Primitive Recursive Arithmetic and its Role in the Foundations of Arithmetic: Historical and Philosophical Reflections

In Honor of Per Martin-Löf on the Occasion of His Retirement

W. W. Tait *

Abstract

We discuss both the historical roots of Skolem's primitive recursive arithmetic, its essential role in the foundations of arithmetic, its relation to the finitism of Hilbert and Bernays, and its relation to Kant's philosophy of mathematics.

1. Skolem tells us in the Concluding Remark of his seminal paper on primitive recursive arithmetic (PRA), "The foundations of arithmetic established by means of the recursive mode of thought, without use of apparent variables ranging over infinite domains" [1923], that the paper was written in 1919 after he had studied Whitehead and Russell's *Principia Mathematica* and in reaction to that work. His specific complaint about the foundations of arithmetic (i.e. number theory) in that work was, as implied by his title, the essential role in it of logic and in particular quantification over infinite domains, even for the understanding of the most elementary propositions of arithmetic such as polynomial equations; and he set about to eliminate these infinitary quantifications by means of the "recursive mode of thought." On this ground, not only polynomial equations, but all primitive recursive formulas stand on their own feet without logical underpinning.

2. Skolem's 1923 paper did not include a formal system of arithmetic, but as he noted in his 1946 address, "The development of recursive arithmetic" [1947], formalization of the methods used in that paper results in one of the many equivalent systems we refer to as PRA. Let me stop here and briefly describe one such system.

*Is paper is loosely based on the Skolem Lecture that I gave at the University of Oslo in June, 2010. The present paper has profited, both with respect to what it now contains and with respect to what it no longer contains, from the discussion following that lecture.

We admit the following *finitist types*[1] of objects:

$$\mathbb{N} \qquad\qquad A \times B$$

when A and B are finitist types. We also admit as *terminal types* the types

$$A \to B$$

of functions from the finitist type A to the finitist type B. The terms of each type are:

Variables $x^A = x$ of each finitist type A

$$x : A$$

.

$$0 : \mathbb{N} \qquad\qquad t : \mathbb{N} \;\Rightarrow\; t' : \mathbb{N}$$

Primitive recursion

$$s : A \;\&\; t : \mathbb{N} \times A \to A \;\&\; r : \mathbb{N} \;\Rightarrow\; PR(s,t,r) : A$$

where A is a finitist type, with the defining axioms

$$PR(s,t,0) = s \qquad\qquad PR(s,t,r') = t(r, PR(s,t,r)).$$

Corresponding to the cartesian product $A \times B$ we have

$$s : A \;\&\; t : B \;\Rightarrow\; (s,t) : A \times B$$

$$p : A \times B \Rightarrow pL : A \;\&\; pR : B$$

with the defining axioms axioms

$$(s,t)L = s \qquad\qquad (s,t)R = t$$

and

$$(pL, pR) = p.$$

Finally, for terminal types $A \to B$ we have

$$s : A \;\&\; t : A \to B \;\Rightarrow\; ts : B$$

and

$$x : A \;\&\; t : B \;\Rightarrow\; \lambda x.t : A \to B$$

with the defining axiom (lambda-conversion)

$$(\lambda x.t)s = t[s/x]$$

The formulas are built up from equations between terms of the same finitist type by means of implication. The logical axioms and rules of inference are those of identity, implication and mathematical induction

$$\frac{\phi(0) \qquad \phi(x) \to \phi(x')}{\phi(t)}$$

[1]The finitist types are called the *finitist types of the first kind* in “Finitism” [Tait, 1981]. The finitist types of the second kind are the types, corresponding to a constant equation, of the computations proving the equation. We need not discuss these here.

It is easy to show that

$$0 = 0' \to \phi$$

for any formula ϕ. So we may abbreviate

$$\neg\phi := \phi \to 0 = 0'.$$

The axiom of double negation elimination

$$\neg\neg\phi \to \phi$$

is then also derivable.

The Dedekind axioms

$$\neg 0 = t' \qquad\qquad s' = t' \to s = t$$

express that the iteration of successor, starting with 0, is 'free"—there are no loops. It is in virtue of that that definition by primitive recursion is valid. Indeed, given definition by primitive recursion, we can derive the Dedekind axioms. Define *sgn* and *pred* by

$$sgn\, 0 = 0 \qquad sgn\, t' = 0' \qquad pred\, 0 = 0 \qquad pred\, t' = t.$$

Then

$$0 = t' \to sgn\, 0 = sgn\, t' \qquad\qquad s' = t' \to pred\, s' = pred\, t'.$$

Using mathematical induction one easily proves the *uniqueness of definition by primitive recursion*: let $s : A, t : \mathbb{N} \times A \to A$ and $u : \mathbb{N} \to A$. Then

$$\frac{u0 = s \qquad un' = t(n, un)}{ur = R(s, t, r)}$$

The converse is also true: mathematical induction can be derived from uniqueness of primitive recursion. (See [Goodstein, 1945; Skolem, 1956; Goodstein, 1957].)

3. Concerning his general philosophy of mathematics or at least of arithmetic, Skolem, again in the Concluding Remark of [Skolem, 1923], expresses his allegiance to Kronecker's finitism. In particular, he refers to *Kronecker's principle* that a mathematical definition (*Bestimmung*), say of a numerical function or property is genuine if and only if it supplies an algorithm for determining its values. In its consequent rejection of infinite quantifiers, this position also stands with Weyl's later 'intuitionism' [1921] and Hilbert's still later 'methodological' finitism [1922; 1923; 1926].

