Another Fall From Paradise:
Counting Gone Crazy

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I am here to write a book in dirty philosophy. Dirty philosophy is when you devote all of your efforts to doing down other philosophers: it tends to leave a bad after-taste. My excuse is that I want to undermine skeptical philosophical arguments about the existence of mathematical objects, the truth of mathematical axioms, the possibility of mathematical knowledge, etc.—to show that such arguments are groundless and just spin their wheels. I want to argue, among other things, that criticism of mathematics must be *internal*: if there are mistakes, they must be seen to be *mathematical* mistakes.

My strategy is in large part to show that the skeptical arguments about mathematics are, formally speaking, the exact counterparts of classical skeptical arguments about the existence of other minds or even of physical objects. To my mind, this shows already that something is seriously wrong with them. But I also want to locate where the arguments go wrong. I have been led by this line of thought (and by that master proponent and practitioner of dirty philosophy, Ludwig Wittgenstein) to a general consideration of cognitive ideas such as meaning, knowing and understanding and, especially, the meaning of existence statements.

But I don't want to talk dirty philosophy to you here now—to such nice people.[1]

There is another conception of philosophy with ancient and honorable claims to the title—and to which the likes of Wittgenstein haven't a clue. It is the domain of Plato's dialectician, with the task of finding the first principles—the right concepts, definitions and axioms—of each field. In this sense, philosophy has no subject matter of its own, but rather refers to a way of looking at all subjects, searching always for clarity and foundations.

Its in this sense that I want now to talk about philosophy.

I suppose that every time a canon breaks down, there is a fall from paradise: from a bounded orderly universe to one relatively limitless and without rules. [And every time, there are, on the one side, the law-and-order types, who seek to preserve the canon, and, on the other, the more adventurous

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[1] Why no one should ever talk dirty philosophy:

> How charming is divine philosophy!/ Not harsh, and crabbed, as dull fools suppose./ But musical, as is Apollo's lute./ And a perpetual feast of nectared sweets,/ where no crude surfeit reigns. J. Milton *Comus*

or foolhardy, who relish freedom and the challenges it entails and ignore the pitfalls—at least until they get old and try to establish new canons.]

There have been some spectacular falls from paradise. Undoubtedly everyone's all-time favorite was the first: the loss of innocence in the Garden of Eden. It had that essential you-can't-go-home-again ingredient that any truly first-rate Fall must have.

Another great Fall, my third favorite, was the adoption, by Galileo and eventually others, of a Copernican cosmology. Here one confronted the orderly world of Dante and the Church, derived from Aristotle's cosmology: the prime mover spinning the sphere of the fixed stars, which in turn create the seasons and, in conjunction with the potentialities of things, account for all the natural motions of our world. Never mind that there were violent motions, such as the trajectory of a spear, not accounted for. More or less, everything had a place, including us, and knew how to behave. To challenge all that, to suggest that the stuff of the stars is the same as Earth's and that their motions may not differ essentially in the laws that govern them from those on Earth, was to open the door to anarchy, both in the cosmos and (certainly more importantly for the Church) in the home. The opposition was fierce. Darwin represents another great Fall, in many ways like the one represented by Galileo. But by Darwin's time, the Church was not so strong and the opposition, although it still persists, has not been so fierce or bloody.

But I want to briefly describe another Fall from Paradise, my second favorite. It is like Galileo's in that it encountered strong human opposition (obviously not a property of the first). But it is in one sense, anyway, more impressive than Galileo's, in that, like in the case of the original exodus from the Garden of Eden, there is no returning home. Of course, after Galileo, man could no longer ever again be at the center of the universe; but let's face it: if we could ask people back then, they would probably say that it wasn't such a great place to be anyway. At least in our understanding of the natural cosmos, the advance in science has brought far more phenomena under the scope of law than was ever conceivable in Aristotlean science, and the hope remains of a single physical theory, able to account for all the forces of nature, turning the specious unity of Aristotelian science into a reality. So, in terms of a harmonious picture of things, maybe in this case we can go home again—and then some.

In the mid-1870's, the world of mathematics was more or less fixed and, at least on the surface, neat and tidy. There was arithmetic and the higher analysis. The central place of Euclidean geometry had been supplanted: points in Euclidean space could be represented by triples of real numbers as in analytic geometry and the properties of the space expressed analytically. So the essential ontology of mathematics seemed to be a closed and orderly world, founded on the system of real numbers. One could consider sets or functions of real numbers, sets or functions of these, and so on; but the base was fixed: write down the axioms governing the real numbers and everything Plato demanded was more or less in place. There could be an endless supply of combinatorially difficult problems to solve, but the foundations were well-understood and Plato's dialectician was out of business.

Of course (as with the Church's cosmos), it wasn't really quite like that: there were for example difficulties with finding the proper definitions of even the basic operations of the calculus and there were conflicts over the use of certain new kinds of 'set theoretic' methods in mathematical construction and reasoning; but maybe it is fair to say that the surface was relatively calm.

