Cantor's *Grundlagen* and the Paradoxes of Set Theory

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Foundations of a General Theory of Manifolds [Cantor, 1883], which I will refer to as the *Grundlagen*, is Cantor's first work on the *general* theory of sets. It was a separate printing, with a preface and some footnotes added, of the fifth in a series of six papers under the title of "On infinite linear point manifolds". I want to briefly describe some of the achievements of this great work. But at the same time, I want to discuss its connection with the so-called paradoxes in set theory. There seems to be some agreement now that Cantor's own understanding of the theory of transfinite numbers in that monograph did not contain an implicit contradiction; but there is less agreement about exactly why this is so and about the content of the theory itself. For various reasons, both historical and internal, the *Grundlagen* seems not to have been widely read compared to later works of Cantor, and to have been even less well understood. But even some of the more recent discussions of the work, while recognizing to some degree its unique character, misunderstand it on crucial points and fail to convey its true worth.

*This paper was written in honor of Charles Parsons, from whom I have profited for many years in my study of the philosophy of mathematics and expect to continue profiting for many more years to come. In particular, listening to his lecture on "Sets and classes", published in [Parsons, 1974], motivated my first attempts to understand proper classes and the realm of transfinite numbers. I read a version of the paper at the APA Central Division meeting in Chicago in May, 1998. I thank Howard Stein, who provided valuable criticisms of an earlier draft, ranging from the correction of spelling mistakes, through important historical remarks, to the correction of a mathematical mistake, and Patricia Blanchette, who commented on the paper at the APA meeting and raised two challenging points which have led to improvements in this final version.

1 Cantor's Pre-*Grundlagen* Achievements in Set Theory

Cantor's earlier work in set theory contained

1. A proof that the set of real numbers is not denumerable, i.e. is not in one-to-one correspondance with or, as we shall say, is not *equipollent to* the set of natural numbers. [1874]

2. A definition of what it means for two sets M and N to have the same *power* or *cardinal number*; namely that they be equipollent.[1878]

3. A proof that the set of real numbers and the set of points in n-dimensional Euclidean space have the same power. [1878]

So, on the basis of this earlier work, one could conclude that there are at least two infinite cardinal numbers, that of the set of natural numbers and that of the set of real numbers, but could not prove that there are more than two infinite powers. (It was in [1878] that Cantor stated his Continuum Hypothesis, from which followed that in mathematics prior to 1883 there were precisely two infinite powers.)

Cantor's clarification of the notion of set prior to 1883 should also be mentioned, especially in connection with his definition of cardinal number. Bolzano, in his *Paradoxes of the Infinite* [1851], seems to have already clearly distinguished a set *simpliciter* from the set armed with some structure: in §4 he writes "An aggregate so conceived that it is indifferent to the arrangement of its members I call a *set*". But when he came to discuss cardinal numbers he seems to have forgotten his definition of set and failed to distinguish between, say, the cardinal number of the set of points on the line and the magnitude of the line as a geometric object. For this reason, he was prevented from resolving one of the traditional paradoxes of the infinite; namely that, e.g., the interval $(0, 1)$ of real numbers is equipollent to the 'larger' interval $(0, 2)$. Of course, even without the confusion of the set with the geometric object, there is still a conflict with Euclid's Common Notion 5, that the whole is greater than the part. But this principle does indeed apply to geometric magnitude, and it is likely that, without the confusion, it would have sooner been accepted that infinite sets are simply a counterexample to it.

In fact, Bolzano's understanding of the notion of set was in general less than perfect. For example, the word I translated as "members" in the above quote is actually the word "Teile" for *parts*, which he generally used to refer to the elements of a set. Throughout his discussion there are signs that he hadn't sufficiently distinguished the element/set relation from the part/whole relation. For example, the last sentence in §3 asserts that it would be absurd to speak of an aggregate with just one element, and the null set is not even contemplated. But in spite of this lack of clarity, it is to him that we owe the identification of sets as the carriers of the property *finite* or *infinite* in mathematics (§11). In particular, his analysis of the 'infinity' of variable quantities in §12 and the observation that this kind of infinity presupposes the infinity of the set of possible values of the corresponding variable preceded (as Cantor acknowledges in §7) Cantor's own discussion in §1 of the *Grundlagen*, where he refers to what he calls the 'variable finite' as the 'improper infinite'.[1]

Cantor himself, in using the notation $\overline{\overline{M}}$ for the cardinal number of the set M, where the second bar indicates abstraction from the order of the elements of M, betrays some confusion between the abstract set M and M armed with some structure. Moreover, both Cantor and Dedekind avoided the null set. (After all, no whole has zero parts.) It is food for thought that as late as 1930, Zermelo chose in his important paper [1930] on the foundations of set theory to axiomatize set theory without the null set (using some distinguished urelement in its stead). The concept of set is no Athena: school children understand it now; but its development was long drawn out, beginning with the earliest counting and reckoning and extending into the late nineteenth century.

But it was nevertheless Cantor who understood it sufficiently to dissolve the traditional paradoxes and to simply confront Common Notion 5 and define the relation of having the same cardinal number in terms of equipollence. The equivalence of these notions had long been accepted for finite sets; but it was rejected, even by Bolzano, in the case of infinite sets. Prior to Cantor, these paradoxes had led people to believe that there was *no* coherent account of cardinal number in the case of infinite multiplicities.[2] It should be noted, too, that, in the face of the long tradition, from Aristotle through Gauss, of opposition to the infinite in mathematics, it was not only a better understanding of the notion of set that Cantor needed to bring to his definition of cardinal number; it required, too, some intellectual courage.

[1] Bolzano was in turn anticipated by Galileo in his *Two New Sciences*, First Day, where Salviati remarks "The very ability to continue forever the division into quantified parts implies the necessity of composition from infinitely many unquantifiables."

[2] In the fourteenth century Henry of Harclay and, perhaps more clearly, Gregory of Rimini seem to have been at least close to Cantor's analysis; but the soil was thin and the idea did not take root.

In the light of these remarks, it is unfortunate that some contemporary writers on philosophy of mathematics and its history insist on referring to Cantor's definition of equality of power as *Hume's Principle*, for the philosopher David Hume, who explicitly rejected the infinite in mathematics.