It is interesting to note that Hilbert, too, studied the foundations of arithmetic in *Principia Mathematica*[2] and, at roughly the same time as Skolem, rejected it, but for a quite opposite reason, namely because with the axioms of infinity and of reducibility, it could no longer claim to be

[2]Concerning Hilbert's flirtation with logicism, see [Hilbert, 1918], his lecture "Prinzipien der Mathematik" in his 1917-18 lectures on logic [Hilbert, 2011, Chapter1] and [Mancosu, 1999; Sieg, 1999a]. Concerning his rejection of it, see his lecture "Probleme der Mathematischen Logik" in [Hilbert, 2011, Chapter 2].

a *logical* foundation. Indeed, it is just as subject to the demand for a consistency proof as the axiomatic theory of numbers, to which Hilbert then returned. The lingering problem for him was to avoid the circle of a proof of consistency of the axiomatic theory that is itself founded on an axiomatic theory. Hilbert and Bernays felt that they solved that problem around 1922 by restricting the methods of proof theory, in which consistency is to be established, to finitist mathematics.

4. For most of us, constructivists or otherwise, arithmetic does not end with Kronecker's principle or with PRA. As Gödel [1958] has taught us, even without the introduction of logic we can extend PRA non-conservatively but constructively by allowing definition by recursion of higher type objects, such as numerical-valued functions of numerical functions, numerical-valued functions of these, etc.[3] If in the definition of the formal system PRA above we abolish the distinction between finitist and terminal types and admit $A \to B$ as well as $A \times B$ to be a type when A and B are, then the resulting system is Gödel's theory T of primitive recursive functions of finite type, provided we add the axioms (so-called *η-conversion*)

$$\lambda x.tx = t$$

when $t : A \to B$ and t does not contain x, and the rule of inference

$$\frac{\phi \to s = t}{\phi \to \lambda x.s = \lambda x.t}$$

when x is not in ϕ.

Actually, this is not quite Gödel's theory, but it agrees with that theory in its arithmetic consequences (i.e. theorems ϕ containing only equations between terms of type $\mathbb{N}$). Objects of types $A \to B$ are interpreted by Gödel to be *computable* (*bereckenbar*) functions from A to B and he interprets equations between objects of this type as expressions of 'intensional' or 'definitional equality'. But, for example, one might prove the equation $sx = tx$, where x is of type $\mathbb{N}$ by induction on x. By the above inference we then have $\lambda xsx = \lambda xtx$ and so, by η-conversion, $s = t$. An application of this is yields the higher-type form of uniqueness of primitive recursive definition:

$$\frac{r0 = s \qquad rn' = t(n, rn)}{r = \lambda nR(s, t, n)}$$

Certainly a notion of equality admitting this inference is not decidable.[4]

But Gödel's conception of T is not entirely satisfactory. Although T is a quantifier-free theory, on his interpretation of it the (domain of

[3] The fact that non-primitive recursive functions such as the Ackermann function, defined by two-fold nested recursions, are definable using primitive recursion of higher type was already shown by Hilbert in [1926].

[4] The difficulty of dealing with equations of higher type was avoided by Spector in [1962] by restricting equations to those between numerical terms and restricting formulas to equations. Equations $s = t$, standing alone, between terms of type $A \to B$ are taken to be abbreviations for $sx = tx$, where the variable x of type A does not occur in s or t. $s = t \to u = v$ is then taken to be an abbreviation for $(1 - sgn|s - t|) + sgn|u - v| = 0$, where $|x - y|$ denotes absolute difference.

objects of) type $A \to B$ is defined only by means of quantification over infinite domains; namely, a computable function of type $A \to B$ is one which, applied to any computable object of type A yields a computable object of type B. If one takes the function to be given by a Turing machine, then the statement that it is computable becomes an arithmetic statement whose complexity grows with the type of the function[5] It seems preferable to accept the notion of function as *sui generis*, to interpret $A \to B$ simply as the domain of functions from A to B, and to understand equations between objects of such a type to mean equality in the usual sense of extensional equality of functions. What makes T constructive is not that it concerns special domains of 'constructive objects' of higher type, but rather that it treats the higher types constructively.[6] This means that Kronecker's principle will be violated and the law of double negation elimination (i.e. classical propositional logic) will not in general be valid for formulas containing equations between terms of higher type; but Gödel's application of T in the *Dialectica* interpretation does not depend on that anyway.

5. Likewise, the requirements of a constructivism more liberal than Kronecker's can be met even with the (non-conservative) introduction of infinite quantification, as in Heyting arithmetic. In this case again decidability of formulas $(\neg\neg\phi \to \phi)$ cannot in general be proved, as Kronecker demands; but it is not assumed as an axiom either. As is well-known, this constructive extension of Kronecker's finitism is from another point of view—a proof-theorist's point of view—just a generalization of Gödel's. For the basic notion of constructive logic is that of a *proof* of a sentence: a sentence is given by stating what counts as a proof of it. So we may think of the sentence as (or at least having associated with it) the *type* of its proofs and we are led to the Curry-Howard theory of dependent types. And when we consider what is to count as a proof of the implication $A \to B$ or quantification $\forall x : A\phi(x)$, we are led to the types $A \to B$ and $\Pi_{x:A}\phi(x)$, respectively. And when we analyze mathematical induction

$$\frac{\phi(0) \qquad \forall x[\phi(x) \to \phi(x')]}{\phi(r)}$$

from this point of view, if s is a proof of the first premise and t is a proof of the second, then the proof p of the conclusion is obtained by primitive recursive definition: $p = PR(s, t, r)$. So, as in the case of Gödel's theory T, we may think of Heyting arithmetic, i.e. the extension of PRA by means of adding infinite quantification, as application of definition and proof by induction to functions and formulas of higher types. This line of thought has of course been developed by Per Martin-Löf, e.g. in [1973; 1998], into a foundation for constructive mathematics. But that is not the direction that I want to take here; rather I want to reflect on the lowest level of the hierarchy of types, the finitist types.

