of course it is precisely what one would expect from someone who is analyzing the notion of finite *ordinal* or counting number that he would define addition as ordinal addition. What is surprising is that Dummett should find, in the case of finite numbers, that 'giving cardinality' is a more 'distinguishing

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[21]Of course Dummett does not mean that Frege and Dedekind were at odds over whether the structure of the system of numbers could be characterized: this was not a question that Frege even considered in (1884). In (1893), he formalizes (without citation) Dedekind's proof of categoricity (after having given a new proof of Dedekind's principle of definition by primitive recursion, again without citation) in his system. (Cf. Heck.(1993))

[22]Stein (1988) raises this question as a mild criticism even of the 'und was sollen" part of the title of Dedekind's monograph.

characteristic' than serving as counting number. He even writes about Frege that "[H]e assumed, as virtually everyone else at the time would have done, that the most general application of the natural numbers is to give the cardinality of finite sets." (p. 293), although he then goes on to point out that Cantor took the ordinal numbers to be primary: in his generalization of the cardinals and ordinals into the transfinite, it is the ordinals that he called 'numbers'. With an apparent reversal of judgement, he also suggests *here* that the notion of ordinal is the more fundamental one.[23] One must also put Dedekind on the side of the ordinal numbers. Kronecker also, in his "Über den Zahlbegriff" (1887), writes "I find the natural starting point for the development of the number concept in the *ordinal numbers*." But anyway, when we are speaking of applications, what about the role of the natural numbers in the foundation of analysis, e.g. in the foundation of the theory of rational and real numbers? Even if we attempt to go the route of Frege (1903) and construct the real numbers as ratios, the natural numbers must function as exterior multipliers on any system of magnitudes? That is, there must be the operation n°x defined for numbers n and quantities x by 1°x=x and n'°x=n°x+x (when n>0)[24]. As Frege himself noted in (1884), §19, in his criticism of Newton's definition of numbers in terms of ratios, the definition of the relation of 'having the same ratio' between pairs of like magnitudes presupposes the operation n°x.[25]

What makes one of these applications of the natural numbers privileged, so that it, rather than others, should be one of their 'distinguishing characteristics'?

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[23]One must question this judgement. For example, the proof in affine geometry that all lines contain the same number of points is not a counting argument.

[24]Actually, it suffices to define the notion that a pair (x,y) of magnitudes are equimultiples of the pair (a,b). We could do this, using Frege's (and Dedekind's) analysis of the ancestral F* of a function F, without introducing the natural numbers. Namely (x,y) is an equimultiple of (a,b) iff (a,b)F*(x,y), where F(c,d)=(c+a,d+b). But then we have essentially introduced the (positive) natural numbers; namely, they are the ratios x:a, where (x,x) is an equimultiple of (a,a).

[25]Dummett (1991), p. 73, writes that Frege "flounders somewhat, and fails to make the simple point as cleanly as he ought", the simple point being that Newton's definition is of the positive real numbers whereas Frege is concerned with the natural numbers. But, of course, the positive real numbers, with the operation +1, form a Dedekind infinity in terms of which a simple infinity may be constructed. So in fact it is essential to Frege's point that Eudoxos' definition already involves the numbers as multipliers.

VII

So far as the application of the numbers as cardinals is concerned, what is wrong with Dedekind's definition (§161) (also given in Kronecker's paper) of 'The set X has cardinal n' as meaning that X is equipollent with the set {1,...,n}? It should be noted that, if this analysis is acceptable, then a certain argument that Frege repeats over and over again fails. The argument appears in "On formal theories of arithmetic" [Frege (1984), pp.112-21] as an *a priori* argument for logicism:

> ... the basic propositions on which arithmetic is based cannot apply merely to a limited area whose peculiarities they express in the way in which the axioms of geometry express the peculiarities of what is spatial; rather, these basic propositions must extend to everything that can be thought. And surely we are justified in ascribing such extremely general propositions to logic. (p. 112.)

Frege's point appears to be endorsed in the foreword to the first edition of Dedekind (1887)

> In speaking of arithmetic (algebra, analysis) as part of logic I mean to imply that I consider the number concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought.

But there is a difference in their arguments. Dedekind's point, echoing Bolzano (who, in turn, quotes Aristotle)[26] , is in effect an expression of unwillingness to admit, in reasoning about numbers, any principles drawn from alien sciences. Frege's argument, on the other hand, is bound up with the idea that the definite description 'the number of x such that f(x)' should apply to any concept f(x) at all and not just to those concerning which we have some special source of knowledge. (We can count anything.) We find this argument repeated many times in Frege (1884). In §14, discussing Kant's conception that arithmetic is founded on intuition, Frege contrasts arithmetic and geometry:

> The fact that this is possible shows that the axioms of geometry are independent of one another and of the primitive laws of logic, and are consequently synthetic. Can the same be said of the fundamental propositions of the science of number? Here, we have only to try

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[26] Bolzano (1817), Preface. (p. 160 in the translation Russ (1980)); *Posterior Analytics* 75a39.

> denying any one of them, and complete confusion ensues. Even to think at all seems no longer possible. The basis of arithmetic lies deeper, it seems, than that of any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs no only the existent (Wirkliche) not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought?

In §19, in the discussion of Newton's definition of number as a ratio between like magnitudes, Frege again presents the argument for logicism from the universal applicability of number. We have already mentioned one objection that he raised to Newton's definition. But his second objection is this:

> [W]e should still remain in doubt as to how the number defined geometrically in this way is related to the number of ordinary life, which would then be entirely cut off from science. Yet surely we are entitled to demand of arithmetic that its numbers should be adopted for use in every application made of number, even although that application is not itself the business of arithmetic. Even in our everyday sums, we must be able to rely on the science of arithmetic to provide the basis for the methods we use. And moreover, the question arises whether arithmetic itself can make do with a geometric concept of number, when we think of some of the concepts in it, such as the number of roots of an equation or of the numbers prime to and smaller than a given number. On the other hand, the number which gives the answer to the question *how many?* can answer among other things how many units are in a length.

In §40, where he is discussing the problem of how one could understand cardinal numbers as sets of 'pure units', he considers the possibility of the units being points in space/time. He again writes

> The first doubt that strikes us about any such view is that then nothing would be numerable except what is spacial and temporal.