It is instructive to compare Cantor's conception of a set prior to his *Grundlagen* with what he writes about it thereafter. As far as I know, his earliest explanation of what he meant by a set is in the third paper [Cantor, 1882] in the series on infinite linear sets of points. He writes

> I call a manifold (an aggregate [Inbegriff], a set) of elements, which belong to any conceptual sphere, well-defined, if on the basis of its definition and in consequence of the logical principle of excluded middle, it must be recognized that it is internally determined whether an arbitrary object of this conceptual sphere belongs to the manifold or not, and also, whether two objects in the set , in spite of formal differences in the manner in which they are given, are equal or not. In general the relevant distinctions cannot in practice be made with certainty and exactness by the capabilities or methods presently availabe. But that is not of any concern. The only concern is the internal determination from which in concrete cases, where it is required, an actual (external) determination is to be developed by means of a perfection of resources. [1932, p. 150]

The latter part of this passage is interesting because it reflects the growing tension within mathematics (and one whose history has yet to be written) over the role of properties which are 'undecidable', i.e. for which we have no algorithm for deciding for any object in the conceptual sphere, whether or not it has the property. Cantor is saying that the existence of such an algorithm is unnecessary in order for the property to define a set. He gives the example of determining whether or not a particular real number is algebraic or not, which may or may not be possible at a given time with the available techiniques. He contends that, nevertheless, the set of algebraic numbers is well-defined. Very likely Cantor took up this issue here because of his proof in [1874] that the set of algebraic numbers is countable and that therefore, in any interval on the real line, there are uncountably many transcendental numbers. This shows that interesting results about a set may be obtainable even when no algorithm exists for determining membership in the set.[3] On

[3]There has been some confusion about the non-constructive character of Cantor's proof

the other end of the ideological scale was Kronecker, who took the view, later associated with Hilbert's *finitism*, not merely that the law of excluded middle should not be assumed, but even more: only those objects which can be finitely represented and only those concepts for which we have an algorithm for deciding whether or not they hold for a given object should be introduced into mathematics. (See [Kronecker, 1886, p. 156, fn. *)].) We shall see that Cantor more fully takes up the defense of classical mathematics against the strictures of Kronecker in §4 of *Grundlagen.*

But to return to the main topic, what did Cantor mean by a 'conceptual sphere'? The answer seems to be clearly indicated by a later passage in the same work, where he writes "The theory of manifolds, according to the interpretation given it here, includes the domains of arithmetic, function theory and geometry, if we leave aside for the time being other conceptual spheres and consider only the mathematical."[1882, p. 152] So I think that Cantor was simply using the expression 'conceptual sphere' to refer to different domains of discourse. Presumably non-mathematical spheres would include that of physical phenomena and of mental phenomena. [4]

Thus I think that in this work and, in fact, up to the discovery of the theory of transfinite numbers, Cantor's notion of a set was the *logical* notion of set, the notion studied in second-order logic, namely of a collection of elements of some *given* domain. This notion of a set would fairly be described

that every interval contains transcendental numbers. His proof that no one-to-one enumeration of real numbers exhausts an interval, while not constructive, is easily reformulated as a constructive proof. His inference from the existence of an enumeration of the algebraic numbers in the interval to one without repetitions is, of course, non-constructive. However, as Howard Stein has pointed out to me, one can constructively enumerate the algerbraic numbers in the interval in a one-to-one manner, although this requires methods, e.g. Sturm's Theorem, that were not available to Cantor.

[4]Cantor's explanation of the notion of set does present another apparent difficulty: his words suggest that equality of objects is relative to the set in question: A set is determined when we have determined what objects of the conceptual sphere are in it and, of two objects in it, whether or not they are equal–in spite of possible formal differences in the manner in which they are given. But here I think that the most reasonable interpretation is that what Cantor actually writes is misleading and that he is thinking of cases such as defining a set of rational numbers by means of a property of pairs of integers or a set of real numbers by means of a property of Cauchy sequences of rational numbers. Thus a Cauchy sequences may have the property in question and so the real number it determines is in the set. What more is required is that we have a criterion for when two such sequences, distinct *qua* sequences, determine the same number and that the property of Cauchy sequences in question respect that criterion.

by saying that a set is, as Frege suggested, the extension of a concept; but, as opposed to Frege's notion of concept, which is defined for all objects, the relevant concept here is one defined only for objects of some given domain. [5] Finally, note that this is the conception of set as it is most often applied in mathematics and that the so-called paradoxes of set theory have nothing to do with it.[6] So much for a summary of Cantor's relevant achievements prior to the *Grundlagen*. I want to turn now to the *Grundlagen* itself.

2 Summary of the Content of the *Grundlagen*

1. The *raison d ètre* of the *Grundlagen* was the theory of transfinite numbers, which Cantor seems to have mentioned for the first time in a letter to

[5] Notice that, in spite of Cantor's avoidance of the null set, it is hard to avoid it on this conception—as Frege pointed out.

[6] It is striking how clear both Cantor's conception of set and his treatment of it are compared to Bolzano's some thirty-one years earlier. In spite of his double abstraction of the set to obtain the cardinal number, mentioned earlier, he in fact is able to distinguish properties of the set in his definition of power from properties that the set may have in virtue of some particular structure on it. Perhaps the explanation of why Cantor was clearer than earlier writers and in particular Bolzano about this lies in his earlier work. The historical confusion was between a geometric object and the set of points constituting it—for example, between a line segment as a geometric object and the set of points on it; and the effect of that confusion was the further one between the measure of the segment (a property of the geometric object) and the number of points on the segment (a property of the set). The explanation I am proposing is that, in his earlier research concerning the uniqueness problem for trigonometric series, he got used to considering sets of points (viz. the domains of convergence of series) to which the notion of measure (certainly as it was understood then) did not obviously extend—in other words, sets devoid of geometric significance. In fact, Cantor didn't even bother to explicitly address the confusion between the set and the geometric object as a problem. He was far more concerned with pointing out the distinction between point sets armed with *topological* structure and abstract sets and with explaining why his proof of the equipollence of point sets of different dimension (e.g., the set of points on the line and the set of points in the plane) does not lead to topological paradoxes. (See [Cantor, 1932, p. 121].) Howard Stein has pointed out to me that, initially, Cantor himself seems to have been not entirely comfortable with his result. In a letter to Dedekind on June 25, 1887 [Nöther and Cavaillès, 1937, p. 35], he expresses concern that it undermines the common assumption that no one-to-one map exists between continuous domains of different dimension. It was Dedekind who pointed out to him in a reply on July 2 [Nöther and Cavaillès, 1937, p. 37] that what had been assumed was that there be no *continuous* such map.

