I will outline an argument that intends to undermine a certain form of skepticism about mathematics. This skepticism hangs on the fact that mathematics is about ideal things—structures and their elements, such as numbers, sets, functions, etc.—concerning the objective knowledge of which our sensory experience is entirely irrelevant. Precisely because of this irrelevance, the skeptic holds that we cannot meaningfully speak of and therefore cannot know anything about ideal things. Prominent expressions of this skeptical argument in relatively recent times are to be found in Paul Benacerraf’s “Mathematical truth” [1973] and Michael Dummett’s “Platonism” [1978a].

I first wrote on this topic in 1986, in the first part of a paper “Truth and proof: the Platonism of mathematics” [1986]. In writing it, I had the sense of diving into deep waters, well beyond my initial interest in foundations of mathematics. When I retired ten years later, further pursuit of the topic was first on my agenda. It, alas, still is.

There have been those, from Aristotle in fourth century B.C. to Willard V.O. Quine and his followers in the twentieth, who have denied that mathematical objects are ideal in this sense, who believe that in one way or another truth about the natural world is relevant to mathematical truth. But, in all of its forms, this view has squared badly with mathematics, both as it was in Aristotle’s time and as it has been ever since, and it seems fair to see it as a failed attempt to avoid the skepticism as applied to mathematics while accepting the skeptic’s argument. Others, such as Poincaré and Brouwer, in more recent times and at least partially influenced by the skeptic’s argument, have attempted to place the objects of mathematics, if not in the domain of our outer experience, at least in the domain of our inner experience, in the mind. There are also those who accept that mathematical objects cannot

1 Of course empirical concerns may lead to the study of this or that ideal structure—e.g. Euclidean space and the number systems. But I am referring to the nature of that study itself.
be reduced in any way to the empirical but for that reason relegate mathematical theories to the realm of pure formalisms or fictions or whatever. The effect of the skepticism has not been confined to philosophers, but has been felt throughout history by mathematicians themselves in the form of reaction to the introduction of new kinds of object—for instance, the introduction of successive extensions of the numbers systems and, especially in the nineteenth and first half of the twentieth century, the introduction of the actual infinite.

1. Ideal Objects. By saying that mathematical objects are ideal I mean that the truth values of mathematical propositions, including existential propositions, are independent of facts about the natural world. This certainly implies that mathematical objects don’t exist in the natural world, but it implies more than that: the human species, for example, does not exist in the natural world, but facts about the natural world can count as evidence for or against propositions about it.

In ancient Greece, where the ideality of mathematics seems to have been first conceived, the language of mathematics included the language of Euclidean geometry and number theory. For our purposes today, we may think of it as the language of first- or second-order number theory or the language of set theory—although there are alternative languages which express the same mathematics.

Both individually and—we can suppose—as a species, language began for us with the everyday language of observable things and events, and it remains basic to our linguistic activity. The explicit introduction of ideal objects seems to have been very late coming in the development of language—as far as we know, in fourth century B.C. Greece. Even the grammatical categories of talk about the ideal derived from the grammar of talk about natural things. But this is an issue concerning priority, not ontological or epistemic legitimacy. The step from animal recognition of sensible objects and sensible properties to language in which we conceptualize and communicate with each other about them and their workings was large and indeed decisive in the development of our species, I would suppose; but, in the search for further patterns in these workings and to state precise laws governing them, we were led to idealize and, from that, inevitably, to the consideration of ideal things.² That was a large step, too, endowing us with, as Richard Dedekind

²Idealization falsifies: it assumes hard edges in place of what Whitehead called ‘the ragged edges of nature’, exemplified for example by the Sorites paradoxes. Idealized truth about nature needs to be understood as truth about the ideal.
wrote [1932, pp. 488-90], a creative power approaching divinity. Dedekind sometimes wrote as though it were the ideal objects that we created, which of course makes no sense: at what precise moment did we create the number 2? What we did create, over time, was a language, a discourse, in which we speak about them.

In characterizing mathematical objects as ideal, I am excluding such ‘mixed’ objects as the set of books on my desk or, in general, the set of $F$’s, where $F$ is an empirical concept. (These are called ‘impure sets’ in [Maddy, 1990].) As sets, these certainly have an element of the ideal, but the question of what their elements are and their properties is an empirical question and inevitably anything nontrivial we say about it will have empirical content. Charles Parsons [2008, p.35] calls such sets ‘quasi-concrete’, drawing on the distinction between what is concrete and what is abstract. But this distinction slices the pie in a quite different way from the one between the ideal and the empirical that I am drawing. Natural numbers, for example, are concrete: there is nothing from which they are abstracted. But since I take as a mark of the ideal that empirical data are irrelevant to the truth of what we say about it, in my sense such ‘mixed’ objects, along with what Quine [1960, p. 233] refers to as ‘abstract particulars’, e.g. the Equator and the North Pole, are natural.