But, before passing on to that, I want to at least mention a difficulty that arises for the theory of dependent types (as opposed to the types

[5] Gödel seems to have tried over and over again (see [Gödel, 1972, footnote *h*]), but I think unsuccessfully, to eliminate this logical complexity.

[6] See [Tait, 2006] for further discussion.

in Gödel's theory T) from the treatment of the concepts of function and higher-order equality that I advocated above in connection with the theory T and say something about its remedy. The relation $s =_{A,B} t$ or for simplicity $s = t$ of extensional equality between objects respectively of type A and type B (unlike that of intensional equality) is definable in the theory itself (see [Tait, 2005a].), but it can happen that the extensionally equal terms s and t are of different (but extensionally equal) types A and B, respectively. For example, it may be that $s = ru$ and $t = rv$, where $r : \forall x : A.\phi(x)$ where x occurs in $\phi(x)$ and u and v are distinct normal terms of type A that are extensionally equal. In that case s and t are of the distinct (but extensionally equal) types $\phi(u)$ and $\phi(v)$, respectively. Because of this, 'substitution of equals for equals' fails for extensionally equality in a strong sense that the substitution may not even be meaningful. For example, if $s : A \to B$, $t : A$ and $t = r$, where r is not of type A, then $st : B$, but sr is not meaningful. However, we can restrict extensional equality to the relations $=_{A,A}$ between terms of the same type and in that case, substitution of equals for equals holds.

6. Let me turn back to the historical question about the source of the idea of founding arithmetic on the 'recursive mode of thought.' Skolem speaks only of Kronecker as a positive influence, but certainly in the passages in which Kronecker expresses his principle concerning mathematical definitions, namely in footnotes in his papers "*Grundzüge einer arithmetischen Theorie der algebraischen Grössen*" [1881, 257] and "*Über einige Anwendungen der Modelsysteme*" [1886, 156, footnote *], there is no specific reference to the recursive mode of thought or, in general, to the systematic foundations of arithmetic. And in his paper "*Über den Zahlbegriff*" [1887] there is no explicit discussion of the principle of proof or definition by mathematical induction at all. In his introductory essay in Skolem's *Selected Works in Logic* [Skolem, 1970, 17 -52], Wang speculates (p. 48) that Skolem might have been at least unconsciously influenced by Grassmann's 1861 *Lehrbuch der Arithmetik* [1904, XXIII]; but although Grassmann, in his axiomatic treatment of the theory of the integers states and uses the principle of *proof* by mathematical induction for propositions about the positive integers and in effect gives the recursive definitions of addition and multiplication of the integers, he does not mention the *general* principle of definition by primitive recursion of arithmetic functions, which is surely the basis of Skolem's paper. [7]

In [1947] Skolem himself states that, as far as he knows, his 1923 paper was the first investigation of "recursive number theory" and he criticizes with some justification Curry's assertion [Curry, 1940] that it had its roots in work of Dedekind and Peano. Peano added to Dedekind's concept of a simply infinite system a framework of formal logic, but he did not even state, much less derive, the principle of definition by recursion as did

[7] Jens Erik Fenstad suggested in conversation that it was perhaps Skolem's close study of E. Netto's *Lehrbuch der Combinatorik* [1901], a work of which he later collaborated in producing a second edition, that led him to his conception of foundations of arithmetic. Again, it is true that the work abounds in proofs by mathematical induction, but nowhere is a function or concept *defined* by induction.

Dedekind. Like Grassmann, he simply introduced as axioms the primitive recursive definitions of the arithmetic operations. As for Dedekind himself, Skolem points out that his foundation for arithmetic had a quite different goal from his own. Indeed, his own motivation, "to avoid the use of quantifiers", was the exact opposite of that of Dedekind who in his monograph *Was sind und was sollen die Zahlen?* [Dedekind, 1888] and along with Frege in his *Begriffsschrift* [Frege, 1879] aimed at the reduction of the concept of finite iteration to logic, employing what we now recognize as second-order logic. Indeed, Dedekind and Frege, rather than Russell and Whitehead, were the real source of the foundation of arithmetic that Skolem was opposing. And Dedekind's set-theoretic and non-algorithmic approach to mathematics was the explicit target of Kronecker's footnotes.