But, of course, on Dedekind's analysis, this line of argument is fallacious: *Whatever* simple infinity we take the system of numbers to be, the question of whether the extension of a concept is equipollent to {1,…,n} makes sense. In §42 Frege continues

> Another way out is to invoke instead of spatial or temporal order a more generalized concept of series, but this too fails in its object; for their position in the series cannot be the basis on which we distinguish the objects, since they must have already been

> distinguished somehow or other, for us to have been able to arrange them in a series.

But this is a strange idea. Why can't we have a series all of whose members are identical? Frege is confusing the notion of a series with that of a linearly ordered set $\langle A,<\rangle$ (where, for x and y in A, $x<y$ implies that x and y are distinct). Why does Frege preclude the series (a,a,...,a) ? His response is:

> When Hankel speaks of our thinking or putting a thing one or twice or three times, this too seems to be an attempt to combine in the things to be numbered distinguishability with identity. But it is obvious at once that it is not successful; for his ideas or intuitions of the same object must, if they are not to coalesce into one, be different in some way or other. Moreover we are, I imagine, fully entitled to speak of 45 million Germans without having first to have thought or put an average German 45 million times, which might be somewhat tedious.

But to regard the number n as an n-element sequence whose members are identical is not to say that the things numbered must be identical. Frege's second point surely betrays his confusion. To say that there are 45 million Germans is to say that there is a set of Germans which is equipollent to {1,...,45,000,000} - and, again, *this is quite independent of how the numbers are defined.*

Schröder's proposal to define the numbers as expressions, i.e. sequences of atomic symbols, is essentially the same as Hankel's proposal, except that Schröder decides against the expressions 11...1 because, for example, three, identified with 111, might be confused with one hundred and eleven. So he wishes to use the expressions 1+1+...+1 instead. This may be silly, but it is not open to Frege's objection in §43 that "This passage shows that for Schröder number is a *symbol*. What the symbol expresses, which is what I have been calling number, is taken, with the words 'how many of these units are present', as already known." Why shouldn't numbers be symbols? 111 is, in any case, a different symbol than 3. Why not take the latter to denote the former? In truth, there is an objection, namely the objection that numbers are numbers and symbols are symbols and that the grammar of the one is different from the grammar of the other. In essence, this is the objection of Dedekind to any reductionist account of the natural numbers or the real numbers. But it cannot be Frege's objection, since he will make numbers be extensions of concepts!

Presumably Dedekind's analysis of "The set X has n elements" won't do for Frege, as Dummett understands it, because it does not make "the use of natural numbers to give cardinality of finite sets ... one of their distinguishing characteristics". But we are owed a definition of 'distinguishing characteristic', as well as of 'intrinsic property'. Since Dedekind's treatment is being contrasted with Frege's in this connection, we should presumably look to Frege to find cases of distinguishing characteristics and intrinsic properties. The two cases with which this presents us are the cardinal numbers and the real numbers. In both instances, the numbers are defined as equivalence classes: A set has the cardinal number n iff it is an element of n and a pair of like magnitudes has the real number r iff it is an element of r. Of course, one problem in the case of cardinal numbers is that this definition is inconsistent for n>0, since there is no set consisting of all sets of power n. But let us restrict the equipollence relation to subsets of some infinite set A, so that n is an equipollence class of the subsets of A. To be sure, for sets X which are not subsets of A, the statement that X has cardinal n will no longer be equivalent to the assertion that it is an element of n; and so it will not be a 'defining characteristic' of n that X has cardinal n. But, even for subsets X of A, why does its being an element of n make the fact that it has cardinal n a defining characteristic or intrinsic property of n? The relation M of set membership (or 'being in the extension') is being given a distinguished role without any indication of why this should be so. For example, suppose that we instead define the cardinal numbers to be the equipollence classes of non-empty subsets of A and define 'X has cardinal n' to mean that Xv{X}Mn (assuming for this that A contains all of its finite subsets). Then {Ø} will no longer be a cardinal of any subset of A, the set of all unit subsets of A will be the cardinal of the null set and, in general, the cardinal of an n-element subset of A will be the equipollence class of all n+1-element subsets of A. Why is the relation XMn between X and n to be preferred to the relation Xv{X}Mn?

The same considerations apply to Frege's rejection of earlier treatments of the real numbers on the grounds that they do not adequately account for applications. Given a (linearly ordered) system of magnitudes , any ratio a:b of magnitudes from that domain is equal to a ratio r:1 in any connected ordered field, and so can be assigned the 'real number' r. It has yet to be made clear why it would be preferable to treat the real number of the pair (a,b)*as* the ratio, i.e. as the equivalence class of all pairs (c,d) (from any system of magnitudes) such that a:b/c:d - leaving aside the question of the consistency of doing so.

Finally, as used by Dummett, the term 'structural property' is misleading.

> *There is no absolute notion of 'structural property'. It is only relative to a specific structure, e.g. «M,e,f», on M that we may speak of the structural properties of the objects of M, namely the properties definable in terms of the structure (in the example, in terms of e and f, with individual quantifiers ranging over M, second-order quantifiers over subsets of M, etc.)*

Thus, when Dummett speaks of "objects or systems of objects lacking the features abstracted from, but not possessing any others in their place" or of Dedekind's natural numbers as having "no properties other than their positions in the ordering determined by their generating operation, and those derivable from them" or of "specific objects having only structural properties", he is guilty of confusion. What is true of the numbers on Dedekind's account is that it is possible to specify a structure on them, e.g. **N** or «N,&» or «N,<», in terms of which they are characterizable to within isomorphism by finitely many axioms. But *this* can hardly be the basis of a metaphysical objection to them.

From this it is clear that the objection of Russell and Dummett cannot reasonably be about the paucity or kind of properties that Dedekind's numbers have nor about the fact that they can be categorically determined in terms of a structure which distinguishes some of those properties. Perhaps the objection is, rather, that they are *given* to us only in terms of that structure. In Dummett's words, the question is "whether it is possible, not merely to characterize the abstract structure of the system of natural numbers, but to identify the natural numbers solely in terms of that structure". But what more is required to 'identify' the numbers, even on Frege's own grounds? The sense of every arithmetical proposition A is fixed: A is true just in case it can be derived from the axioms for a simple infinity. Moreover, we lack no account of the application of the numbers on this foundation. The idea that the numbers can be identified or, perhaps, further identified in terms of some particular application of them is, as we have seen, neither a very clear idea nor a desirable one.