In contrast with the set of $F$’s, the number of $F$’s, when it is well-defined, is a mathematical object, $n$, even when $F$ is an empirical concept. Quine [1953, footnote 1] attempted to break down the distinction I am drawing here by noting that the statement that there is no ratio between the number of centaurs and the number of unicorns has empirical content. One might question this, as does Saul Kripke [1972, 157] implicitly, on the grounds that the non-existence of mythical creatures is a non-empirical truth; but then Quine’s example can be replaced by ones that do have empirical content—the number of honest politicians and the like. From my point of view, though, the relevant objection to Quine’s argument is that there is an ideal, i.e. non-empirical, content of the statement, namely that division by 0 is undefined: this is a mathematical fact. The statement about centaurs and unicorns, or about honest politicians, is an application of this fact to mixed objects, the set of unicorns, etc. There might have been an honest politician and, if one disbelieves Kripke’s opinion concerning mythical creatures, an animal sufficiently endowed to be called a unicorn; and in that case there would be a ratio. But it is not possible that 0 has a reciprocal. Even when the number of $F$’s is a well-defined number $n$, we can separate the mathematical properties of $n$ from the empirical fact that the number of
I'm not sure who first introduced the expression “abstract object” to cover what I am calling ideal objects, subsuming the issue over the existence of ideal objects—call it the Platonism/antiPlatonism issue—under the medieval Realism/Nominalism debate over the existence of universals. Quine and Nelson Goodman used the term in this way in their “Steps toward a constructive nominalism” [1947] and it has become the standard terminology; but let me explain why I prefer to speak of ideal rather than ‘abstract’ objects. As Frege [1884, §34] noted, abstract objects, properly speaking, ought to be abstracted from something; they are formed by taking something away. And the truth of what I say about the abstract object is derived from the truth about the objects from which it is abstracted; and so abstraction from empirical things yields only empirical objects. I believe that a central ontological issue between Plato and Aristotle concerned whether the objects of geometry, say, are ideal or are abstract. Certainly if one takes them to be abstract, then one must reject the existence of such things as points, lines and surfaces, which we regard as basic geometric objects but which cannot be regarded as abstracted from anything; and indeed, Aristotle and his followers in the middle ages did reject them. But the Realism/Nominalism debate in the middle ages was not about the existence of ideal objects such as these, but rather of abstract objects, in particular of universals. The issue there was whether in the sentences “Socrates is human” and “Plato is human”, there is some one thing, a universal, being denoted by “human”. Of course these abstract objects are not ideal and, in particular, are not mathematical objects. We can replace the universal by the ‘class’ of things falling under it (assuming, contrary to fact in at least most cases, that the class is well-defined), and it is in this way perhaps that the issue of the existence of abstract objects joined the twentieth century debate over the existence of mathematical objects. But the class corresponding to the universal, well-defined or not, is not an ideal object either. Like the case of the set of books on my desk, it is empirical. Abstraction occurs within mathematics, too, as when we abstract from a ring its additive group or the set of its elements—it is this latter abstraction that gives significance to the expression “abstract set”. It follows that the distinctions empirical/ideal and concrete/abstract cut across each other and so both of them should be retained.

\[3\] Names of the form “the number of F's” seem to be unique as names of mathematical objects the reference of which depends upon empirical facts. Expressions such as “the length of \(L\)”, “the temperature of \(W\)”, etc., where \(L\) and \(W\) are empirical objects of the suitable sort (as opposed to idealizations of such objects), obviously do not actually denote real numbers.
2. **Objective Meaning.** The skepticism to which I am referring results from a general view that conceptual reference to objects and more generally conceptual knowledge about them presupposes that they be given to us in some way non-conceptually or at least are of a kind that can be given in this way. In contrast, for example, the natural numbers can be given to us ‘conceptually’: we can characterize the system of numbers to within isomorphism as simply infinite and each number can then be explicitly defined relative to that system. But obviously, the skeptic would not appreciate this brand of ‘givenness’.

I should say that I am using the term “conceptual” here in the objective (Fregean) sense; concepts are something that we can share equally and communicate about with each other; so when I speak of being able to refer to things of a kind M I mean that we can refer to them by name or describe them or refer to them collectively by expressions such as “some M’s” or “all M’s”, etc. in our common language. And by conceptual knowledge about them I mean propositional knowledge. So the skeptic is insisting that linguistic reference to and objective propositional knowledge of objects presupposes that the objects are of a kind that can be given to us non-linguistically. My focus on objective meaning and objective knowledge runs counter to much of the current discussion of language and knowledge in the philosophical literature, in which the focus is on personal meaning and personal knowledge—what I mean by what is said and what I know. My switch from private to public does not reflect simply a desire to change the subject; rather, it is motivated by the conviction, not only that the public cannot be reduced to the private, but that private meaning and knowing ultimately cannot be understood in isolation from the public.