7. Nevertheless, there is an element of truth in Curry's assessment, for there are two themes in Dedekind's monograph. One is indeed the reduction of definition and proof by induction to logic. That reduction can be criticized not only by extreme constructivists such as Kronecker or more moderate constructivists such as Brouwer, but also by those, like Poincaré and (in his predicativist phase) Weyl, who rejected impredicative definitions. For the core of Dedekind's (and Frege's) reduction is his definition of the least set containing a given object and closed under a given function as the intersection of all such sets; and this is the paradigm of an impredicative definition[8] But the other theme of Dedekind's monograph makes it, I believe, a legitimate ancestor of Skolem's paper. For Dedekind was first to explicitly recognize the central role in the foundations of arithmetic of proof and definition by mathematical induction, the first of which he built into his notion of a simply infinite system and the latter of which (in the form of definition by iteration) he derived from it and then used both to define the arithmetic operations and to prove the categoricity of the axioms for a simply infinite set. Indeed, if one replaces the second-order axiom of mathematical induction in his theory of a simply infinite system by the rule of induction in the form

$$\frac{\phi(0) \qquad \phi(x) \to \phi(x')}{\phi(t)}$$

for arbitrary numerical predicates $\phi(x)$ of numbers and the principle of definition by iteration

$$I(s,t,0) = s \qquad\qquad I(s,t,r') = t(I(s,t,r))$$

[8] It is worth noting that in Dedekind's case the impredicativity is limited to (1) constructing from a Dedekind infinite system $\langle D, 0, ' \rangle$ (i.e. where $0 \in D$ and $x \mapsto x'$ is an injective operation on D whose range does not contain 0) the simply infinite subsystem $\langle \mathbb{N}, 0, ' \rangle$, and (2) defining the ordering $<$ of $\mathbb{N}$. According to his well-known letter to Keferstein [1890], (1) was simply part of his proof that the theory of simply infinite systems is consistent. His development of arithmetic from the axioms of a simply infinite system together with the axioms of order ($<$), in particular his familiar bottom-up derivation of the principle of definition by recursion (as opposed to Frege's [1893] top-down derivation), is predicative. (2) is worth noting because $<$ is primitive recursive, but the natural definition of its characteristic function is not by iteration; and the principle of definition by induction that Dedekind proved was restricted to definition by iteration. Dedekind defined the ordering $n < x$ essentially as the least set which contains n' and is closed under $x \mapsto x'$. Dedekind needed the relation $<$ in his derivation of the principle of definition by iteration.

where the operation $I(a, f, n)$ is defined for an arbitrary domain of objects D with $a \in D, f : D \to D$ and $n \in \mathbb{N}$, then we have a logic-free foundation of arithmetic for which Dedekind's proof of categoricity goes through exactly as stated. (We will see that the general principle of definition by primitive recursion is derivable in this framework.) The 'open-endedness' of this system—*arbitrary* numerical predicate $\phi(x)$ and *arbitrary* domain D—may seem to contrast with Dedekind's characterization of the numbers in second-order logic. But, as we know, the range of the second-order quantifier is itself open-ended: by going to ever higher orders, we introduce new sets of numbers into its scope.

An aside: Although Dedekind and Frege shared in the second-order analysis of the notion of a finite iterate of a function (or, in Frege's case, of a binary relation),[9] their conceptions of what would constitute a foundations of arithmetic were profoundly different. In answer to the question "What are the numbers?" Dedekind recognized their *sui generis* character and sought only to uniquely characterize the *system* of numbers, while Frege, focusing on the role of the numbers as finite cardinals, sought to *reduce* them to something else, namely to extensions of concepts. One consequence of his approach was that the principle of definition by recursion does not play an essential role in Frege's foundation, since he defined the arithmetic operations on numbers in terms of the corresponding operations on extensions. It was only in [Frege, 1903] that Frege derived this principle and in effect proved the categoricity of the theory of simply infinite systems.[10]

8. As we just noted, Dedekind proved the principle of definition by iteration, not the general principle of primitive recursive definition. As Peter Aczel pointed out in conversation, the latter principle does seem to have appeared explicitly for the first time in [Skolem, 1923]. It is the special case of iteration $I(s, t, n)$ that is immediately justified on the basis of the notion of the numbers representing the 'free' finite iterations (i.e. without loops): the iteration

$$0, 1 \ldots, n$$

of $'$ starting with 0 is imaged by the iteration

$$s, ts, \ldots t^n s$$

[9][Frege, 1879] and [Dedekind, 1888]. It is worth noting that a version of Dedekind's monograph under the same title can be found in his notebooks dating, as he himself indicates in the Preface to the first edition, from 1872-1878. A reference to this manuscript can also be found in a letter to Dedekind from Heinrich Weber, dated 13 November 1878. (See [P.Dugac, 1976].)

[10]It is usual to say that, whereas Frege understood the numbers as cardinals, Dedekind took them to be ordinals. Although there is some justice in this, there is also an objection: when Cantor introduced the concept of ordinal number, it was as the isomorphism types of well-ordered sets, just as he introduced the cardinals as isomorphism types of abstract sets. Just as Cantor did *not* refer to his transfinite numbers in [1883a] as ordinal numbers (see [Tait, 2000] for further discussion of this), Dedekind did not define the (finite) numbers as order types nor did he refer to them as ordinals. A significant thing about both systems of numbers, Cantor's transfinite numbers and Dedekind's finite numbers, and as opposed to Frege's, is that they are defined intrinsically, without reference to the domain of either sets or well-ordered sets.