At the end of the day, I think that, for Russell and for Dummett, the objection to Dedekind's treatment of the natural numbers is that, for Dedekind, they are just numbers and not something else as well. For Dedekind, the question "What are the numbers?" could only be answered by exhibiting their structure. For many writers since Frege, the question has rather meant: "What *besides* numbers are the numbers?" This becomes clear in the case of Dummett in his discussion of 'structuralism', when he writes

> On the stronger interpretation, structuralism is the doctrine that mathematics in general is solely concerned with structures in the abstract sense, that is, with systems left no further specified than as exemplifying the structure in question. This doctrine has, again, two versions. According to the more mystical of these, mathematics relates to *abstract structures*, distinguished by the fact that their elements have no non-structural properties. The abstract four-element Boolean algebra is, on this view, a specific system, with specific elements; but, for example, the zero of the algebra has no other properties than those which follow from its being the zero of that Boolean algebra - it is not a set, or a number, or anything else whose nature is extrinsic to that algebra. This may be regarded as Dedekind's version of structuralism: for him the natural numbers are specific objects; but they are objects that have no properties save those that derive from their position in 'the' abstract simply infinite system (sequence of order type w.)

VIII

There is an aspect of Cantor's abstractionist conception of ordinal and cardinal numbers that I have so far ignored and which was the main target of Frege's attack. Namely, the cardinal number of a set is to be itself a set, equipollent to the given set, and its elements are to be 'pure units'; and the order type of an ordered set is to be itself an ordered set of 'pure units', isomorphic to the given ordered set.

Consider just the case of cardinals. Clearly the idea of a cardinal as a set of pure units - call it a *cardinal set* - is inessential to the foundation of the theory of cardinals on logical abstraction, at least as that kind of abstraction is described in §V. But what did Cantor actually mean by speaking of pure units? There are two ways in which we might try to understand his idea of the cardinal set corresponding to the set M as a set of 'pure units'. One is that the units, the elements of a given cardinal set, should be obtained by abstracting from the particular properties that distinguish the elements of M from one another and thus should be indistinguishable in some sense from one another. That is the way in which Frege understands Cantor. The other interpretation is that the abstraction concerns, not the individuating properties of the elements relative to one another, but rather the individuating property of the set itself, for example the concept of which it is the extension. On this interpretation, the cardinal set C corresponding to a set M is to be constituted of unique elements, specified in no way other than that they are the elements of C and that C is equipollent to M. Thus, the cardinal sets are not sets of points in Euclidean space or of numbers or of sets, or of apples or etc. Once we decide in the first place that the cardinal number of M should be a set equipollent to M, then the argument for Cantor's conception of the cardinal set, on this interpretation, is exactly Dedekind's argument that the system of numbers is a 'creation of the human spirit': The cardinal set corresponding to M should not be a set of points or of numbers or of apples or of sets (as in the case of the initial von Neumann ordinals). The things that we may say about these other kinds of sets would be ungrammatical when speaking of cardinal numbers. Now, the role that Cantor would have his doctrine that cardinal numbers are sets of pure units play in the theory of cardinals is to infer the equivalence of Card(M)=Card(N) and the equipollence of M and N, and only this. It is clear from this that it is the *second* interpretation of his doctrine that is correct, and that Frege has entirely misunderstood him when he refered to his *Grundlagen* for a refutation of Cantor's doctrine.

As a matter of fact, Cantor's motivation for the conception of cardinals as cardinal sets is weak: one is introducing one cardinal set by 'abstraction' corresponding to each equipollence class, but one is not analyzing the notion that M is equipollent to N *in terms of* the notion of cardinal number. Abstraction, as Frege says, is strong lye. To abstract the cardinal set corresponding to M from M, we must specify what it is from which we are abstracting. The only possible answer is that we are abstracting from all properties and relationships of M except those which respect the equipollence relationships of M. So we do not derive the equipollence of M and N from Card(M)=Card(N); rather, the former notion is built into the latter. (Frege makes essentially the same point in criticizing Husserl's formulation of Cantor's doctrine.) Moreover, Dedekind's argument from grammar applies to Cantor's doctrine of cardinals as sets, itself: it is ungrammatical to ask whether a particular object is an element of a particular cardinal number. So I think that Cantor's view that cardinal numbers are sets of pure units is ill-conceived.

But Frege's claim was that it is *incoherent*. But certainly, on the second interpretation, which I think that we must accept as the correct one, the theory of cardinal-sets is perfectly coherent. We may take the pure units in a cardinal-set to be atoms (i.e. non-sets or *urelements*). If k and l are distinct von Neumann cardinals (i.e. initial von Neumann ordinals), we may even assume that the corresponding cardinal-sets C_k and C_l are disjoint sets of atoms. So the question is whether or not we can coherently assume that, corresponding to every von Neumann cardinal k, there is a set C_k of atoms which is equipollent to k such that, for distinct von Neumann cardinals k and l, $C_k V C_l = \emptyset$. But we can construct a standard model **M** of this assumption together with the axioms of second order set theory with urelements from a standard model of second-order set theory with a strongly inaccessible cardinal.[27] Moreover, any permutation of the elements of the cardinal set C_k induces an automorphism of **M** in an obvious way. For example, all of the cardinal sets C_l and all pure sets (i.e. whose transitive closures contain no atoms) are fixed points of this automorphism. So, in any reasonable sense, even on Frege's mistaken interpretation of Cantor's doctrine, it perfectly coherent. Of course there is a property that distinguishes the element b of C_k from all of the other elements, namely the property x=b. But, however Frege was reading Cantor, surely it would have been unreasonable of him to suppose that Cantor either overlooked this or would have denied it.