3. **Middle-sized Empirical Objects.** There are elements of irony in taking the existence of the things of the natural world, of tables and chairs, as unproblematic while questioning the existence of ideal things. Some of the so-called Sorites paradoxes strike right at the consistency of our notion of physical object: consider a piece of chalk S. I rub it ever so lightly on

---

4. Of course, there are ideal objects that cannot be ‘given’ in this way: for example, the points in Euclidean space.

5. Frege wrote

   What is objective . . . is what is subject to laws, what can be conceived and judged, what is expressible in words. What is purely intuitable is not communicable. ([1884, §26])

Tyler Burge [2010] catalogues a wide range of uses of the term “objective” of which this is just one
the chalkboard. Is the resulting object \( S' \) identical with \( S \)? Remembering that, if you are going to be stubborn about it, my light rubbing could be replaced by one even lighter or even by softly blowing on \( S \), you are going to have to agree that \( S = S' \); otherwise, the invaluable notion of physical object as we have known it is lost to us, along with the whole language of ordinary physical objects. But then, by the same token, when I rub again, \( S' = S'' \). The point is clear: continuing the sequence of rubs, we eventually obtain some \( S^{(n)} \) which is now just a dab of chalk which no-one would agree was identical with \( S \). So we have \( S = S' = S'' = \cdots = S^{(n-1)} = S^{(n)} \), but \( S \neq S^{(n)} \).

Besides inconsistency, one might challenge the existence of the objects of ordinary sense experience on physical grounds. When we magnify the visual field a billion times, what we call a physical object in the ordinary sense disappears and is replaced by a loosely organized cloud of atoms, with new atoms joining the cloud and others flying off. Precisely what set of atoms corresponds at this precise moment to my body? Of course, if we are willing to sacrifice our rather useful traditional notion of a physical object, the atoms themselves at this level can be taken as the new ‘physical objects’. But scaling even larger, the parts of atoms are revealed, for which the notion of a physical object, with a definite location and motion, ceases to be applicable. The point of this story is not, of course, to undermine talk about the existence of tables and chairs, or of atoms, but rather to shake up a bit, at least in a preliminary way, the grounds upon which their existence is taken to be unquestionable while the existence of ideal things is questioned.

4. Semantics. A natural view of language and of how it works seems to reinforce the skepticism. It is the view of language as representation of objects, events, facts, etc., and involves the idea that words name things and that the relationship among the words in an elementary (‘atomic’) sentence expresses a certain relationship among the corresponding named things—the sentence being true when the latter relationship obtains. I will refer to this as the representational concept of language. As long as we are considering language only in its static descriptive function, which is indeed our primary interest, there is nothing wrong with this picture as a program for describing the semantics of a language. The truth definition relative to a model for a formal language in the framework of predicate logic [Tarski, 1936] shows how it works for those straight-laced fragments of language that can be ‘regimented’ into such a framework. Of course language serves other functions
in our lives than this representational one; but it is a mistake to think that, even where it applies, this semantics provides an ultimate account of the meaning of words in the sense of indicating what the individual needs to know in order to use or understand our common language. In “General semantics” David Lewis makes the point as follows:

I distinguish two topics: first, the description of possible languages or grammars as abstract semantic systems whereby symbols are associated with aspects of the world; and, second, the description of the psychological and sociological facts whereby a particular one of these abstract semantic systems is the one used by a person or population. Only confusion comes of mixing these two topics. [1970, p. 19]

The view that an account of meaning in our sense is provided by a semantics assumes that this or that domain of discourse is given to us as a well-discriminated structure, upon the elements of which we hang our words; and it is given to us extra-linguistically, since it is already presupposed for the understanding or use of language. A clear example of this point of view may be found in Paul Benacerraf’s “Mathematical truth” [1973], which was one of the explicit targets of my “Truth and proof” paper [Tait, 1986]. Benacerraf writes

One of its [i.e., the standard Platonistic accounts] primary advantages is that the truth definitions for individual mathematical theories thus construed will have the same recursion clauses as those employed for their less lofty empirical cousins. (p. 669)

and

I favor a causal account of knowledge in which for \( X \) to know that \( S \) is true requires some causal relation to obtain between \( X \) and the referent of the names, predicates and quantifiers of \( S \). I believe in addition in a causal theory of reference, thus making the link to my saying knowingly that \( S \) doubly causal. (p.671)