My second most favorite Fall from Paradise was brought on by Georg Cantor (1845-1918), the son of a Danish father, who became a successful merchant in St. Petersburg, and a Russian mother. They moved to Germany when Cantor was quite young and, though he apparently always felt himself out of place, he lived there the rest of his life. He spent his academic career at the university in Halle, a place from which, after a short while, he desperately wanted to escape; but, partly because of opposition of those in power to those of his ideas that we are about to touch on, was unable to. He began his career working in number theory and then in analysis. In the course of the latter work, he was led to ideas that radically changed mathematics—he is generally reckoned to be the founder of the theory of sets. He was also in many ways unstable and is thought to have been a manic-depressive. He spent much time in his later life in institutions. Perhaps some indication of his lack of stability was his obsessive conviction that Bacon wrote the Shakespearean plays—a belief to which he devoted a good deal of time and energy.

In the 1870's, in the course of solving a certain rather important problem of the day in analysis, Cantor introduced an operation which, applied to certain sets P of real numbers, yields another such set P'. P' is obtained from P by discarding certain numbers from P (numbers that are 'isolated' in relation to the other numbers in P) and so every number in P' is already in P:

$$P \supseteq P'$$

It can happen that no numbers are discarded in forming P'–there are no isolated numbers— and so $P' = P$: that is the good case and we can stop there. But in general, P has isolated numbers and, in passing to P', relationships have changed and new numbers may become isolated. In that case we have to go on and apply the operation to P', forming P''. If we continue applying this operation, we obtain the nested sequence

$$P \supseteq P' \supseteq P'' \supseteq \cdots \supseteq P^n \supseteq P^{n+1} \supseteq \cdots$$

Sometimes, after some finite number n of steps, P^n is the same set as $(P^n)' = P^{n+1}$; and then, again, we can stop: Cantor has the set P^n that he needs. But sometimes this isn't so, and each P^{n+1} is distinct from the preceding P^n. If this is so for each n, then it happens to be the case that after we have run through all the stages, there still remain numbers that are in all the P^n's. We should give the set of all these remaining numbers a name—call it P^ω (and don't ask my why—the last letter of the Greek alphabet?, the bottom half of a lazy 8?—I dunno)—so we have

$$P \supseteq P' \supseteq P'' \supseteq \cdots \supseteq P^n \supseteq \cdots \supseteq P^\omega$$

But now, in the aid of the solution to Cantor's problem, we must go on and apply the operation to P^ω, obtaining $P^{\omega+1}$, and so on.

$$P \supseteq P' \supseteq P'' \supseteq \cdots \supseteq P^n \supseteq \cdots \supseteq P^\omega \supseteq P^{\omega+1} \supseteq \cdots$$

In this way, if we now forget the original operation and just consider the superscripts, we are led purely formally to the series of 'numbers'

$$\begin{array}{l}
0, 1, \ldots, n, \ldots, \\
\omega, \omega+1, \ldots, \omega+n, \ldots, \\
\omega+\omega = \omega \times 2, \ldots, \omega \times 3, \ldots \omega \times n, \ldots, \\
\omega \times \omega = \omega^2, \ldots, \omega^3, \ldots, \omega^n, \ldots \\
\omega^\omega, \omega^{(\omega^\omega)}, \ldots
\end{array}$$

Is this counting gone crazy, or what?[2]

But you haven't seen anything yet!

In fact, the implicit scheme used in generating these transfinite numbers eventually peters out (where does that expression come from?), even before reaching all the numbers needed for Cantor's problem.

But later, in 1883 in one of the very greatest classics of philosophy, and one unread by an overwhelming majority of philosophers, *Grundlagen einer allgemeinen Mannigfaltigheitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*, Cantor introduced his general theory of transfinite numbers—and, my goodness, how simple it seems to be!

$$\text{If } \Sigma \text{ is any initial segment of numbers, then there is a least number } S(\Sigma)$$

$$\text{which is greater than all the numbers in } \Sigma$$

To see how this works, note that

$$S(\text{empty segment}) = 0, \qquad S(0, \ldots, \alpha) = \alpha + 1 \qquad S(0, 1, 2, \ldots) = \omega$$

and if we take Σ to be the initial segment of numbers sketched above, then we can still form $S(\Sigma)$, which has a name: ϵ_0, and go on—$\epsilon_0 + 1$, etc.. Clearly, we can go on and on: any time we have an initial segment of numbers Σ, we can just take its supremum $S(\Sigma)$, and continue on.

—*OR CAN WE?* What about the totality of all numbers?

It is an initial segment of numbers; but if we admit S(all numbers) as a number, then we have to conclude, absurdly, that it is less than itself

$$S(\text{all numbers}) < S(\text{all numbers})$$

In other words,

there is a fundamental problem of distinguishing among those initial segments Σ of numbers to which upper bounds $S(\Sigma)$ can be assigned and those to which not.