The suggestion of incoherence of the notion of the cardinals being sets of pure units seems to arise for Frege from two sources. One source is Leibniz's Principle of the Identity of Indiscernibles: If nothing distinguishes between two 'pure units' in the cardinal-set C, then they must be the same; and so every cardinal number must be 0 or 1. For example, recall that, in his review of Cantor, Frege writes of "those unfortunate Ones which are different even though there is nothing to distinguish them one from another" (p. 270). He goes on to say: "The author evidently did not have the slightest inkling of the presence of this difficulty, which I deal with at length in §§34-54 of my *Grundlagen*." Unfortunately, the discussion in these sections is interwoven with a discussion, beginning at §29, of quite another notion, that of a 'unit', as well as with Frege's view of abstraction. But, concerning the issue at hand, he writes in §35

> We cannot succeed in making different things identical simply by dint of operations with concepts. But even if we did, we should then no longer have things in the plural, but only one thing; for, as

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[27]Take the domain to be V_{k+k}, where k is the least inaccessible cardinal. The sets of rank $\leq k$ are taken to be atoms, except for the null set, which is taken to be the null set. The sets in V_{k-k} of rank $>k$ thus become the non-null sets of rank $<k$ over the set of atoms. The elements of C_m, for m a cardinal $< k$, are the atoms $m \approx \{m\}$.

> Descartes says, the number (or better, the plurality) in things arises from their distinction.

The first sentence of this passage refers to his argument in §34 that abstraction cannot produce new objects ('make distinct things identical'). His correction of Descartes, choosing the term 'plurality' over 'number', illustrates another source of confusion in his discussion of other authors. By a number, Descartes meant essentially a set and, as we shall see below, that was a common usage of the term in earlier times. Moreover, the passage cited in Descartes' *Principia* (Part I, §60) in no way supports Frege's point. 'Distinct' there means only 'not identical', not 'distinguished by some property', and Descartes is distinguishing three different kinds of 'distinction': real, modal and rational. Nowhere in this discussion (§§60-62) does he imply that non-identity of P and Q requires there to be some property possessed by one of them and not the other. But there can be no doubt that Frege himself subscribed to this principle: In *Grundlagen* §65 he takes as his own definition of identity (being the same object) Leibniz's:[28]

> Things are the same as each other, of which one can be substituted for the other without loss of truth.

But, of course, this principle doesn't tell us anything until we know in which propositional contexts F(x) the substitution may occur. For example, any two distinct objects will be distinguished by the context 'xMM', where M is a set which contains one but not the other.

Frege himself might defend against this by insisting that 'xMM' is not a primitive concept and must be replaced by the concept F(x) of which M is the extension. And, presumably, F(x) cannot itself be defined in terms of the identity relation, since otherwise x=b distinguishes the object b from all other objects and Leibniz's principle is trivialized. I have already expressed the view that it would have been entirely unreasonable for Frege to have supposed Cantor to have rejected this trivial form of Leibniz's principle or to have overlooked it in formulating his doctrine of cardinal numbers as sets of pure units. So Frege's objection would then be that there is no such concept F(x), not already involving the identity relation, which distinguishes between two 'pure units' of the cardinal-set.

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[28]He later rejected this as a *definition* (cf. Frege (1984) p. 200), but he continued to affirm its validity.

But of course this whole line of argument presupposes the validity of the non-trivial form of Leibniz's principle, and, in Frege's case, it seems hard to defend. Certainly no point in Euclidean space is distinguished from any other by a concept, unless that concept itself is defined by reference to specific points, since the space is homogeneous. But there is a difficulty with individuating points by means of concepts which themselves refer to points. For example, the points p and q may be distinguished by the concept 'x is between r and s', which is satisfied by p and not by q. But then q satisfies a corresponding concept 'x lies between t and u, e.g. where (p,r,s) is congruent to (q,t,u). The problem is that our grounds for calling these two concepts distinct is only that (r,s) and (t,u) are distinct (i.e. non-identical) pairs of points: the individuation of such concepts presupposes the individuation of pairs of points and so, ultimately, of points. Hence, there is a circle.[29] Of course, since Frege believed that Euclidean geometry is the science of physical space, he may have believed that any two points are distinguished by empirical, non-geometric, concepts, e.g. by physical scalars. But it would be hard to accept the non-trivial form of Leibniz's principle as a metaphysical principle based upon such a belief.

The other source of the appearance of incoherence in the notion of a cardinal number as a set of pure units is the argument that, if cardinal numbers are sets at all, then, for example, 1+1 must equal 1. In §35 Frege writes

> Jevons goes on: "Whenever I use the symbol 5 I really mean
> 1+1+1+1+1
> and it is perfectly understood that each of these units is distinct from each other. If requisite I might mark them thus
> 1&+1&&+1&&&+1&&&&+1&&&&&."
> Certainly it is requisite to mark them differently, if they are different: otherwise the utmost confusion must result. ...
> ... The symbols
> 1&, 1&&, 1&&&
> tell the tale of our embarrassment. We must have identity - hence the 1; but we must have difference - hence the indices; only unfortunately, the latter undo the work of the former.

Frege and Jevons collaborate in a confusion here. Jevons problem is that he is thinking of 5 as a set of five elements on the one hand and as 1+ ... +1 on the other. If the different occurrences of '1' do not

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[29]Leibniz did not face this difficulty. For him, the principle of identity of indiscernibles applies to substances (which have no real relations), and not to ideal things such as points in space.

denote different unit sets, then Jevons worries that 1+ ... +1 will not yield a set of five elements. *But that is because he thinks that, if cardinals are sets, then their addition is just their union.* But even if we take cardinal numbers to be sets, m+n does not denote union of m and n; it denotes the cardinal of their disjoint union. Frege (§38) somewhat misses the point here: He thinks that Jevon's problem arises from confusing 'unit' with 'one' and of treating numbers as 'agglomerations'. He is right that one cannot take the units in 5 to be the unit in 1, since there are five of the former and only one of the latter. But Jevons' equation 5=1+1+1+1+1 leads to this identification, not because he thought that numbers are sets (after all, the von Neuman cardinals are sets), but because he confused the addition of cardinals with the union of sets. Frege's argument, were it valid, would not simply be an argument against cardinals being sets of pure units, *it would be an argument against the cardinal of a set M being a set equipollent to M at all.* I am persuaded by Dedekind's grammatical argument that numbers are not sets; but Frege's line of argument would exclude even the *representation* of cardinals by initial von Neumann ordinals.