An obvious difficulty with the truth definition as an account of meaning in our sense is that it is given in a metalanguage. But what is a metalanguage of English, for example? In fact, on Tarski’s definition of truth, for a given sentence \( S \), the sentence ‘\( S \) is true” is simply a translation of \( S \) into the metalanguage—one is just repeating \( S \), but in another language. Indeed, on any account of truth, the grounds for asserting “\( S \) is true” should be
the same as the grounds for asserting \( S \). This of course does not mean that
the predicate “is true” is superfluous, since there are meaningful sentences
such as “Not every sentence is true” in which the predicate is ineliminable.
Within mathematics there is a role for the notion of truth, namely truth in
a given mathematical structure \( M \); but in this case we are restricting the
notion of truth to the language associated with the particular structure and
the associated metalanguage is the language of mathematics.

Wittgenstein, who himself in his earlier *Tractatus Logico-Philosophicus*
offered a version of the representational concept of language, refers it in the
*Investigations* to a passage in Augustine’s *Confessions*, Book 1, §iv, in which
the author describes his learning of language as a child. Wittgenstein quotes
the passage in §1 and then remarks

> These words, it seems to me, give a particular picture of the
> essence of human language. It is this: the words in the language
> name objects—sentences are combinations of such names.—In
> this picture of language we find the roots of the following idea:
> Every word has a meaning. This meaning is correlated with the
> word. It vis the object for which the word stands.

In §32 Wittgenstein sums up what is wrong with the representational con-
ception as an account of meaning:

> Augustine describes the learning of human language as if the
> child [learning the language] came into a foreign country and
> did not understand the language of the country; that is, as if he
> already had a language, only not this one.

When we think about the relation of language to reality (i.e. semantics), we
are such children perforce. Armed with my command of English, I view and
can describe the natural world as having a more or less definite structure,
consisting of objects of this or that sort, having this or that property and
entering into this or that relationship. So faced with questions of the re-
ference of words of some other language, I can answer by referring to kinds
of objects, properties or relations in my own: I can translate. Of course,
the other language might fail to make distinctions that can be made in our
language; then we have Quine’s ‘indeterminacy of translation’ [Quine, 1960].

But when instead of a foreign country we are at home and the language
is our own—when in Quine’s terms we go ‘domestic’ *(op. cit.)*—we are in a
circle: we can no longer be thinking about translation, and the questions
we ask may become silly: the reasonable question about what the native
speaker of an exotic language means by the word “gavagai” becomes the odd question of what we mean by the word “rabbit”. Its as though we have another language, buried in our psyche, to which we refer in our usage of the word.

The purpose of Wittgenstein’s reference to the passage from the Confessions was not to focus on language learning and the issue of what the infant innately brings to the table in the process; it is rather about the nature of language that the passage (according to Wittgenstein) reveals or at least suggests. But with the reference to language learning there is a possibility of misunderstanding the point I want to make in quoting §32—and here I am not certain about Wittgenstein himself, what he believed with respect to this issue. The issue arises also in connection with what Wilfred Sellars had in mind in what appears to be a gloss on §32):

...unless we are careful, we can easily take for granted that the process of teaching a child to use a language is that of teaching it to discriminate elements within a logical space of particulars, universals, facts, etc., of which it is already undiscriminatingly aware, and to associate these discriminated elements with verbal symbols. [Sellars, 1963, §30]

This passage perhaps contains the suggestion that the child is in no sense able to focus on objects in its environment, re-identify them, recognize properties and changes of properties of them, prior to the learning of language. If that is what Sellars meant—and I do think that there is some ground for thinking that Wittgenstein did mean it—then they are running up against rather convincing evidence that there is significant cognition of objects and their properties preceding the learning of language and in fact essential to it; and I certainly do not want to follow them there. But maybe on the most plausible reading of Sellars, he is warning against the view that learning language presupposes that the child is already ‘undiscriminatingly aware’ of elements of a logical space of particulars, universals, facts, etc., qua elements of that logical space; that given the names, the child already knows their use. I would warn in this connection that the point at which the child has entered the logical space of particulars, universals, facts, etc.—i.e. the point at which it has learned its native language—is ill-defined: at what point are the sounds it makes really words and its gestures endowed with linguistic meaning?