Cantor, who was well aware of this problem from the very beginning (but hid mention of it in an endnote!), later on called the segments which have no upper bound *inconsistent multiplicities.* In the later literature, they are called *proper classes* and are distinguished from the segments which

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[2] One of Aristotle's arguments against there being 'actual' infinite totalities is that, to be a totality, it has to be capable of being counted through, and, if counted through, it is finite. For some reason, Cantor agreed with the premise; but the above construction is his counterexample to the conclusion.

do have upper bounds and are called *improper classes* or *sets*. But this is, of course, just naming the problem, not solving it.

What Cantor did was to discover a possibility of interplay between *objects* and *sets of objects*. Unlike speaking about sets of object of some given fixed domain, such as the real numbers, in the case of the transfinite numbers, the notion of set of numbers figures dynamically in the definition of the notion of number, itself. In general, to know what a set of objects of some kind is, you have to know what the objects of that kind are. But, in the case of the transfinite numbers, the converse is also true: to know what the numbers are, you have to know what the sets of numbers are. The situation just reeks of the possibility of inconsistency.

One way to understand the problem—my way—is to see that no precise notion of "all numbers" exists because there is no one precise mathematical characterization of what a *set* or *consistent multiplicity* of numbers is which is adequate.

And once one makes the definition of the numbers precise by imposing some precise condition C on initial segments, so we admit the $S(\Sigma)$ only when Σ satisfies the condition C, then there is no longer any obscurity, and we arrive at the unproblematic totality of all numbers obtained by the above definition so modified:

$$\text{If } \Sigma \text{ is any initial segment of numbers satisfying the condition } C,$$

$$\text{then there is a least number } S(\Sigma) \text{ which is greater than all the numbers in } \Sigma$$

Call the numbers obtained in this way the *C-numbers*. Unlike the situation before, there is no apparent contradiction in admitting

$$S(\text{all } C\text{-numbers})$$

All we can conclude about the totality of all C-numbers is that it itself doesn't satisfy the condition C—If it did, then S(all C-numbers) would be a C-number and we would have the contradiction

$$S(\text{all } C\text{-numbers}) < S(\text{all } C\text{-numbers})$$

just as before.

So we should be willing to introduce S(all C-numbers) as a number; but if we do, *then we can't do it on the basis of the condition C: we have to do so on the basis of a new condition, say C^+.* We can then proceed to consider the totality of all the C^+-numbers and, on the basis of a further new condition, say C^{++}, take *its* least upper bound—and so on. *But this process never ends*: whatever conditions C—whatever axioms—we introduce to obtain new numbers, we will always be led to the well-defined totality of C-numbers, for which it always seems reasonable to take its least upper bound and continue on.

Obtaining new and ever-stronger conditions C or axioms is one of the foremost problems of contemporary set theory. To give some indication of how large the numbers are that have already been obtained, let me say that there are classes of numbers called successively "inaccessible", "huge", "totally indescribable" and "ineffable", all of which have members which now rank as relatively small numbers.

(Its like the attempted decimal rating $5.1 - .10$ of technical rock-climbs: this now includes 5.10 a)-d)-5.14 a)-d), 5.15a; and, for all I know, 5.16's are now being climbed by human flies.)

The theory of these numbers is essentially incomplete: no matter what axioms we adopt, always new and stronger ones will suggest themselves. It is this open-endedness that distinguishes this Fall from that initiated by Galileo and puts it on the plane—in that respect at least—with Adam and Eve.

I should also say that there are many philosophers and mathematicians who don't like the theory of transfinite numbers: its open-endedness makes them uneasy.

And besides, what can you do with such big numbers? They go well beyond Cantor's original counting problem. He himself, a deeply religious man, believed that anything we humans could coherently dream up, God would figure out a use for (to put it somewhat crudely)—a version of Leibniz's principle of plenitude, I suppose. But, if so, God is, so far, way behind on his invention. Not even the smallest inaccessible number plays any role in contemporary empirical science. Another direction current research in this field takes is to look for low-down mathematical applications of these large numbers—say to solve problems about the integers or the real numbers. There are some results of this kind concerning sets of real numbers, but so far leaving open important questions about such sets.

But, anyway, no matter how useless or irritating the transfinite numbers are to some, you just can't make them go away. Skeptical philosophy and other forms of denial may enable one to ignore them; but, they have been introduced and, when we open our eyes, they're still there to taunt us—forever, a barrier to the possibility of any final closure of the mathematical universe.

There's no way back to Paradise. But, like most Falls from Paradise, its a whole lot better than its alternative: In our case, the good side is that Plato's dialectician will never again run out of work.

For irony, try:

> No one shall expel us from the paradise that Cantor has created for us.—David Hilbert

Hilbert was one of the greatest mathematicians at the end of the 19th century. He happily ate the fruit and wrongly thought that we could still go home again.