Of course, Frege might have made the argument that, since the identification of cardinals with sets does not admit the identification of cardinal addition with set-theoretic union, then there is no point in regarding cardinals as sets at all. But this is not the argument that he gave.

IX

One problem with reading the literature on the number concept prior to Cantor, Dedekind and Frege is that, aside from Bolzano, the authors generally have not fully distinguished the notion of a set and tended to subsume it under a more general notion of a 'multitude' or 'plurality'. Anything with proper parts was regarded as a plurality - a line segment, Socrates, a heap of stones, a flock of sheep. It was understood from the time of Plato that number does not unambiguously apply to pluralities. Socrates is one but has many parts, the flock of ten sheep also includes a plurality of twenty sheeps' eyes and a plurality of forty sheeps' legs, etc. Aristotle explicitly understood that assigning number to pluralities in this sense requires a prior choice of the parts to be numbered.[30] He referred to this as a choice of 'unit': To assign numbers to line

[30]Aristotle, *Metaphysics*, Book X, Ch. 1 and 2.

segments, for example, we must first choose a particular line segment as the unit of measurement. This conception, that what can be numbered, the ériymÒw, is some object (in a generalized sense that admits flocks of sheep, the aggregate of planets, etc.) relative to a partition - a choice of unit -, survived even into the late nineteenth century in the form of the rejection of the null set: no object can be partitioned into zero parts. There was also a surviving conceptual difficulty with unit sets, which is reflected in the sometime rejection of the number 1 by the classical Greeks: if what can be numbered is an object X relative to a choice of unit, the unit set of X would be X *qua* part of X, which was indistinguishable from X.[31].

Indeed, Frege reveals some of this confusion about the concept of a set in *Grundlagen*, §28, where he is discussing the notion that a number is simply a set. He writes

> Some writers define the number as a set or multitude or plurality. All of these views suffer from the drawback that the concept will not then cover the numbers 0 and 1. Moreover, these terms are utterly vague: sometimes they approximate in meaning to "heap" or "group" or "aggregate" ("*Aggregat*"), referring to a juxtaposition in space, sometimes they are so used as to be practically equivalent to "number", only vaguer.

In *Paradoxien des Unendlichen* (1851), Bolzano had already made a distinction between what he called an '*Inbegriff*' and a set:

> There exist aggregates (*Inbegriffe*) which agree in containing the selfsame members, and nevertheless present themselves as *different* when seen under different aspects or under different conceptions, and this kind of difference we call 'essential'. For example: an unbroken tumbler and a tumbler broken into pieces, considered as a drinking vessel. We call the ground of distinction between two such aggregates their *mode of combination* or their *arrangement*. An aggregate whose basic conception renders the arrangement of its members a matter of indifference, and whose permutation therefore produces no essential difference, I call a *set* ... (§4)

Bolzano's 'Inbegriff' seems to be Frege's 'Aggregat' - a plurality with the unit chosen. But it is fair to say that even Bolzano had not entirely liberated the notion of a set from that of an aggregate and, in particular, could not accommodate unit sets, since in §3 he writes "For if A were identical with B, it would of course be absurd to speak of an aggregate composed of the objects A and B." Moreover,

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[31] *Metaphysics* 1052^b31. Also see the second quote below from Bolzano's *Paradoxien des Unendlichen*.

Bolzano uses the term 'Teil' to refer to elements of a set, suggesting, too, that the element/set relation is not entirely distinguished from the part/whole relation. Notice also that Cantor did not clearly distinguish between Bolzano's notions of Inbegriff and Menge in the quote above from (1895-97), p. 282, in as much as he speaks of obtaining the power of the *set* (Menge) M by abstracting both from the nature of its elements and *from the order in which they are given*.[32] Yet, when he speaks of a set as "already completely delimited by the fact that everything that belongs to it is determined in itself and well distinguished from everything that does not belong to it", it would seem that he intends the notion of a set to be independent of the order of its elements.

A further difficulty with reading the pre-twentieth century literature on the number concept is manifested in Frege's "[these terms] are so used as to be practically equivalent to 'number', only vaguer". The fact is that *the term "number", in the sense of the whole numbers, often really did just mean a (finite) set* - in the somewhat confused sense of 'set' that we have just discussed. So a 'number' is given only relative to the choice of unit. This meaning goes back to the Greek meaning of ériymÒw which came to be translated as "number". Thus at the beginning of Book VII of the *Elements*, Euclid has these definitions

1. A unit is that with respect to which each of the things that exist is called one.
2. A number is a multitude composed of units.

In (1884), §29, Frege writes that Euclid "seems to mean by the word "monãw" sometimes an object to be numbered, sometimes a property of such an object and sometimes the number one." But the meaning is clear. In Definition 1 Euclid says that numbering begins with the choice of unit, of what is to be counted - what is to be called one. Definition 2 then defines a 'number' (an ériymÒw) to be a set of such units. It is true that 'unit' in Definition 1 refers to a property and in Definition 2 to objects having that property. But that is a frivolous objection. We use the word 'man' sometimes to refer to a property and sometimes to men. So things may be called 'one' after the choice of unit in the same way that men may be called 'man'. It is because Frege does not understand that 'one' is a *common* name, applying to all units (once the unit has been chosen), that he misunderstands Euclid. Similarly, also in §29, he misunderstands

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[32]Of course, Cantor had been concerned with point sets in Euclidean space, and this is what he had in mind in this passage.

Schröder (1873), p. 5, when he quotes "Each of the things to be numbered is called a unit" and then goes on to object:

> "We may well wonder why we must first conceive of the things as units, instead of simply defining number right away as a "set of things", which would bring us back once again to the view just discussed.

Here he is referring to his discussion in §28, partly quoted above. But he can't have it both ways: if he believes that the term 'set' can only refer to heaps and the like, then he should also agree that number can be assigned only after the choice of unit.

Hume also uses the term 'number' to mean a set. In (1884), §63, discussing how the notion of equinumerosity should be defined, Frege writes

> Hume long ago mentioned such a means: "When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them as equal." This opinion, that equality of numbers must be defined in terms of one-to-one correspondence, seems in recent years to have gained widespread acceptance among mathematicians.