5. The Intervening Mind. Beginning at least with Aristotle’s On Interpretations (16a1 – 8), the investigation of the notions of meaning and
understanding (grasping meaning) has been bound up with the idea of mind as the intermediary between words and that to which they refer: meaning is conferred on my words by what (in some sense) I have in mind when I utter them, and what I have in mind is a result of my interaction with whatever it is—object, property, fact—to which I am referring. And we have (more or less) successfully communicated when that meaning coincides (more or less) with what you have in mind when you hear them.\(^6\)

On the view of meaning as located in the intervening mind, meaning and understanding are the same thing: my meaning when I am speaking and your meaning when you are understanding; so to talk about meaning and understanding is to talk about individual meanings internal to the agent, whether speaker or auditor. From this point of view, and quoting one of its prominent contemporary advocates, “the philosophy of language is a branch of the philosophy of mind.” [Searle, 1983, p. vii]. It draws considerable apparent support from what we may call the view from inside, from our introspective sense of autonomy, that the meaning of the words I speak is the meaning with which I endow them. (Of course, this would seem to imply—wouldn’t it?—that the meaning of your words is the meaning with which you, not I, endow them.) One might also call this view that meaning is in or conferred by the mind, in all of its forms, the Aristotelian point of view, not only because of the source in On Interpretations, but because of Aristotle’s insistence on the autonomy of substances: there are no relations among substances that are essential to any of them. Just as Aristotelian physics held that a change of state (including ‘natural’ locomotion) of a substance is something entirely internal to the substance and not something defined only relative to its environment, the position in question holds that the meaning of what one says is something entirely internal to the agent. If I speak meaningfully or understand what is said to me, this is a fact about me, in its essence independent of what is external to me. Being rational is somehow about me essentially, and is not to be explained by my functioning as a well-behaved unit in vast networks of interactions and communication. One inevitable consequence of this view is that languages are essentially private idiolects: we each have our own language and communication amounts to a handshake across idiolects.

A crucial difficulty with this view is that it fails to account adequately for

\(^6\)Augustine’s actual conception of language was in fact a part of a tradition, stemming from On Interpretations, according to which there is a natural language of the mind, in contrast with the conventional languages by means of which we communicate. On this conception, learning the conventional language is learning to translate between it and the language of the mind.
the objective norms that describe and sometimes govern our use of language in communication. If the only thing I know about Groucho Marx is that he wrote *Das Capital*, then my mind is a poor reference for the meaning and so truth-value of the sentence “Groucho Marx wrote *Das Capital*.” In fact, when my error is pointed out to me, I blushingly stand corrected—not for the sake of a handshake, but because I quickly learn that I was simply wrong in my use of a name in our common language. We may adjust to each other’s idiosyncrasies in speaking (say) English, but the foundation of our communication is that we *are* both English speakers: you speak to an audience in English and the members of the audience are listening to English.

One might want to argue that the misplacing of names is a minor transgression from the public norms: I merely translated a name incorrectly from the internal language of my mind. But Wittgenstein’s discussion of rule-following, which has far wider application than just the following of rules, cuts much deeper. I will say more about this soon; but Saul Kripke clearly states the conclusion of Wittgenstein’s argument in the case of a particular example of rule-following:

> When I respond in one way rather than another to such a problem as “68 + 57”, I can have no justification for one response rather than another ... there is no fact about me that distinguishes between my meaning plus [by “+”] and my meaning quus. Indeed, there is no fact about me that distinguishes between my meaning a definite function by + (which determines my responses in new cases) and my meaning nothing at all.

Kripke counts this as a statement of a skeptical paradox; but it is so only if one begins with the assumption that ‘some fact about me’ (other than the fact that I am a speaker of the relevant language) justifies my response. In fact, we shall see that Wittgenstein’s argument should be read as a refutation of exactly that assumption: he writes at §201:

> This was our paradox.

The emphasis is mine; but it is clear from the passage that no abiding paradox was intended. It should be read as a part of an extended argument for a quite different way of thinking about meaning and the rights and wrongs of linguistic activity.

6. The View From Outside. In opposition to the view from the inside, and drawing, as I have said, on a thread of arguments running through the
Investigations, I want to support a view from outside: our language is not a system of idiolects but a social activity, and what words mean is a question ultimately, not of what goes on in individual minds, but of the role the words play in linguistic interchange, of how they are used in the linguistic community; and that is something that is there for all to observe. Thus I am accepting a version of the formula

\[ \text{Meaning is Use}. \]

But note that when I speak of ‘use’, I am not referring in any way to mental states. I am referring only to the extensional behavior of the language users. And whether or not a person understands the words or, better, the degree to which they are understood is also there for everyone to see: for it is a question solely of the extent to which the person’s usage is in accordance with the meaning—with the communal usage. Let me emphasize that behavior is the criterion, not because the person’s usage is evidence of an internal state, say an Intentional state, but because understanding the words is nothing but the disposition or propensity to use and react to them appropriately—and the criterion for that is the person’s actions, and nothing more. Again, it is important to note here that I am using the term “act” in the extensional sense: turning on the lamp and throwing the switch are the same act. This distinguishes the view that I want to support from that of others, for example John McDowell (see [1981; 1984; 1994]), who would accept the above formula but would infuse the notion of linguistic act, and therefore of linguistic use, with an intentional element. (Don’t be confused by the contrast ‘extensional’/‘intentional’: it arises because the intensional element of McDowell’s non-extensional acts are intensions.)