$= I(s, t, n)$ of $t : D \to D$ starting with $s : D$. In [1981] I argued directly that definition by primitive recursion followed from the notion of number as the form of finite sequences; but behind the argument was the following reduction of primitive recursion to pure iteration.[11] To carry out the reduction, we need to use the uniqueness of the recursion equations for the identity function $\lambda x : \mathbb{N}.x$ on $\mathbb{N}$ in the form.

$$\frac{r0 = 0 \qquad rn' = (rn)'}{rs = s}$$

For the reduction of primitive recursion to iteration the admission of types $A \times B$ may also be essential. As far as I know, this question remains open. Let $s : A$ and $t : \mathbb{N} \times A \to A$. Define $\bar{s} : \mathbb{N} \times A$ and $\bar{t} : \mathbb{N} \times A \to \mathbb{N} \times A$ by

$$\bar{s} = (0, s) \qquad \bar{t}(n, x) = (n', t(n, x)).$$

and write $f = \lambda n I(\bar{s}, \bar{t}, n)$. ($f$ is a term of terminal type containing any variables that might be in s or t.) Thus

$$f(0) = \bar{s} \qquad f(n') = \bar{t}(f(n)).$$

Note that

$$f0L = 0$$

$$fn'L = \bar{t}(fn)L = \bar{t}(fnL, fnR)L = ((fnL)', t(fnL, fnR)L = (fnL)'.$$

So $\lambda n fnL$ satisfies the same iteration equations as the identity function $\lambda n : \mathbb{N}.n$ on $\mathbb{N}$ and so

$$fnL = n$$

and

$$fn'R = \bar{t}(fn)R = \bar{t}(fnL, fnR)R = \bar{t}(n, fnR)R = (n', t(n, fnR))R = t(n, fnR).$$

So $PR(s, t, n)$ can be defined to be $gn = fnR = I(\bar{s}, \bar{t}, n)R$.

9. This might be a good place to mention the axiom

$$(pL, pR) = p$$

when $p : A \times B$ or $p : \exists x{:}A.B(x)$, which we used in the above reduction of primitive recursion to iteration with $p = fn$. Like the corresponding

$$\lambda x fx = f$$

when $f : A \to B$ or $f : \forall x : A.B(x)$, it is not usually counted as following from the notion of *definitional equality*. This is certainly right historically, but I am not clear on the principle by which it is excluded. Shouldn't it be part of the characterization of $A \times B$, for example, that every object $p : A \times B$ has the form (a, b) with $a : A$ and $b : B$? If so then the equation $p = (pL, pR)$ seems mandatory. (On the other hand, the analogous argument that every object of type $A \to B$ should be of

[11] Such a reduction is carried out in [Robinson, 1947] for the case in which D is finitary type, without the introduction of product types $A \times B$. Instead, she introduced the primitive recursive coding of pairs of natural numbers and then showed that all other primitive recursions could be reduced, using this coding, to iteration.

the form $\lambda x.t$ is perhaps not so compelling.) I would welcome some insight on this. The notion of definitional equality derives from Gödel's *Dialectica* paper [Gödel, 1958], where in footnote 7 he states that "identity (*Identität*) between functions is to be understood as intensional or definitional equality." Clearly that statement in itself does not force the narrow meaning of 'definitional equality' that was later established (by me as well as others). If, as Gödel intended, classical propositional logic is to be applied to equations of higher type, this notion of equality or identity needs to be decidable, but that certainly leaves latitude for our equation $(pL, pR) = p$ as well as the corresponding $\lambda x.tx = t$.[12] Is there a natural notion of intensional equality? Whatever answer this question deserves, I don't see any principled grounds for rejecting the equations in question as intensional equations.

However, there is another problematic ingredient in the reduction of primitive recursive definition to iteration, namely the uniqueness of the iteration equations for $\lambda x : \mathbb{N}.x$, and this is a different matter. Equations obtained from uniqueness of iteration are no more intensional than equations obtained by mathematical induction; indeed, as we noted, the two principles are equivalent. So the equation

$$PR(s, t, .n) = I(\bar{s}, \bar{t}, n)R$$

must be understood as an extensional equality.

Of course, in the case of our present concern, PRA, which deals only with numerical terms, the intensional meaning of an equation $s = t$ between closed terms, namely that s and t compute to the same numeral, and the extensional meaning, that they denote the same number, agree.

10. Closest to Skolem's conception of the foundations of arithmetic among his predecessors seems to be that of Poincaré, especially in [1894], although Skolem makes no mention of him in this context and Poincaré stops far short of Skolem's detailed development of the subject[13]. But as Skolem subsequently did, he explicitly founds arithmetic on what he calls 'reasoning by recurrence', i.e. definition and proof by induction. He doesn't state a general principle of definition by recursion and the only definitions by recursion he actually gives are of the operations of addition and multiplication; so it is not clear whether he had in mind definition by iteration or the more general principle of definition by primitive recursion. Indeed, given his attitude towards formality, it seems possible, even likely, that he never explicitly considered the difference.

Interestingly, both Poincaré and Skolem simply take the initial number, lets say 0,[14] and the successor function $n \mapsto n'$ to be given. "The notions 'natural number' and 'the number $n + 1$ following the number

[12] By the Strong Normalization Theorem, definitional equality means having the same normal form. This theorem is preserved when the conversions of (pL, pR) and λxtx to p and t, respectively, are admitted.

[13] "Here I stop this monotonous series of reasonings." §IV.

[14] Many of the writers that I mention, including Poincaré and Dedekind, in fact take the least natural number to be 1. For the sake of simplicity, since nothing of relevance for us is really at stake in the choice, I will pretend that everyone starts with 0.

n' (thus the descriptive function $n+1$) as well as the recursive mode of thought are taken as basic." [Skolem, 1923]. Poincaré in fact assumes that they are defined and then observes that "these definitions, whatever they may be, do not enter into the course of the reasoning." [Ewald, 1996, 974]. What is striking about this is that both of them were reacting to the logicism of Russell. But for Russell, following Frege, giving a foundation for arithmetic required *defining* the natural numbers, and the logical complexity and in particular the need for the infinitary quantification that so displeased Skolem and enraged Poincaré resulted precisely from the attempt to define them (namely as the extensions of concepts).