Dedekind in November of 1882. He defines the numbers to be what can be obtained, starting with the initial number and applying the two operations of taking successors (the *first principle of generation*) and taking limits of increasing sequences (the *second* principle of generation). (See §3 below.) He in fact took the initial number to be 1, but it will make no difference if we adopt the now more common practice of starting with 0.[7]

2. The transfinite numbers—and henceforth I will often just speak of the *numbers*, both finite and transfinite—were stratified into the number classes. (See §4 below.) Whereas only two infinite powers were known to exist prior to *Grundlagen*, the number classes represent an increasing sequence of powers or cardinal numbers, in one-to-one correspondance with the numbers themselves. Cantor believed the number classes to represent all powers; and it was this application of his theory of transfinite numbers to the theory of powers that he mentioned in his letter to Dedekind and that he mentions first in the *Grundlagen*. In 1890-1, Cantor introduced his diagonal argument, proving that the set of two-valued functions on a set is of higher power than

[7]The date I assign, November 1882, may seem to contradict Cantor's own statement in §1 that he had already had the notion of transfinite number in earlier works [Cantor, 1880], where he wanted to iterate the operation of taking the derived set (the set of limit points) of a set of real numbers into the transfinite. He introduced what he called the "definite defined infinity symbols", which, in contemporary terms, we would call a *system of ordinal notations*. In fact, Cantor's symbols constitute a very familiar system of ordinal notations, one which represents precisely all of the ordinals $< \epsilon_0$–an ordinal in the second number class. The essential difference between this system of notations and Cantor's later definition of the second number class is not, as it is often stated, that the former is a system of *symbols* and the latter of *numbers*. Rather the important thing is that *any* system of ordinal notations, being countable, will be bounded by some number in the second number class. In fact, in a reasonable sense of the notion of a system of ordinal notations, there is an absolute bound on the ordinal of any such system–namely, the least non-projectable ordinal, which is still in the second number class. To obtain *all* of the second number class, much less the higher number classes, a new idea was needed, namely the idea of introducing numbers as limits of *arbitrary* increasing sequences of numbers (ω-sequences in the case of the second number class). As far as I know, at no time before his letter to Dedekind did Cantor explicitly introduce this idea. So Purkert and Ilgauds, who, in their fine book [1987], claim that the origin of set theory was in Cantor's invention of the the transfinite numbers, a claim with which I entirely agree, may be making a mistake in giving the date of this origin as 1870 (p. 39), when, apparently, Cantor first thought of the ordinal notations. But this may be a matter for more historical investigation. Cantor, himself, in the letter to Dedekind explains that he calls the transfinite numbers *numbers*, in contrast with his earlier name 'infinity symbols', because he has introduced the fundamental arithmetic operations on them.

the set itself[1891], and thus providing another, cofinal, sequence of powers; but in 1883 the number classes were the *only* examples that existed of higher powers.

3. In §2, Cantor analyzed the notion of a counting number, for which he used the term *Anzahl*, and extended it to the transfinite. He saw that counting a set determines a (total) ordering of it and, indeed, a well-ordering. This notion of well-ordering was introduced into mathematics here for the first time. As he noted, the requirement of well-ordering had been obscured by the fact that, in the case of finite sets, *all* total orderings of the set are well-orderings and are isomorphic to one another.

He noted, too, that the set of predecessors of any transfinite number form a well-ordered set and that every well-ordered set is isomorphic to such a proper segment of the numbers. For this reason, his transfinite numbers have come to be called *ordinal numbers* in the literature on the *Grundlagen*, although Cantor himself refers to them simply as *numbers* (*Zahlen*) or as *real whole* (*reale ganze*) numbers. He seems to have wanted to distinguish the numbers themselves from their application as measures of well-ordered sets, just as we may consider the finite whole numbers as they are in themselves, independently of applications as counting numbers or as measures of finite sets.

Failure to see this has led to a minor mystification about Cantor's use of the term *Anzahl* and how it is to be distinguished from his use of the term *Zahl* when speaking about his transfinite numbers. [Hallett, 1984] translates '*Anzahl*' as 'enumeral', which seems a reasonable alternative to 'counting number'; but he interprets the term *Zahl* as referring to ordinal numbers—and then worries about the distinction. An ordinal number for Cantor, when he later introduced the term, is the order type of a well-ordered set, just as a cardinal number is for him the equipollence type of an abstract set. It is true that the numbers (*Zahlen*) represent ordinal numbers, in the sense that every well-ordered set is measured by some number; but the numbers must first be regared as given before the proper initial segments of them can be taken to be measure sticks of the well-ordered sets. Among philosophers, especially, the confusion has been exacerbated by the influence of Frege, who used the term *Anzahl* for the cardinal numbers. Contrary to the general tendency in the late nineteenth century on foundations of arithmetic, Frege considered the natural numbers primarily in their role as cardinals.

The view that Cantor regarded the transfinite numbers essentially as ordinals stands out rather strongly in Hallett's book, where Cantor's actual definition of the numbers seems to be counted as a mistake, and it is only their role as measures of well-ordered sets that gives the numbers substance. One difficulty with this view, aside from the obvious ones that there is nothing wrong with Cantor's definition in the *Grundlagen* as it is and that he does *not* identify the numbers as ordinals there, is that if the numbers exist essentially as measures of well-ordered sets, then how is one to understand the numbers in the higher number classes? The problem is that, at that time, as we have already noted, the only infinite well-ordered sets known to be not isomorphic to well-orderings of the natural numbers or the continuum were those represented by proper segments of the system of transfinite numbers themselves. In response to this, Hallett attempts to construe the construction of the sequence of numbers as a procedure whereby, having constructed a segment of the numbers and recognized it as well-ordered, one then may introduce its order type (p. 57). But this baroque construction is not at all the way in which Cantor introduces the numbers.[8]

Certainly Hallett is right that Cantor believed the application of the transfinite numbers as measures of well-ordered sets constituted an argument for admitting them into mathematics–of 'legitimitizing' them, if you like. Cantor himself mentions this at the beginning of §2 of *Grundlagen*.[9] But, it should

[8]Hallett (pp. 50-51) refers to letters to Kronecker and Mittag-Leffler, both in 1884, in which Cantor seems to be supporting the view that the the foundation for the transfinite numbers should really be their application as measures of well-ordered sets, i.e. that they should be identified with the ordinal numbers. In the letter to Kronecker he writes "I have for some time had a foundation for these numbers which is somewhat different from that given in my written works, and this will certainly suit you better." He goes on to describe a number as the "symbol or concept" of an order type of a well-ordered set. But notice that it is *Kronecker* that the new foundation would suit better. Cantor was unjustifiably optimistic in thinking that Kronecker, who resisted the introduction of the irrational numbers in terms of ω-sequences of rationals, would be interested in even the second number class, that is the totality of all well-orderings of the rationals; but surely his optimism did not extend to thinking that he could interest him in the theory of the higher number classes. But it is connection with them, as we have noted, that the conception of number as order type would have been inadequate. The letter to Mittag-Leffler refers to a manuscript which Cantor never published, but is most likely the one published in [Grattan-Guinness, 1970]. But in this manuscript, Cantor is interested in the *general theory of order types* and in this context, certainly, it is reasonable to identifiy the numbers with the ordinal numbers.