It follows from this that speaking about meaning makes an implicit reference to a linguistic community, and this is of course a very elastic notion: the relevant boundary of such a community depends upon circumstance and may be drawn variously along geographical, temporal or cultural lines. It may even cut cross the borders of natural languages, as when mathematicians or scientists communicate with at most a rudimentary knowledge of each other’s native language. In such cases, there will be a technical language resting on relatively primitive fragments of the native languages. In consequence of the fact both that what constitutes the linguistic community and that, within such a community, what counts as correct usage are in general only loosely determined, the question of the meaning of a given expression, of what counts as its correct usage, often has no precise answer. Indeed, precision of this kind belongs only to the domain of the ideal. So
henceforth, when I speak of a ‘language’, I will have in mind a linguistic community whose boundary is (roughly) determined by the context.

Also, I don’t hesitate to confuse somewhat the boundary between language and theory. Our ordinary language about physical objects has a theoretical side: it is implicit in our usage that these objects have a unique position, for example, and that if one object \( A \) is taller than an object \( B \), and \( B \) is taller than \( C \), then \( A \) is taller than \( C \). And, in the other direction, one meaningfully speaks of mathematics, for example, as a language.\(^7\)

Language about observable things—physical objects or events, people, etc.—unlike language about ideal objects, can involve pointing. But as Wittgenstein argues in §§28-30, these ostensive gestures are a part of language, not a prelude to it. We indeed use such gestures to name things; but the language, including the practice of pointing, needs to be already in place for the gesture to be meaningful and unambiguous. Am I pointing to the object?, its color?, its time-slice?, its surface?, a direction?, etc. Normally the ambiguity is avoided by the linguistic context of the gesture: “That object”, “that color”, etc.\(^8\)

In contrast with the view from inside, there is no problem about communication on the view from outside: whatever the physical, psychological or social mechanisms that drive it, a language is a system of communication.

I have noted that, on the idiolect conception of language, meaning and understanding (grasping meaning) are essentially the same thing: speaker’s meaning versus hearer’s meaning. According to the view from outside, they are of course significantly different. Given our position that linguistic meaning is ultimately explained in terms of use (in the extensional sense) in a community, the second part of our thesis is forced on us—that the only criterion for whether or not an agent understands a linguistic expression, i.e. grasps its meaning, is what he or she does or would do, again in the extensional sense, \textit{qua} user of the language. There is of course an implicit holism here: understanding an expression presupposes some level of understanding of the language. And whether or not one understands is a question of what one does or would do. Wittgenstein sums up:

\(^7\)Of course, we are not using either of the terms ”language” and ”theory” in the sense in which they are used in logic.

\(^8\)I don’t mean to imply that such gestures do not also have a (call it) ‘natural meaning’, like hieroglyphs, that are in some sense language-independent: for example, pointing, both by the infant and its parent, plays a role in the process of language learning.
To understand a sentence means to understand a language. To understand a language means to have mastered a technique. §199

Clearly there is no threshold level of understanding. Understanding, whether it is of an expression, a rule, a language, or a theory, is a matter of degree. We understand it to the extent that we use and react to it in suitable ways. There are no Cartesian absolutes here. What counts as ‘suitable’ in this respect obviously defies easy description: at what stage does an infant really understand the words that it is beginning to utter? (In this respect, see the discussion in §§156-70 of ‘reading’.) How high does the student have to score on a test in arithmetic to be said to understand the rule for adding numbers? And what counts as a suitable reaction to the command “Stop what you are doing and come here immediately!”? Surely in the latter case it will depend upon circumstances; for example we would certainly not judge in all circumstances that a person who failed to obey the command failed to understand it.

Thus I am taking the position that understanding language is simply being disposed to use and react to it more or less correctly. Viewed solely as a state internal to the agent, linguistic competence is nothing but a system of dispositions or propensities to act and react in certain ways. The sole criterion for whether or not one has such a disposition is how one acts—in the extensional sense of “acts”: this is what Wittgenstein meant when he wrote in §201 of the Investigations that

For what we thereby show is that there is a way of grasping a rule which is not an interpretation, but which, from case to case of application, is exhibited in what we call “following the rule” and “going against it.” §201

The specific issue in this passage is understanding rules; but the larger context is the understanding of linguistic expressions and language in general and the only criterion for understanding is what in fact the agent has done, does or would do.