Frege goes on to say that the definition of numerical equality in terms of one-to-one correspondence raises certain "logical doubts and difficulties." In particular, he writes "It is not only among numbers that the relation of equality (Gleichheit) is to be found. From which it seems to follow that we ought not to define it specially for the case of numnbers." The ambiguity of the German "Gleichheit" as between "equality" and "identity" helps (along with Austin's translation) to hide a confusion here: It is quite clear, even just from the passage in Hume's *Treatise* that is quoted (Bk. I, Part III, Sect. I), that by 'number' Hume is referring to finite sets and that, when he speaks of equality of numbers, he is not referring to the identity relation but to the relation of equinumerosity, *which indeed is to be defined specially for the case of 'numbers', i.e. sets*. The 'logical doubts and difficulties' were created by Frege's incorrect reading, not by Hume's conception.

Bolzano (1851) also used the term 'number' for sets. In §8 he attempts to characterize "*finite* or *countable multitudes*, or quite boldly: *numbers*".

Certainly the term 'number' did not *always* refer to a set. Thus people spoke of 'the number ten' and of the set (and number) of prime numbers less than n, etc. But, on the one hand, there seems to be no obstacle to understanding number in these contexts as refering to species of finite sets - just as one might speak of the animal, man, or of the number of species in a certain genus. Into the nineteenth century, every theorem of number theory could be understood as a statement about an arbitrary finite set, free from any assumption of the existence of an infinite set. Moreover, there was always the resource (exploited in analytic number theory) of regarding the natural numbers as embedded in the system of real numbers. The move towards treating the natural numbers as forming an autonomous system of objects dates from later in the century. It required the explicit admission of the actual infinite into mathematics, and the motivation would seem to be the arithmetical foundation of the system of real numbers. Rational numbers are constructed from natural numbers and real numbers are, for example, sets of rationals. If the natural numbers are not to form an autonomous system of objects, then it is hard to make sense of this construction.[33]

X

In (1884), §62, Frege raises the question of the meaning of "the number which belongs to the concept F is the same as that which belongs to the concept G". We have already quoted his attribution to Hume of the definition of this concept as meaning equipollence. He goes on to cite Schröder, Kossak and Cantor (1883) paper as well. Dummett writes "By the time that Frege wrote *Grundlagen*, the definition had already become a piece of mathematical orthodoxy, though Frege undoubtedly gave it its most exact formulation and its most acute philosophical defense." (pp. 142-43) But Frege's citation of earlier authors is misleading. We have already noted his misunderstanding of Hume. But there is another and more compelling respect in which his citation is misleading and, in particular, slights Cantor's contribution. I do not mean the fact that Frege refers to Cantor's 1883 paper rather than to his earliest

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[33]Dummett's explanation (pp. 133-34) of why *Frege* required that the numbers be objects is unconvincing. His explanation is that 0,…,n have to be objects in order to prove that n has a successor, namely the number of {0,…,n}. In other words, we need a set (i.e. extension of a concept) with n+1 elements. But, whether or not we count the object, call it n*, which Frege takes to be the number n *as* the number n, it is an object in his system and so the set {0*,…,n*} still exists. Indeed, eliminating the middle man and making no commitment at all about the nature of the numbers, we may replace k by the von Neuman ordinal $N_k=\{N_0,\dots,N_{k-1}\}$ in Frege's argument.

published definition of equality of power, Cantor (1878).[34] Rather, I refer to the fact that *the other authors cited were concerned only with finite 'numbers'*, i.e. sets, and not with the general notion of cardinal number which applies to infinite sets. Frege and Cantor, on the other hand were concerned with the general notion of cardinal. The extension to this general notion was not a trivial matter. As late as 1851, in his monograph just cited, Bolzano, who did understand the notion of set to include infinite sets, had argued that the characterization of numerical equality in terms of one-to-one correspondence, though correct for finite sets, could not be applied to infinite sets. His reason was the traditional obstacle to a coherent theory of infinite numbers: It would then happen that infinite sets could be numerically equal to proper subsets of themselves. (§23) *It was Cantor, in (1878), who took this bull by the horns and forever separated the notion of proper subset from that of proper numerical inequality*. [See Cantor (1932), p.119)] Quite clearly the *general* analysis of the concept of having the same cardinal number, finite or infinite, can be attributed only to Cantor.

XI

One must wonder why Dummett wrote that Frege gave the definition of equinumerosity in terms of one-to-one correspondence of sets "its most exact formulation and its most acute philosophical defense". (pp. 142-43) The definition is quite clear in the above quote from Hume once one understands that by 'number' he means the thing to be numbered, the set. But anyway it is stated with admirable clarity by both Bolzano and Cantor, even though the former did not accept it as the definition of equinumerosity in the case of infinite sets. As for a defense, who was attacking it and why was defense needed, philosophical or otherwise? Certainly one attack that needed to be answered was Bolzano's, which we have already mentioned. Cantor responded to this and Frege made no mention of it at all.[35] Another attack came from Cantor himself, who in (1883), §2, distinguished the two conceptions of number, cardinal and ordinal, which essentially coincide in the finite case

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[34] Not, as Dummett cites, (1874). In this paper, Cantor does not define the general concept of equipollence, although he does produce the first significant result concerning infinite powers.

[35] Although both Cantor and Dedekind cite Bolzano's fundamental work on the nature of the infinite, I have found no reference at all to him in the published works of Frege.

but not in the infinite. On the ordinal conception, it is not abstract sets but well-ordered sets to which number applies. Again, this analysis was undertaken by Cantor, not by Frege.[36]

What *did* Frege contribute to this question? The problem as he saw it is formulated in (1884), §39, as a dilemma:

> If we try to produce a number by putting together different distinct objects, the result is an agglomeration (Anhäufung) in which the objects contained remain still in possession of precisely those properties which serve to distinguish them from one another; and that is not a number. But if we try to do it in the other way, by putting together identicals (Gleichem), the result runs perpetually together into one and we never reach a plurality.