[9]§2 begins

> Since this concept [i.e. of *Anzahl*] is always expressed by a completely determinate number of our extended domain of numbers and since on the other

be noted that the application to well-ordered sets was not the only application of the transfinite numbers that Cantor had in mind: as we have already noted, it was not even the first. The first application that he mentioned in the *Grundlagen* was to the theory of powers. Moreover, in his earlier letter to Dedekind, he also wrote about the issue of legitimatizing his theory; but it was in terms of the application to the theory of powers.[10] But this matter of legitimitization, which we will discuss below, is distinct from the question of whether the notion of transfinite number depends on that of a well-ordered

> hand the concept of *Anzahl* has an immediate objective representation in our inner intuition (*Anschauung*), then through this connection between *Anzahl* and number, the reality that I stress for the latter, even in the determinate-infinite case, is proven.

The reference to inner intuition is surprising. Later on, in the discussion in §10 of the idea of a continuum, Cantor rejects the idea that this can be founded on the concept or intuition of time or on spatial intuition; rather he argues, following Dedekind in [1872] and, indeed, [Bolzano, 1817], that just the reverse is true: our adequate conception of space and time depend upon the analysis of the mathematical idea of a continuum. But he says even more: " even with the help of this latter [i.e. an independent concept of continuity] [time] can be conceived neither objectively as a substance, nor subjectively as a necessary *a priori* form of intuition." Thus he seems in this passage to be rejecting Kant's doctrine of inner intuition, at least in so far as it is identified with time. But, it seems, he is not rejecting it as a basis of the concept of *Anzahl*. On the other hand, note that he is not asserting that it is the basis of the concept of *number*, either finite or infinite. So in this respect he is not in conflict with the view of [Frege, 1884] and [Dedekind, 1887], that arithmetic can and should be developed purely logically, without reference to inner intuiition. But what does he mean when he writes that the concept of *Anzahl* has an objective representation in our inner intuition? My surmise–and it can only be that: I have found no other place in which he discussed this–is that he is referring to the intuitiveness of the idea of iterating an operation, even into the transfinite. Certainly it was that idea, in the case of iterating the operation of taking the derivative of a set of points on the real line, which led him to the transfinite numbers in the first place.

[10] "In this way, by complying with all three elements [i.e. the two principles of generation and the inhibiting principle] one can with the greatest certainty succeed to ever new number classes and powers; and furthermore the new numbers obtained in this way are all entirely of the same concrete determinateness and reality as the old [i.e. the finite numbers]. Thus I truly don't know what holds us back from this process of constructing new whole numbers, so long as it is shown that, for the progress of science, the new introduction of these uncountably many number classes has become desirable or indeed indispensible. And the latter appears to me to be the case in the theory of sets–and perhaps also in a wide range of other cases–at least, without this extension [of the domain of numbers], I can make no further progress and, with it, I obtain much that is entirely unexpected."

set–whether numbers need to be *explained*, as Hallett puts it, as enumerals. And in fact Cantor's definition of the numbers stands on its own feet and is entirely independent of their application e.g. as measures of well-ordered sets.

4. Not only did Cantor introduce the notion of a well-ordered set in *Grundlagen*, but at the beginning of §3, he proposed the Well-Ordering Principle, that every set is well-orderable, as a fundamental law of thought. It follows from this principle that every infinite power is represented in the sequence of number classes. (See §4.) Zermelo, in 1904, deduced the Well-Ordering Principle from the Axiom of Choice, which seems to have been (implicitly) regarded as a part of logic by earlier writers such as Cantor and Dedekind.

5. The *Grundlagen* also introduces for the first time the distinction between sets and what later came to be called *proper classes*. Every well-defined set has a power (§1), but, as we shall see, Cantor recognized that there are totalities, such as the totality of all whole numbers or of all powers, which have no power. (See *Grundlagen*, Note 2 and §3 below.) Often we use the term 'proper class' in a relative sense, to refer to the subsets of the domain a model of set theory which are not coextensive with some element of the domain. But we also use it in the absolute sense to refer, for example, to the totality of all ordinals, independently of the (well-founded) models in which they are represented. It is in this latter sense that Cantor discovered the distinction between a set and a proper class—his distinction between the determinate infinites (represented by the number classes) and the absolute infinite (represented by the totality of transfinite numbers or the totality of the number classes or powers).

6. In §8, in defending the introduction of the transfinite numbers, Cantor gives what may be the first statement and defense of the autonomy of what we would call pure mathematics and which he prefers to call 'free' mathematics. I am not referring to the thesis that reasoning in mathematics should proceed purely deductively, without reference to empirical phenomena, but rather the thesis that pure mathematics may be concerned with systems of objects which have no known relation to empirical phenomena at all. Especially when one remembers how long it took for the various extensions of the number system, 0 and the negative numbers and the complex numbers, to be accepted, it is not remarkable that Cantor felt the need to discuss this matter. Let me quote Cantor on free mathematics:

> Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, established by definitions, to those concepts which have previously been introduced and are already at hand and established. In particular, in the introduction of new numbers it is only obligated to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to the older numbers that they can in any given instance be precisely distinguished. As soon as a number satisfies all these conditions it can and must be regarded in mathematics as existent and real.

We are justified in regarding the numbers as real in so far as the system of transfinite numbers has been consistently defined and integrated with the finite numbers.

Cantor's argument for the 'freedom' of mathematics and the reality of the transfinite numbers is based on his distinction between 'immanent' or 'intrasubjective' reality and 'transient' or 'transsubjective' reality. Of the former he writes

> First, we may regard the whole numbers as real in so far as, on the basis of definitions, they occupy an entirely determinate place in our understanding, are well distinguished from all other parts of our thought and stand to them in determinate relationships, and thus modify the substance of our minds in a determinate way.

So the reality that he has claimed for the numbers is immanent reality. Concerning transient reality he wrote:

> But then, reality can also be ascribed to numbers to the extent that they must be taken as an expression or copy of the events and relationships in the external world which confronts the intellect, or to the extent that, for instance, the various number classes are representatives of powers that actually occur in physical or mental nature. [11]

[11]It is more or less clear what Cantor meant by "*intrasubjektive*" or "*immanente Realität*" and by "*transsubjektive*" or "*transiente Realität*". But where do these *terms* come from?