For Poincaré the principle of reasoning by recursion is a synthetic *a priori* truth. This thesis was explicitly a rejection of both the logicism of Russell (and Couturat) and Hilbert's axiomatic conception of mathematics. As far as I know, he makes no explicit mention in his discussions of philosophy of mathematics of Dedekind's foundations of arithmetic or of Frege's; but in the first decade of the twentieth century (and so prior to the publication of *Principia Mathematica*, he devoted a series of papers [1905; 1906a; 1906b; 1906c; 1909] to an attack on Russell's logical foundations of arithmetic. Unlike the later rejection by Hilbert and Skolem, many of his objections, especially in the earlier papers, were based on a faulty understanding of the new logic which formed the framework for attempts at logical foundations[15] and added nothing useful to the discussion. But that is not so of his critique in general: two of his objections, in conjunction, are quite telling and indeed were taken over some years later by Hilbert. The first is his critique of impredicative reasoning and his recognition that Russell's foundation of arithmetic, once the Axiom of Reducibility is introduced (as it must be), involves impredicative definitions. As we have already noted, Hilbert was later to conclude that this means that the foundation is no longer a *logical* foundation: quantification over sets of numbers cannot be eliminated in the way that Russell claimed in his 'no-class' theory. The second objection now comes into play and it was equally applied by Poincaré as a criticism of Hilbert's initial foray [1905] into proof theory: The axioms of Russell's theory, like Hilbert's, are subject to the demand for a consistency proof. But Poincaré recognizes, as Hilbert at that point did not, a circularity problem: for example, a proof of syntactic consistency of a system containing mathematical induction would itself inevitably need to employ mathematical induction. This is the circle that Hilbert only confronted many years later, e.g. in [1922] and, as he at least felt, avoided by restricting the proof theory to finitist methods of reasoning.

11. Poincaré's critique applies also to Dedekind, although as we noted, he makes no explicit reference to him. Dedekind's foundation starts with the assumption that there is a Dedekind infinite system, i.e. a set D with an injective function $f : D \to D$ and an element $e \in D$ that is not in the range of f. His claim to be giving a logical foundation of arithmetic was based upon the fact that, in constructing a simply infinite system from a Dedekind infinite system and developing arithmetic in a simply

[15] For a discussion of this see [Goldfarb, 1988].

infinite system, he eliminated the role of inner *Anschauung*. But, as he made explicit in [Dedekind, 1890], he believed, with Hilbert and Poincaré, that a proof that the concept of a Dedekind infinite system is consistent is required.

We have with Dedekind and Poincaré an interesting contrast and, perhaps, the polar opposites in foundations of arithmetic. For the latter, the concept of finite iteration is unanalyzable, given to us in intuition. For Dedekind it has a logical analysis and he believes, contrary to Poincaré and Skolem, that this logical analysis is not just a transformation of intuitive truths into grotesque 'logical' constructions, but that, when we reason by recursion, his logical analysis actually plays out in our minds. In the Preface to the first edition of "*Was sind und was sollen die Zahlen?*" he writes

> . . . I feel conscious that many a reader will scarcely recognize in the shadowy forms which I bring before him his numbers which all his life long have accompanied him as faithful and familiar friends; he will feel frightened by the long series of simple inferences corresponding to our step-by-step understanding, by the matter-of-fact dissection of the chains of reasoning on which the laws of numbers depend, and will become impatient at being compelled to follow out proofs for truths which to his supposed inner intuition (*Anschaaung*) seem at once evident and certain. On the contrary in just this possibility of reducing such truths to others more simple, no matter how long and apparently artificial the series of inferences, I recognize a convincing proof that their possession or belief in them is never given by inner intuition but is always gained only by more or less complete repetition of the individual inferences. [Dedekind, 1963, 33]

The first sentence of the preface is

> In science, nothing capable of proof ought to be accepted without proof.

But presumably Poincaré's answer would be that an argument based on impredicative definitions is not a proof.

12. We have mentioned Poincaré's (well-vindicated) belief that arithmetic is founded on a synthetic *a priori* principle, reasoning by recurrence:

> This rule, inaccessible to analytic demonstration and to experience, is the veritable type of a synthetic *a priori* judgement. [Poincaré, 1894, §VI], [Ewald, 1996, 979]

He goes on to write

> Why then does this judgement force itself upon us with an irresistible evidence? It is because it is only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of the same act when once this act is possible. The mind has a direct intuition of this power, and experience can only give occasion for using it and thereby becoming conscious of it.

And he continues two paragraphs later:

> Mathematical induction, that is, demonstration by recurrence ... imposes itself necessarily because it is only the affirmation of a property of the mind itself.

Poincaré's use of the term "intuition" is in general quite broad (see [Poincaré, 1900]) and it is not entirely clear that he intends his usage in connection with the principle of reasoning by recurrence to coincide with Kant's. He certainly understands himself to be defending a general Kantian point of view: for example

> This is what M. Couturat has set forth in the work just cited; this he says still more explicitly in his Kant jubilee discourse, so that I heard my neighbor whisper: "I well see this is the centenary of Kant's *death*.
>
> Can we subscribe to this conclusive condemnation? I think not, ... [Poincaré, 1905], [Ewald, 1996, 1023]

But it is not clear whether he believed himself to be following Kant in his use of the term "intuition", or even how aware he was of Kant's precise doctrine. There is indeed a difference, but I think one can argue that, at the end of the day and in the case of foundations of arithmetic at least, the difference doesn't matter.