But distinguish two questions which, in his discussion of earlier authors, Frege tended to confuse: What are the things to which number applies? And, what are numbers? The first horn of his dilemma concerns the first question. And the things to be numbered are not 'agglomerations' but sets, which indeed arise by 'putting together' different distinct objects. These sets *were* called 'numbers' by some of these authors, and this is one source of Frege's confusion. The second horn of Frege's dilemma can concern only the view that the numbers themselves are sets of pure units. The argument is that, if the units are not distinguished by their properties, then they will be identical. We have already discussed this view and concluded that the notion of cardinal number as a set of pure units, though unattractive, is by no means incoherent. Distinct units are indeed distinguished by their properties; but when from a set of two cats, one white and one black, we 'abstract' the number two as a set of pure units, the units are not white and black, respectively, and they are not cats.

But also, unlike Frege, I find no sign of the conception of number as set of pure units in the quotes from other authors that he offers in (1884) (in contrast to Cantor in his later writings and to Husserl's book). His reading seems to me to have been misdirected by two related things: his interpretation of "*gleich*" to mean identity and his failure to understand the historical use of the term "number" to mean what is numbered. For example, in §34, where he begins the discussion of whether the units are *gleich*, he refers to the *very same page* (p. 5) of Schröder (1873) from which the quote "Each of the things to be numbered is called a unit" (cited in §28) came. He now paraphrases Schröder as giving as the reason we call things

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[36]Indeed, from the discussion in Frege (1884), §86, it seems likely that he did not even understand Cantors concept of a well-ordered set at that time.

"units" that it ascribes "to the items that are to be numbered the necessary identity (Gleichheit)". Clearly Schröder could not mean here that the things are being identified with each other. He means rather that they are being identified *as* the things to be counted. Indeed, in §54, Frege quotes p. 7 of the same work on the notion of a unit: "This generic name or concept will be called the denomination of the number formed by the method given, and constitutes, in effect, what is meant by its unit." As further evidence of Frege's confused reading of other authors, let me repeat part of the quote cited above from (1884), §34:

> For suppose that we do, as Thomae demands, "abstract from the peculiarities of the individual members of a set of things", or "disregard, in considering separate things, those characteristics which serve to distinguish them:. In that event we are not left, as Lipschitz maintains, with "the concept of the number of the things considered"; what we get is rather a general concept under which the things considered fall. The things themselves do not in the process lose any of their special characteristics.

If we put together Thomae's "abstract ...set of things" with Lipschitz's "the concept of the number...", then it seems that we are abstracting from the particular nature of the elements of a *set* to obtain the number of the set. But if we read only what Lipschitz wrote, then we are, in considering separate things, disregarding those characteristics which serve to distinguish them and are left with the concept of the number (anzahl) of the things in question. But by "anzahl" here Lipschitz means the set. So Frege is in complete agreement with Lipschitz: The set (or concept whose extension is the set) is obtained by abstracting from the differences among the elements of the set.

The citation of Thomae is equally misleading. *His* ultimate concern is with the theory of analytic functions and so with the complex numbers. He is sketching the construction of these numbers from the natural numbers. His account is 'formalistic', in the sense that he treats the natural numbers as signs and the real numbers as infinite sequences of signs. This account is by no means as defective as Frege makes it out to be (e.g. in (1903), pp. 96-139);[37] but that is not my point here. The numbers, being for Thomae signs, are certainly not sets of pure units. The context of the above quote from Thomae is this:

[37]And Frege's treatment of the 'formalism' of Heine (1872) in (1903) is totally unjustified.

> We assume that one can count, i.e. that one is in the position to abstract from the peculiarities of the individuals in a set of objects and assign successive distinct names to distinct such sets of objects. Each individual of the set is called a unit, and, as a consequence of the required abstraction from all distinctive peculiarities, one may replace any unit by any other. *The units are equal to one another*. [Thomae (1880), p.1.]

The units are *equal*, not identical. The sense in which they are equal is explained: one can be substituted for any other without altering the name assigned, i.e. the number.

Frege's proposed solution to his dilemma is to be found in §§46-48 and in §54: The things to which number applies are concepts or extensions of concepts. This is consistent with his view that Cantor's notion of a set can only be understood as the extension of a concept. Concerning the notion of a unit, Frege points out (§54) that the term was used with two senses: For the concept and for the objects that fall under the concept. Indeed, we saw that this was true of Euclid. But Frege seems to have thought that this was a source of confusion, whereas one would think that it would have been a commonplace observation: "Man" is sometimes used to denote a particular man and sometimes to denote the property of being a man. The difference between 'man' and 'unit' in this respect is that the meaning of the former is fixed and the meaning of the latter is relative and must be specified in any context - as meaning 'man in the room' in one context, 'sheep in the flock' in another, etc. It is to this relativity that Schröder refers when he speaks of it as a *generic* name or concept.

XII

Ultimately, Frege's contribution with respect to the definition of equinumerosity was to replace Cantor's sets as the objects of number attributions by concepts. Indeed, his proposal in his review of Cantor is that we should understand the term 'set' to refer to extensions of concepts. Perhaps it is this idea that Dummett thinks renders the definition more precise. *In the case of finite numbers*, Frege's proposal would indeed have clarified the discussion of number among his contemporaries: the meaning of choosing a unit and of all units being equal is well analyzed in terms of choosing a concept, in something like Frege's sense of concept. The number of people in the room is an attribute of the concept 'x is in the room', where x ranges over people. It is only a pity that he needed to so discredit other authors, from Euclid to Thomae, in order to make his contribution.

But notice that, in the above example, we do not really have Frege's notion of concept, since x ranges over people and, for Frege, the variable ranges over all objects. In the example given, the appropriate concept for Frege would be 'x is a person in the room'. But, when we admit Fregean concepts in general, that is concepts of the form F(x), where x ranges over 'all objects', then Frege's idea that number is attributable to concepts goes wrong. In the case of *infinite* numbers, the fact is that Cantor had already noted in his (1883) that there are concepts, for example the concept that x is an ordinal (or cardinal), which do not have a power. In his note to §4 he writes

> [E]ach of the number classes, and hence each of the powers, is associated with an entirely determinate number of the absolutely infinite totality of numbers, ... Thus the different powers also form an absolutely infinite sequence.