But he further argues in §8 that mathematics–that is 'free' mathematics–is constrained *only* by the requirements of immanent reality.

It should be noted that, up to this point in his argument, the application of the transfinite numbers either as measures of well-ordered sets or to the theory of powers plays no role at all. But I think that we may see that role in the 'legitimitization' of the transfinite numbers in the following passage in §8:

> It is not necessary, I believe, to fear, as many do, that these principles [admitting into mathematics objects satisfying the criteria for immanent existence] present any danger to science. For in the first place the designated conditions, under which alone the freedom to form numbers can be practiced, are of such a kind as to allow only the narrowest scope for discretion (*Willkür*). Moreover, every mathematical concept carries within itself the necessary corrective: if it is fruitless or unsuited to its purpose, then that appears very soon through its uselessness and it will be abandoned for lack of success.

Quite simply, the two applications of the transfinite numbers were important to Cantor in establishing his theory of transfinite numbers as a legitimate part of mathematics because it was part of his argument that the theory is fruitful. *This has nothing to do with the internal logic of the theory, only with the question of whether it is worth pursuing.* In particular, it has nothing to do with the immanent reality of the numbers.

It should also be noted that the issue of legitimacy is separate from that of transient reality: the applicability of the transfinite numbers to the theory of higher powers or well-ordered sets is no guarantee that they have any empirical application.

As a matter of fact, Cantor expresses his faith that whatever has immanent reality also has transient reality:

> there is no doubt in my mind that these two forms of reality always occur together in the sense that a concept said to exist in the first sense also always possesses in certain, even infinitely many, ways a transient reality. To be sure, the determination of this transient reality is often one of the most troublesome and difficult problems in metaphysics, and must frequently be left to the future, when the natural developement of one of the other sciences will uncover the transient meaning of the concept in question.

Metaphysics here seems to refer to the inventory of the basic structures of the empirical world–since it may be left to the other sciences to uncover them. I take Cantor to be simply expressing an article of faith here, but not one on which his theory of transfinite numbers in any sense rests. This could be questioned on the grounds that he goes on to write

> The mention of this connection [between the two realities] has here only one purpose: that of enabling one to derive from it a result which seems to me of very great importance for mathematics, namely, that mathematics in the developement of its ideas has only to take account of the *immanent* reality of its concepts ....

It might then seem that Cantor believes that in some sense the validity of free mathematics depends on this article of faith. But in the ensuing discussion he writes

> If I had not discovered this property of mathematics [i.e. its freedom] by means of the reasoning I have described, then the entire development of the science itself, as we find it in our century, would have led me to exactly the same opinions.

I think that the point of these remarks is not to qualify the autonomy of free mathematics; rather it is to argue that, *even if what one is interested in is metaphysics, i.e. the basic structures of the natural world*, one should allow mathematics to proceed freely, because what it develops freely will in the end turn out to be instantiated in nature.

I especially mention this because Hallett takes a quite different stance with regard to this discussion in §8 and one according to which Cantor's view is identified with an unintelligible doctrine that is often attributed to Plato. He writes [1984, p. 18] "And crucially, immanent and transient reality are intimately connected." After quoting Note 6, in which Cantor suggests that his view that what is immanently real will also turn out to be transiently real is in agreement with Plato, he writes

> As Cantor himself says ..., what he proposes is a Platonic principle: the 'creation' of a consistent coherent concept in the human mind is actually the uncovering or discovering of a permanently and independently existing real abstract idea.
>
> It is now clear why Cantor considered mathematics as so free. It does concern itself with objective truth and an independent

> (Platonic) realm of existents in so far as its objects of study are transiently real. But it need not attempt to investigate this transient reality directly, or even worry about the precise transient 'significance' of a concept. All that mathematics need worry itself with is 'intrasubjective' reality, and once this is established it is *guaranteed* that the concepts are also transiently real

It is evident that, for Hallett, perhaps because he was misled by Cantor's use of the term 'metaphysics', transient reality refers, not to the instantiation of the concept in nature, but to an 'independent (Platonic) realm of existents'. But that is not the point of Cantor's reference to Plato: rather he seems to be assuming that Plato also advocated the free development of mathematics and believed that what it created freely would as a matter of fact turn out to be exemplified in the natural world. This reading of Plato, though better than what we usually get, is incorrect; but it *is* the basis of Cantor's note.

Purkert [1989, p. 58] seems to share Hallett's view: he cites as evidence a letter to Everhard Illigens from May, 1886 in which Cantor writes "If I have known the internal consistency of a concept that represents a being, then I am forced to believe by the idea of the omnipotence of God that the being which is stated by the concept under discussion must be realized in some way. With regard to this, I call it a *possible* being but this does not mean that it is realized somewhere and some time and somehow in reality." Purkert concludes from this that "For Cantor's Platonistic ontology of mathematical objects, consistency was a necessary but not a sufficient condition." But, as I understand it, this is not at all what Cantor is saying. He is saying that his theological beliefs lead him to believe that the immanent being must be empirically realized in some way. That it could be so realized is the respect under which he refers to it as 'possible'; but calling it 'possible' does not mean that it has been, is or will be realized, nor does it mean that it lacks immanent being if it fails to be so realized.

It is simply impossible to make sense of the reading of Hallett and Purkert, according to which immanent reality is in some sense wanting and transient reality means 'really exists' in some 'Platonic' sense: Cantor speaks of the further development of the sciences (other than mathematics) uncovering the transient reality; and he refers to the development of function theory as an instance of mathematics proceeding freely, without having first secured the transient reality of its concepts in mechanics, astronomy, etc. Hallett doesn't ignore this; rather he introduces another kind of reality: there is not only immanent and transient reality, but there is also having 'physical applications'. He writes (p.18) "There may be all kinds of ways in which transient reality is manifested; in particular, concepts might be represented or instantiated in the physical world." Thus, being transiently real is manifested by, but not identical with, having empirical application (rather like a calvinist being among the elect is manifested by, but not identical with, having a prosperous farm). But what in the text or in sweet reason prevents Hallett from identifying the latter kind of reality with transient reality? [12]