The most important distinction in the usage of "intuition" is between Kant's, according to which intuition is the (non-propositional) *intuition of*, and the meaning according to which it is (propositional) *intuition that*.[16] In the latter sense, intuitive truths historically are those with which reasoning must begin, when premises have been pushed back until no further reduction is possible. It is in this sense, for example, that Leibniz used the term.[17] Poincaré clearly uses the term in the latter, propositional, sense and, on this understanding, given his rejection of impredicative reasoning, it would seem that he is absolutely correct on his own terms in calling the principle of reasoning by recurrence an intuitive truth.

But this is a case of intuition *that*. However much Kant may have on occasion used the term "*inuitus*" or "*Anschauung*" in the propositional sense sense of *intuition that*, it is a fundament of his philosophy to distinguish sensibility, the faculty of intuition, from understanding, the faculty of concepts, and there is no doubt but that, in this context, intuition is intuition *of*, the unique immediate mode of our acquaintance with objects:

[16] It has sometimes been suggested that the difference between these two meanings of "intuition", Kant's and the particular sense of 'intuition that' that we are discussing, deriving from "*intuitus*", was created by translating Kant's "*Anschauung*" into English (and French) as "intuition". But what Kant referred to as "*Anschauung*" in the *Critique of Pure Reason*, he sometimes parenthetically called "intuitus" and also referred to exclusively as *intuitus* in his earlier *Inaugural Dissertation*, written in Latin. (See for example §10.) Thus, in using the term "*Anschauung*", he was merely translating the Latin into German: no new meaning was created by our translation of "*Anschauung*"; it was already there in *his* own use of the term "*intuitus*." An interesting question, which I won't attempt to answer here, is why Kant adapted the term *intuitus* in the way he did.

[17] Thus, the intuitive truths in this sense are the *a priori* truths (i.e. the 'first principles') in the original sense of that term.

All objects are represented in sensible intuition. Abstracted from its empirical content the intuition is just space (pure outer intuition) and time (pure inner intuition). He also speaks of (sensible) intuitions of objects to refer to their representations in intuition. But an intuition by itself is not knowledge: The latter requires recognizing that an object represented in intuition falls under a certain concept or that one concept entails another. *A priori* knowledge of the latter sort, that all S are P, may be analytic, namely when P is contained in S. But, although the truths of mathematics can be known *a priori*, they are not in general analytic. When they are not analytic, the connection between subject and predicate is mediated by *construction.* The demonstration of the proposition begins with the 'construction of the concept' S. Thus, to take one of Kant's examples, to demonstrate that the interior angles of a triangle equal two right angles, we first construct the concept 'triangle'. We then construct some auxiliary lines, and then compute the equality of the sums of two sets of angles, using the Postulate "All right angles are equal" and the Common Notions "Equals added to (subtracted from) equals are equal". The construction of a concept is according to a rule, which Kant calls the *schema* of the concept. In the case of geometric concepts these are or at least include the rules of construction given by Euclid's 'to construct' postulates, Postulates 1-3 and 5. Of course, these rules are rules to construct objects from given objects. For example, given three points A, B, C, we can construct the three lines joining them and thereby, assuming that they are non-collinear, construct the triangle ABC. About constructing the concept Kant writes

> For the construction of the concept ... a *non-empirical* intuition is required, which consequently, as intuition, is an *individual* object, but that must nevertheless, as the construction of the concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept.

The nature of these 'non-empirical intuitions' remains one of the main issues in the study of Kant's critical philosophy. When, in the *Discipline of Pure Reason in Dogmatic Use* (B 741-2), he actually gives the above example of the proof that the interior angles of a triangle equal two right angles, he speaks of constructing an empirical figure or one in imagination, where in the former case (at least) one abstracts from everything we do not intend to be part of the figure.

Kant had in fact very little to say specifically about arithmetic, in the *Critique of Pure Reason* or elsewhere; and what he did say is subject to different readings. He identifies *number* as the schema of magnitude, including both quantity and geometric magnitude. (In the latter case, he has in mind the fundamental role of number in measurement, i.e. in defining ratios in Book V of Euclid's *Elements.*[18]) From his discussion of it in the *Schematism*, number seems to be identified with the rule of representing something in intuition as a finite sequence of objects, and so a particular number, say 5, is the property of a representation of an object in

[18]If A, B and C, D are pairs of like magnitudes, then $A : B \leq C : D$ if and only if for all positive numbers m and n, $mB \leq nA$ implies $mC \leq nD$.

intuition as a sequence of 5 things. Presumably, reasoning about numbers begins with 'constructing' one or more in pure inner intuition (time). But if, in analogy with the case of geometry (see the passage quoted above), the construction is to be 'of the concept of a general representation', then reasoning would seem to begin with 'constructions of *arbitrary* numbers. The development of this conception might indeed lead to a theory of number founded on the the principle of reasoning by induction (see [Tait, 1981]);[19] and this is what I meant by suggesting that, at the end of the day, in spite of the difference between Kant's and Poincaré's use of "intuition", they are essentially in agreement about the foundations of arithmetic. But it is certainly a stretch to think that Kant anticipated such a development or even had a clear idea of arithmetic as opposed to algebra.[20]

13. In name at least, Kant plays a significant role in the ultimate response by Hilbert and Bernays to Poincaré's charge of circularity against Hilbert's earlier approach to proof theory. The role of finitary reasoning in Hilbert's program, as it developed in the 1920's, was this: In order to be assured that the axioms, say of first- or second-order number theory, indeed do define a structure, we must prove them consistent. In non-trivial cases, such a proof would itself seem to involve non-trivial mathematics. If the mathematics involved in the consistency proof were itself founded on a system of axioms, we would be in a circle: Poincaré's circle. Therefore, **a different conception of mathematics** needs to be invoked in founding the methods used in consistency proofs. These methods must themselves be immune to the demand for consistency proof.