Of course, having a language is more than just having a complex system of dispositions to use and react to sounds and marks. The dispositions need to add up to a competence, say a competence in English. But—and here my anti-Aristotelianism is showing—what makes it a competence is nothing internal to the agent; rather it is that the dispositions are in conformity with the communal linguistic practice. Linguistic competence is not a property of the agent in isolation; it makes essential reference to the linguistic community.
It is possible that the sheer complexity of the network of dispositions to act and react that are involved in what we would judge to be linguistic competence has led many to postulate something in the mind that is outside the scope of ordinary natural science, intentions perhaps, to account wholesale for our linguistic competence—just as the sheer complexity of many systems that have evolved in nature has led people to postulate that they did not naturally evolve at all but are the creatures of an Intelligent Designer. But the black box of Intentionality, like the black box of Intelligent Design, blocks off the search for a truly informative account of the complex phenomena.

Notice that I say “has led many to postulate” here. Surely the idea that there is something in the mind that accounts for the meaningfulness of our linguistic behavior, something other than simply the dispositions to behave thus and so, is suggested by the fact that we sometimes are consciously ‘meaning’ or ‘intending’ something. But, however that conscious meaning or intending is to be understood, it is far from accounting for very much of our linguistic behavior. Searle himself pointed out [1983, p. 2] that, in order to provide a full account of rationality in terms of intentions, we must admit of intentions of which we are unaware. For example, in reading these lines, there are no conscious ‘meanings’ or ‘intentions’ behind my reading: I simply read them—and in the same sense, I simply wrote them. When I am balancing my accounts, I am generally not consciously intending to do arithmetic, to follow the rules of addition; if I am thinking about anything at all, its more likely to be about how I managed to spend so much money. Once in awhile, one gets confused and consciously appeals to the rules for adding or, more likely, subtracting numbers in order to get back on track. And once in awhile we appeal to the rules to convince others, e.g. students. But for the most part, when we act linguistically, we are acting unreflectively—even when we are conscious:

When I follow a rule, I do not choose. I follow the rule **blindly**.

§219

So the intentionality hypothesis is just that: it is *postulating* something that *lies behind* our linguistic activity and is intended to explain it. But what have these behind-the-scenes meanings or intentions to do with the conscious sense of meaning or intending that was their origin?

7. **Linguistic Rights and Wrongs.** I pointed out earlier that a difficulty with the conception of common language as founded on idiolects is
its inability to account for objective norms. Interestingly, early commentators on Wittgenstein’s later philosophy, such as Michael Dummett [1959], Barry Stroud [1965], Jonathan Lear [1982] and Saul Kripke [1982], held that Wittgenstein’s position undermined the possibility of objective norms. Although they understood more or less correctly what he had to say about meaning and understanding, the difficulty they had with his position was that it seemed to commit him to an empiricist view of the nature of grammatical/logical norms: if the warrant for my linguistic behavior is not internal to me but rather resides in the communal practice, then what is right or wrong linguistically or logically is an empirical question, the answer to which is to be found by investigating how people actually behave linguistically. It is this conclusion that has led to the view that Wittgenstein, himself, was more or less a skeptic.

McDowell has rejected this empiricist interpretation of Wittgenstein, attributing to him a view that McDowell himself defends. He agrees that, suitably understood, meaning is use and that understanding is a competence; but for him “use” is no longer to be understood extensionally and “competence”, viewed as a property of the agent, no longer refers simply to dispositions to act (in the extensional sense) in suitable ways. For him, “use” here must mean ‘mindful’ use, where the mind shares in a ‘second nature’, something specifically human and achieved by suitable training. Under McDowell’s conception, objective meaning is brought into contact with private meaning because all the agents in question are acting in the light of second nature. My inner compulsion to make a particular linguistic or logical move is an expression of objective norms because the compulsion is triggered by our common second nature.9

For McDowell, second nature lies outside the domain of natural science, including cognitive science. He draws here on Sellars’ distinction between the logical space of causes and the logical space of reasons. It is only with reference to the later space that we can make sense of talk about linguistic and logical norms; and on McDowell’s view, it is in virtue of our second nature that we gain entry to the logical space of reasons. In contrast, from our point of view, to be in the logical space of reasons, which would be better named as the space of the meaning-dependent, simply is to be disposed to interact in suitable ways linguistically. Thus the logical space of reasons remains in the natural world: one is in it in virtue of one’s (extensional) relationships to other natural objects. And contrary to McDowell’s position, second nature is a part of nature. It is certainly true that linguistic and

---

9I have discussed this view at greater length in “The myth of the mind” [2002].
rational competence is more complex than other kinds of abilities, such as gymnastic abilities, that we may acquire by training (and it was this contrast that was originally made by Aristotle [Nicomachean Ethics, Book 2] when he coined the expression “second nature”), but as I have already suggested, complexity does not argue for extra-natural. It seems already quite clear that, however complex the learning of linguistic skills may be, it is in a continuum with the learning of other skills, cognitive and non-cognitive, and is within the domain of natural science.