7. In §4, Cantor defended what has come to be called classical mathematics, in particular the methods in function theory associated with Bolzano, Cauchy, Riemann and Weierstrass, as well as his new theory of transfinite numbers, against the opposition of Kronecker. We noted that he had already in [1882] defended the use of the law of excluded middle in reasoning in arithmetic, geometry and function theory. But in *Grundlagen* the emphasis is more on a related, but somewhat different stricture of Kronecker, namely concerning the kinds of objects that should be admitted into mathematics. For Kronecker, all genuine mathematical propositions must ultimately be interpretable as statements about the natural numbers. Let me quote Cantor:

> In this manner a definite (if also rather prosaic and obvious) principle is recommended to all as a guidline; it should thereby serve to confine the playing out of the passion for speculation and conceptual invention in mathematics within the true boundaries,

[12] Perhaps some hint of his thinking about this is contained in his remarks on the above quote from *Grundlagen* concerning inner intuition. He writes " one can make sense of the passage as follows. We invent the concept of (ordinal) number and even postulate that such numbers exist; but the concept obtains legitimacy and significance and we realize that the postulated objects actually *do* exist by recognizing that they correspond to enumerals of well-ordered sets. This is the explanation of the perhaps hazily understood concept of number in terms of, for Cantor, the clear concept of enumeral" (pp.53-4). Well, *Cantor* indeed did invent the concept of transfinite whole number; but I doubt that it would have occured to him to *postulate* that they exist: on the basis of their definition, he would *prove* that some exist with this or that property. The concept did indeed obtain legitimacy and significance for Cantor by the recognition that the numbers measure well-ordered sets. But this is *not* an *explanation* of the concept of number in terms of that of an enumeral. In whatever sense the concept of number is hazily understood, namely, because (as we will see) it is essentially open-ended, the fact that the numbers measure well-ordered sets does not in any way disperse the haze. And, moreover, as we noted, being enumerals of well-ordered sets does not bestow transient reality on the numbers, anyway, unless the well-ordered sets in question have transient reality.

> within which it runs no danger of falling into the abyss of the "Transcendent", in which, it is said in order to inspire dread and wholesome terror, "everything is possible". It is uncertain (who knows?) whether it was not just from the point of view of expediency alone that the originators of this doctrine decided to recommend it to the soaring powers, which so easily endanger themselves through enthusiasm and extravagance, as an effective regulation for protection against all error; but a *fruitful* principle cannot be found in it. For I cannot accept the assumption that the originators of this view themselves, in the discovery of new truths, started from these principles. And I, no matter how many good things I may cull from these maxims, must strictly speaking regard them as *erroneous*: no real progress has stemmed from them, and if science had proceeded precisely in accordance with them, it would have been retarded or at least confined within the narrowest of boundaries.

This precedes by four years the well-known footnote in [Dedekind, 1887] in which the author challenges Kronecker to justify his constraints on mathematics.

This concludes my summary of the content of the *Grundlagen*. Given such a rich assortment of original material and given the prominence anyway of the problem of the infinite in the history of philosophy, one would *a priori* have expected the *Grundlagen* to be regarded as one of the great philosophical classics of all time; but in fact, until recently, even in discussions of Cantor's work, it has been largely neglected and, when considered at all, has tended to be viewed through the window of his later papers, leading to serious misunderstanding. This is especially so when, as is often the case, Cantor's theory of sets as a whole is interpreted as 'naive set theory', the ground for Frege's later inconsistent *Begriffsschrift*–as a somewhat imprecise formulation of naive intuitions which Frege, at tragic cost to himself, merely made precise. Of course, the exposition in the *Grundlagen* has also contributed to the lack of appreciation of it. Cantor's exposition of technical arguments is generally quite lucid; but this paper, with its wealth of conceptual, philosophical analysis, does not share that property; and it easily falls prey to those who (following the model of Frege) too quickly attack the choice of words without sufficiently searching for the intended meaning.

Recent scholars such as Michael Hallett [1984], Walter Purkert [1989] and Shaughan Lavine [1994], have recognized the special nature of *Grundlagen*, even within Cantor's *œuvre* on set theory, and to varying degrees have rejected the myth of Cantor's 'naive' set theory; [13] but interesting and enlightening as they are, they still leave without satisfactory answers a number of questions concerning the text. Indeed, each of the three cited works, while advancing our understanding of the *Grundlagen* on the whole, introduces interpretations on important points which, as in some cases I have already made clear, seem to me entirely wrong.

3 The *Grundlagen* and the paradoxes of Set Theory

There is a significant change in Cantor's conception of a set in the *Grundlagen*. In Note 1 he writes:

> By a 'manifold' or 'set' I understand any multiplicity which can be thought of as one, i.e. any aggregate [inbegriff] of determinate elements which can be united into a whole by some law.

The most notable changes in this from his earlier explanation of the concept of set is the absence of any reference to a prior conceptual sphere or domain from which the elements of the set are drawn and the modification according to which the property or 'law' which determines elementhood in the set "unites them into a whole". But I think that a convincing explanation for this change can be found in his theory of the transfinite number; and it will provide a natural transition to the question of the relation of the *Grundlagen* to the paradoxes of set theory. In introducing the transfinite numbers, Cantor employs the notion of set in an *entirely new way*: numbers are defined in terms of the notion of a set of numbers. Essentially, he introduces the transfinite numbers as follows:

$$X \text{ is a subset of } \Omega \Longrightarrow S(X) \in \Omega$$

Here, Ω denotes the class of all numbers and, let me emphasize: X ranges over *sets*. $S(X)$ is intended to be the least number greater than every number in X. So assuming the existence of the null set and unit sets of numbers

[13] The opening paragraphs of Lavine's book are especially pleasing.

$$S(\emptyset) \text{ should be the least number } 0$$

$$S(\{\alpha\}) \text{ should denote the successor of } \alpha.$$

and, if X is a set of number with no greatest element, then $S(X)$ is the limit of the sequence of numbers in X arranged in natural order. We omit the definitions (which Cantor also doesn't bother to give) of what it means for two numbers to be equal and for one number to be less than another. Cantor also took it to be implicit in his definition of the numbers that there are no infinite descending sequences

$$\alpha_0 > \alpha_1 > \alpha_2 > \cdots$$

so that Ω is well-ordered by $<$.

Cantor's definition of Ω has the familiar look of an inductive definition; but that is deceptive: An inductive definition picks out a subset of some *given* domain of objects by means of some closure condition. The definition of the transfinite numbers, on the contrary, is intended to introduce a whole *new* domain of objects, not a subcollection of a given domain.