The 'absolute infinite' is contrasted with a determinate infinite, which has a power. His much maligned review of Frege's *Grundlagen* in 1885 may also be read as spotting immediately what is wrong with Frege's conception. He writes

> The author comes upon the unfortunate idea - and it appears that he is following in this respect a suggestion of Überweg in his *System der Logic*, §53 - of taking what is called in Scholastic logic (Schullogik) the 'extension of a concept' as the foundation of the number concept. He entirely overlooks the fact that the 'extension of a concept' in general may be quantitatively completely indeterminate. Only in certain cases is the 'extension of a concept' quantitatively determinate: Then it has of course, if it is finite, a definite number or, in the case it is infinite, a definite power. For such a quantitative determination of an 'extension of a concept' the concepts of 'number' and 'power' must already be given from another source, and it is a *reversal of direction* if one undertakes to found the latter concepts on that of 'extension of a concept'. (Cantor (1932), p. 440)

Although neither Frege nor Cantor's editor, Zermelo,[38] understood him in this way, it is reasonable to believe that the concepts he had in mind, whose extensions are quantitatively indeterminate, are 'x is an ordinal' and 'x is a cardinal'. One might question this on the grounds that he does not explicitly name these concepts; but, since he doesn't give *any* examples, that would be an objection to *any*

[38]Cantor (1932), p.441.

attempt to interpret his remark that there are quantitatively indeterminate extensions of concepts. Moreover, he had recently published his (1883) and Frege (1884) referred to it; so it would have been not unreasonable to assume that the reader of the review and, in particular, Frege would know that the totalities of ordinals and powers have no power. But most importantly, other interpretations - such as that he is referring to non-sortal concepts, such as 'water', or to concepts whose extensions are indefinite, such as 'living creature' - make no sense of Cantor's subsequent remark about reversal of direction. The point of that remark, it seems to me, must be that the number classes are in place to serve as a measure for an infinite totality being a determinate infinity, i.e. having a power. The number classes would play no conceivable role in distinguishing sortals from non-sortals or well-defined concepts from ill-defined ones. Again, Cantor's reference to Überweg's *System der Logik*, §53, although it does not seem to be particularly apt, lends some support for my interpretation of his review. Überweg is speaking, not of concepts there, but of representations (Vorstellungen). Stripped of Überweg's psychologistic conception of them, they come closest to what Frege means by a 'concept'. Concepts for Überweg, on the other hand, are of a restricted kind of representation (§56). The extension of a representation (§53) consists of other representations. Thus the relation between representation and elements of the representation conforms not at all to Frege's sharp distinction between concept and object. An individual is in an extension by way of having its individual representation in the extension. Concerning number, Überweg writes

> The formal relation of the subordination of many representations under the same higher one leads to the concept (Begriff) of number which, in its original sense (as Anzahl), is the determination, by means of a unit, of the plurality (Vielheit) of individuals of the extension. (§53)

I do not understand the role of the unit in this conception. Shouldn't the plurality of individuals in the extension be determined already by the representation itself? But anyway, in an earlier passage, and perhaps closer to Frege's conception, Überweg writes

> Only on the basis of concept formation can *numerals* be understood; for they presuppose the subsumption of similar objects under the same concept. (§47)

I can't find any grounds in this for thinking that Cantor understood Frege's concepts to include non-sortals or concepts whose extensions are not well-defined.

It is easy to misunderstand Cantor's review because, for many, the primary question is to be formulated by asking whether a given totality is a set. If it is, then it has a cardinal number. Indeed, in his later papers, in which he seems to have abandoned his autonomous theory of ordinals and powers, this question is primary. But in the 1883 paper and in his review of Frege, Cantor's understanding was that the theory of the number classes is primary and with it, the theory of infinite cardinals *as the powers of the number classes*. Our question of whether a totality is a well-defined set *is* precisely his question of whether it is 'numerically determinate' - in other words, whether it is finite or equipollent to one of the number classes. It is for this reason that Cantor speaks of a reversal of direction. One cannot define the cardinals simply to be the extensions of second-order concepts of the form 'equipollent to F', since not every concept F has a cardinal.

Frege's reply [(1984), p. 122] to Cantor's review shows that he thinks that Cantor's remark is aimed at his definition of N_xFx, the number of F's' as the extension of the concept 'equipollent with F', and is pointing out that *this* concept might or might not be numerically determinate. Frege responds that is of no consequence, since this concept does not have to have a number in order to *be* the number N_xFx. But Cantor means that F might itself be quantitatively indeterminate, so that N_xFx does not exist. On explanation for Frege's interpretation of Cantor's criticism, and one which lends some support to it, is Cantor's assertion that Frege took the notion of the extension of a concept as the foundation of the number concept. My suggested reading of this requires that we understand him to be saying that, for Frege, it is the extension of concepts which have number; but Frege's actual definition in (1884) attributes number to the concept and it is the number itself which is an extension of a concept. On the other hand, although Frege does not explicitly speak of higher level concepts in (1884), his official definition of $N_xF(x)$ makes it the extension of a second level concept, so that its elements are themselves first level concepts. It is not at all unreasonable to suppose that Cantor would understand this by taking the elements of $N_xF(x)$ to be the corresponding extensions of the concepts.

Of course, even with the misunderstanding, Frege should have taken heed. If there is *any* concept whose extension does not have a cardinal, then Frege's analysis fails. *Even if it were concepts to which number should be attributable, one would need to distinguish, on some other grounds, as Cantor insisted, those concepts which do have number from those which don't.* Frege might have responded that the concept 'equipollent with F' is a second-level concept, and that it is only first-level concepts to which number is attributable. But in (1893), which implicitly postulates as an object an extension of each first-level concept, this response fails; for every second-level concept induces the first-level concept of the corresponding extensions In particular, the second-level concept 'equipollent with F' yields the first-level concept 'equipollent with the extension of F'.

There tends to be a picture of Frege as a tragic victim of fate: by his very virtue, namely his insistence on precision, he committed himself explicitly to a contradiction that was already implicit in mathematical thought. But in fact his assumption in the *Grundgesetze* that every concept has an extension was an act of recklessness, forewarned against by Cantor already in 1883 and again, explicitly, in his review in 1885.

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