For this, Hilbert and Bernays went back to an older conception of mathematics, which is indeed Kantian, according to which mathematics is construction and computation—a conception which, if one didn't look too closely at least, worked quite well in Kant's time and indeed so long as $\epsilon-\delta$ arguments—i.e. logic—could be successfully hidden behind the use of infinitesimals. Of course, $\epsilon-\delta$ arguments were there early in Greek mathematics, in applications of the method of exhaustion. Moreover, Newton was explicitly aware that the use of infinitesimals was just a shorthand and had to be backed up with an $\epsilon - \delta$ arguments. (Leibniz may not have thought that the elimination of infinitesimals was necessary, but he explicitly believed that it was always possible.) However in Kant's time in the eighteenth century, the *calculus* truly reigned. If one looks at Euler's books on function theory, *Introduction to the Analysis of the Infinite* or the *Foundations of Differential Calculus*, after a very brief indication of the justification for using infinitesimals in the Preface, the text looks to be entirely logic-free calculation.

But of course for Hilbert and Bernays, it wasn't all of mathematics that needs to be founded in this way on computation and construction.

[19] One would have, on Kant's behalf, to admit, given the construction of a number $f(X)$ from the arbitrary number X, the iteration $f^Y(X)$ of this construction along the arbitrary number Y.

[20] In his discussion of mathematical reasoning in contrast with philosophical reasoning in the *Discipline of Pure Reason*, he speaks of geometric reasoning and algebraic reasoning but indicates no awareness of the special character of reasoning about the natural numbers.

And this was the appeal of their conception over the severely restrictive view of Kronecker, that *all* of mathematics must be finitist, or the less restrictive view of Brouwer but whose intuitionism would nevertheless still reject much of the analysis developed in the nineteenth century. For Hilbert and Bernays, only the discrete mathematics that is involved in the consistency proofs needed to be founded on this 'Kantian' conception. Once consistency of the formal axiom system was established, the full range of methods coded in it would be available.

So in their quest for a foundation of proof theory, Hilbert and Bernays did indeed turn to Kant, at least in the sense that they returned to a conception of mathematics that was prevalent in Kant's times and that was, indeed, embraced by Kant. But their claims to Kant's authority go beyond that: Hilbert wrote in "On the Infinite" [1926]

> Kant already taught [. . .] that mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone;. . . . Rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

Bernays, in "The philosophy of mathematics and Hilbert's proof theory" [Bernays, 1930–31] endorses what he takes to be

> Kant's fundamental idea that mathematical knowledge and also the successful application of logical inference rests on intuitive knowledge

while distinguishing this from the "particular form that Kant gave to this idea in his theory of space and time" and he sketches a theory of such intuitive knowledge in terms of his notion of 'formal abstraction' and a 'formal object'.

But what has been abandoned, in addition to Kant's particular views about space and time, is his Schematism, the idea that the mind is equipped with rules that govern the application of concepts, i.e. our reasoning about formal objects—our computations or constructions. In [Hilbert, 1926] we are given only a negative injunction that is essentially Kronecker's principle, that the concepts we use should be algorithmic—so, for example, we must reject infinitary quantification in general. But nothing is said about where reasoning about these objects is to begin. Bernays [1930–31, Part II, §1] argues that definition and proof by recursion are valid on this finitist conception when we take the natural numbers to be the signs (formal objects) $||\ldots|$; but his argument for this is in essence simply the usual one, based not on the particular nature of the individual formal objects (the particular nature of 0 and the successor operation $n \mapsto n|$), but on the way that they are generated—by iterating the successor operation finitely often. (See §8 above.) But *the concept of a formal object does not contain this notion of finite iteration.* The gap in Bernays' argument becomes most evident when he acknowledges our 'empirical limitations', the fact that arithmetic concerns numbers such as $10^{10^{20}}$ which are unlikely to occur any way in physical reality. He writes "But intuitive abstraction is not constrained by such limits on the possibility of realization. For the limits are accidental from the formal standpoint. Formal abstraction finds no earlier place, so to speak, to make a principled distinction than at the difference between finite and infinite." But our projection from the number-signs that can be perceived by us, much less empirically realized, to those that cannot is mediated by the concept of finite iteration ("We go on-and-on like that"); and it is *this* concept that is the essence of arithmetic. As Poincaré put it: "these definitions, whatever they may be, do not enter into the course of the reasoning."

14. Kronecker's principle, stated above, allows us to introduce a function only when its definition yields an algorithm for computing its values. But as we know, the question of whether or not the definition of the function actually yields such an algorithm is itself in general a nontrivial arithmetic problem whose solution may depend upon what methods of proof we are willing to admit. On what basis do we accept that the algorithm works—that the definition is legitimate? We might agree that what *can* be proved *should* be proved; but obviously proof has to start somewhere. So, unless we abandon the idea of absolute proof in arithmetic, there must be some principles of arithmetic reasoning that are immune to the demand that we prove legitimacy. These must be the principles that follow from the very conception of the natural numbers and are, as I argued in my paper "Finitism" [Tait, 1981], precisely the principle of definition and proof by induction.

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