But he is not only introducing a new domain of objects, he is introducing it in terms of the notion of a *set* of objects *of that very same domain*. That is, unlike his previous notion of a set, according to which a set is a set of objects from some *given* domain, already well-defined, here the notion of an object of the domain and that of a set of objects of the domain are dependent on each other. This is an entirely new context for the notion of a set. [14]

A symptom of the problem that arises from the interdependence of the notions of (transfinite) number and set of numbers is the

THEOREM. Ω is not a set.

The proof is simply that, otherwise, $S(\Omega)$ would be a number and so $S(\Omega) > S(\Omega) > \cdots$ would be an infinite descending sequence of numbers.

[14] That Cantor speaks of *sequences* of numbers rather than sets is inconsequential, since the sequences in question are in their natural order and so are determined by the set of their members.

Cantor clearly knew this simple theorem. This, I suggest, accounts for the fact that, in his later explanations of the notion of set, including the one in *Grundlagen*, Cantor does not simply take the notion of set to refer to the extension of a concept in some conceptual sphere. For in the conceptual sphere of transfinite number theory, this leads to a contradiction.

There has been much discussion of whether or not he knew the Burali-Forti paradox, the paradox of the greatest ordinal; but behind that discussion lies a misconception that we have already noted, namely of thinking of the transfinite numbers as ordinals. This has lead some commentators to forget Cantor's actual definition of the numbers in the *Grundlagen* and to think of them simply as order types of well-ordered sets. Thus, although I entirely agree with [Purkert, 1989] that it would be difficult to believe that Cantor was not aware that it would be contradictory to assume that Ω is a set; I disagree with him on exactly what contradiction Cantor was likely to have discerned. Purkert cites essentially the Burali-Forti paradox in this connection (p. 57). This paradox involves showing that the totality of all order types of well-ordered sets, if a set O, is a well-ordered set under the natural ordering but one whose order type θ cannot be in O, since otherwise $\theta + 1$ would be in O and so be $< \theta$. But given Cantor's actual definition of the numbers, reference to well-orderings is grotesquely prolix. The contradiction we described above is based, *not on the property of ordinals as order types of well-ordered sets*, but directly on Cantor's definition of the numbers, which admits the more direct and immediately evident argument. It is this latter argument, and not the Burali-Forti paradox, that I am reasonably convinced could not have escaped Cantor's eye. Here I think is one of many instances in the literature on Cantor where a failure to read *Grundlagen* on its own terms rather than through the window of later works somewhat distorts the picture; though it is far from being the worst such instance.

Whether or not Cantor was aware that it would be contradictory to assume that Ω is a set, we have already noted that in Note 2 in *Grundlagen* he had certainly excluded Ω and the totality of all powers as sets. Yet Cantor's comments in this note do not refer to a contradiction as the ground for rejecting the totality of numbers or powers as sets. Rather he writes

> The absolute can only be acknowledged but never be known–and not even approximately known. For just as in the [first number class] every finite number, however great, always has the same power of finite numbers greater than it, so every supra-finite num-

> ber, however great, of any of the higher number classes is followed by an aggregate of numbers and number-classes whose power is not in the slightest reduces compared to the entire absolutely infinite aggregate of numbers, starting with 1.

In fact, this is a strange argument which ultimately makes no sense. For not only the first number class and the totality of all numbers or alephs, but *any* number class N has the property that it has the same power as the set M of all of its elements $\geq$ a given one α. For the function $\beta \mapsto \alpha + \beta$ is a bijection of N onto M. Moreover, when γ is a fixed point $\gamma = \omega_\gamma$ of the initial ordinal function $\alpha \mapsto \omega_\alpha$, then for any α in N_γ the function $\beta \mapsto N_{\alpha+\beta}$ is a bijection from N_γ onto the set of number classes N_β for $\alpha \leq \beta < \gamma$). So, when Purkert refers to a letter to Hilbert in 1897 in which Cantor writes that "Totalities that cannot be regarded as sets (an example is the totality of all alephs as is shown above), I have already many years ago called absolute infinite totalities, which I sharply distinguish from infinite sets", he is almost certainly right in taking the reference to be to Note 2 in *Grundlagen*; but his claim that this is evidence that Cantor at that time already knew the paradox of the greatest aleph is less well-founded.

4 What Numbers/Sets of Numbers Are There?

It was Cantor's construction of the system of transfinite numbers employing the concept 'set of numbers' which opened a Pandora's Box of foundational problems in mathematics, namely, the question of what cardinal numbers there are. One can in a way understand the resistance to Cantor's ideas on the part of the mathematical law-and-order types—in the same way that one can understand the church terrorizing the elderly Galileo: in defense of a closed, tidy universe. In that respect Hilbert's reference to Cantor's 'Paradise' is ironic: it was the Kroneckers who wanted to stay in Paradise and it was Cantor who lost it for us—bless him. I should note, though, that Kronecker went some way beyond the rejection of just Cantor's theory of transfinite numbers. His brand of finitism would have cut back much of the Garden of Eden itself—not just classical analysis, but even constructive analysis in the sense of Brouwer or Bishop.

The latter point is significant because there are many mathematicians who will accept the Garden of Eden, i.e. the theory of functions as developed in the 19th century, but will, if not reject, at least put aside the theory of transfinite numbers, on the grounds that it is not needed for analysis. Of course, on such grounds, one might also ask what analysis is needed for; and if the answer is basic physics, one might then ask what that is needed for. When it comes down to putting food in one's mouth, the 'need' for any real mathematics becomes somewhat tenuous. Cantor started us on an intellectual journey. One can peel off at any point; but no one should make a virtue of doing so.

The question of what numbers $S(X)$ exist is precisely equivalent to the question of what sets X of numbers exist. Before proceeding further, I want to remark on how this fact bears on Lavine's account in [Lavine, 1994] of the notion of set in *Grundlagen*. He understands Cantor to have *defined* a set to be a totality in one-to-one correspondance with a proper initial segment of the numbers (see Definition 2.5 on p.81); but this can't be right. In the first place, in §3 Cantor describes the Well-Ordering Principle as a "*law of thought*"—"that it is always possible to bring any *well-defined* set into the form of a *well-ordered* set–a law which seems to me fundamental and momentous and quite astonishing by reason of its general validity." For some reason, Lavine quotes this very passage as confirmation of his view; but it is a strange use of language to count a definition as a law of thought. Moreover we have already quoted a passage from Note 1 in which Cantor explained what he meant by a set without any reference to the notion of well-ordering. Lavine again takes this passage as confirmation of his view because "Cantor's typical use of the word 'law' in the [*Grundlagen*] is 'natural succession according to law', which suggests quite a different picture [from the one in which 'law' simply refers to some property]": a "law is, for Cantor, a well-ordering or 'counting' ". But the note in question attaches to the first sentence of §1 and the notion of a well-ordering is introduced *for the first time ever* in §2. Moreover, there are, according to my own count, exactly 19 uses of the term "Gesetz" or its plural in the *Grundlagen* besides the one in question and the occurrence of "Denkgesetz" mentioned above and counting one occurrence of gesetzmässig"; and not one of them supports Lavine's contention. [15]

[15] Of the 19, 14 unambiguously refer to theorems or laws of arithmetic. The occurrence of "gesetzmässig" is in a discussion of subsets of R^n defined by any law. Here Cantor clearly is including the case of sets defined by equations; and so the usage is not what Lavine suggests. That leaves four occurrences to consider. 2 of them, and these are possibly the 2 that Lavine considered, are contained in the discussion of well-ordered sets and refer to 'laws of counting' or laws of succession, according to which a set can be counted. But there is no implication in these passages that the set is to be *initially* given by such a

In the second place, and much more decisive: The question of whether an initial segment X of numbers is a proper segment is, as we have noted, *precisely* the question of whether it is a set: it is proper just in case $S(X)$ exists. So, in the particular case of transitive classes X of numbers, Lavine's proposed definition is circular.[16]

Lavine goes on to account for the changes in Cantor's later views by pointing to his discovery in 1891 of the hierarchy of powers arising from iterating the operation of passing from a set M to the set of 2-valued functions on M, starting with the set of finite numbers. His point is that, since it is not clear that these function spaces can be well-ordered, Cantor had to give up Lavine's Definition 2.5 of the notion of set. But that doesn't really make sense. Cantor already knew of the totality of real numbers, and any problems that he had with the well-orderability of the set of two-valued functions on a set he would already have faced in the case of the real numbers. But he explicitly speaks of sets of real numbers and subsets of R^n.

Lavine expresses the view that the alternative to his reading of the explanation of the notion of set in Note 1 is to hold that Cantor's set theory in *Grundlagen* is *naive* set theory.[1994, p.85] But, it is only with his introduction of the transfinite numbers in terms of sets of transfinite numbers that the notion of set became problematic; prior to that, naive set theory—viz. the Comprehension Principle, that every property determines a set—was perfectly valid, since property meant property of objects of some conceptual sphere. With the introduction of the transfinite numbers, though, Cantor immediately recognized that the notion of set was problematic, to the extent of understanding that not *every* property of numbers 'unites the objects possessing it into a whole', thereby determining a set. So, even if we read the passage in Note 1 in the most natural way, he was not naive. Let me note again that defining sets to be totalities equipollent to some proper initial segment of the numbers in no way eliminates this problem of set theory, not for Cantor and not for us, since the question of whether or not an initial segment of the numbers is proper is precisely the question of whether or not it is cofinal with a set.

law. Of the final 2 occurrences, one refers to the natural numbers in their natural order 'according to law' and the other refers to an arbitrary subset of the second number class, determined by some law.

[16]Moreover, the case of classes of numbers is the crucial case. On the usual conception of set theory, every set has a rank and a class of sets is a set just in case the class of ranks of its elements is a set of numbers.

But Pandora's Box is indeed open: Under what conditions *should* we admit the extension of a property of transfinite numbers to be a set—or, equivalently, what transfinite numbers are there? No answer is final, in the sense that, given any criterion for what counts as a set of numbers, we can relativize the definition of Ω to sets satisfying that criterion and obtain a class Ω' of numbers. But there would be no grounds for denying that Ω' is a set: the argument above that Ω is not a set merely transforms in the case of Ω' into a proof that Ω' does not satisfy the criterion in question. So $S(\Omega')$ is a number, and we can go on. In the foundations of set theory, Plato's dialectician, searching for the first principles, will never go out of business.

Cantor himself offered the first answer to the question of what sets exist. For totalities M and N, let

$$M \preceq N$$

mean that there is a function defined on N whose range of values is all of M—i.e. M is of power $\leq$ that of N. Cantor repeatedly took as a sufficient condition for the sethood of M that $M \preceq N$ for some set N, which is essentially the Axiom of Replacement. Relativizing the notion of set in the definition of Ω to totalities of power $\leq$ that of N, we obtain the set Ω', which we denote by $\Omega(N)$. Cantor proved (assuming the Well-Ordering Principle), that $\Omega(N)$ represents the next highest power after N. By iterating the operation $\Omega(-)$ starting with the finite numbers, we obtain (essentially) Cantor's number classes:

$$N_0 \text{ is the set of finite numbers}$$

$$N_{\alpha+1} = \Omega(N_\alpha)$$

and for γ a limit number

$$N_\gamma = \bigcup_{\alpha<\gamma} N_\alpha$$

So the power of N_α is $\aleph_\alpha$.

But, of course, Cantor's answer is only the first in an open sequence of non-trivial answers to the question of what numbers exist. Suitably understood, Cantor's hierarchy yields all the numbers less than the least weakly inaccessible number, i.e. the least regular fixed point $\alpha = \omega_\alpha$ of the initial ordinal function. Namely, if C is the least transitive class of numbers which contains 0, is closed under successors, includes N_α for all $\alpha \in C$ and contains $Lim_{\alpha_i<\gamma}\alpha_i$ when γ and the α_i's are in C, then the least upper bound of C is the least weakly inaccessible number.[17]

[17] I don't know to what extent Cantor understood the open-ended character of the notion

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of transfinite number. In particular, did he believe that the class C that we just described contains all numbers? I don't think that the evidence so far is decisive. At the end of note 2 he writes that the smallness of γ compared to ω_γ, when γ is a successor number, 'mocks all description; and all the more so, the greater we take γ to be'. By excluding the case of limit numbers γ, he excludes the case of fixed points $\gamma = \omega_\gamma$; but let γ be any number and δ a fixed point greater than γ. Then $\gamma \leq \omega_\gamma < \delta$. So the 'smallness of γ compared to ω_γ is hardly increasing the greater we take γ and the difference between γ and ω_γ hardly mocks description. So, despite his exclusion of limit numbers in his statement, it isn't clear that he understood the behavior of the initial ordinal function–or of normal functions in general (i.e. order-preserving number-valued functions defined on the numbers which are continuous at limit numbers)–and, in particular, knew that it had fixed points. Much later, in [1897], he discussed the fixed points of the normal exponential function $\alpha \mapsto \omega^\alpha$ (the so-called *ϵ-numbers*), where α ranges over the second number class. But given the date of this work and the restriction to the second number class, this isn't conclusive.

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