But anyway a second nature, outside the ‘logical space of causes’, is not needed to account for norms and the role that they play in our linguistic lives or, therefore, to save Wittgenstein’s position from skepticism about norms. Within the linguistic community, there simply is agreement in use of language: that is what makes it a linguistic community. But—and here the overlap of language and theory comes into play—an element of this agreement in use of language is agreement about the use of language, a ‘second-order’ level of agreement about various linguistic rights and wrongs—about what we should or should not do linguistically: about correct word usage, grammatical and logical construction, and so on. But this agreement is not the result of an empirical investigation of what members of the community actually do—indeed, as we who have been teachers well know, in many cases of linguistic rights and wrongs, it runs counter to what the majority of members of the community actually do. Knowing the rights and wrongs of language, being able to evaluate linguistic behavior, is just another level of knowing the language itself, of being a member of the community. We not only can add numbers, but we can check whether a computation is correct. We not only go 2, 4, 6, . . . 1000, 1002, 1004, . . ., but we know that 2, 4, 6, . . . 1000, 1002, 1006, . . . is wrong. That this second-order knowledge of rights and wrongs of linguistic behavior conforms, to the rough extent that it does, with our ‘first-order’ linguistic practice is simply a fact of nature, perhaps to be explained in part by the need for instruction in linguistic behavior in order for the language to survive from one generation to the next.

Following Wittgenstein, I have already suggested that the role of norms in our linguistic and rational life tends to be severely overblown. We are not walking rule books. Once in a while we consult the rules or at least think about them, either to guide ourselves or to convince others; but for

---

\[10\] A simple adding machine can do the first, but not the second. This suggests a sense in which the reflection principle “S knows P implies S knows that she knows that P” can be questioned.
the most part we simply act, unreflectively, more or less in accordance with
them. There well may be—indeed it would be shocking to me if there were
not—laws accounting for our action; but these would be causal laws, not
normative ones.

Language is of course an empirical phenomenon and what counts as right
or wrong in linguistic usage is a contingent matter—it might have been
otherwise. One might for that reason feel that laws of logic or arithmetic,
for example, are contingent, that they might have been otherwise. But there
is a confusion here: what we may conclude from Wittgenstein’s argument
to be contingently true is not this or that sentence but rather the fact that
we use the constitutive terms in the sentence in the way we do. Indeed, the
fact that we use any expression in the way that we do is always a contingent
matter. But this in no way implies that, in using the constituent expressions
in the way that we do, we are not expressing a necessary truth. Of course,
there has in fact to be some constancy in usage in order for there to be
grammatical and logical norms: without some such constancy, there would
not be a viable notion of truth (applied to sentences)—there would not be
a language. In this way, we may say that grammatical and logical necessity
are founded on what is contingent; but this does not destroy their necessity.
It is an empirical contingent fact that “2 + 2 = 4” is a sentence and expresses
a truth, but it is not an empirical or contingent fact that 2 + 2 = 4. This is
at least part of what Wittgenstein meant when he wrote

So you are saying that human agreement decides what is true
and what is false?”—It is what human beings say that it true
and false; and they agree in the language they use. That is not
an agreement in opinions but in a form of life. §241

It is an empirical fact that we can learn to reason and communicate; that
with suitable preparation, the student will continue the series

\[0, 2, 4, \ldots, 1000, 1002, 1004, \ldots\]

and not

\[0, 2, 4, \ldots, 1000, 1002, 106, \ldots\]

and will add numbers according to plus and not Kripke’s quus [Kripke,
1982]; that with suitable prompts we will interpret the native speaker’s
“gavagi” as referring to rabbits and not time slices of rabbits or whatever
[Quine, 1960]; and that when we witness lots of green things having the
property \(P\) and no green things without that property, we will take this as
evidence that green things are \(P\) and not evidence that grue things are \(P\)
[Goodman, 1973]. These are examples of the kinds of empirical conditions under which our kind of language and reasoning are possible. Language and science build on top of this.

8. Ontology. My argument has been that meaningful language about objects does not presuppose a language-independent interaction with them. Physical interaction with objects obviously played a crucial role both in the historical evolution of languages and in our individual language learning. But that interaction should not be confused with reference, an objective relation that linguistic terms have to objects. The interaction in question is between a language user and an object; reference is between a term in the language and an object. The ontological corollary of the formula “meaning is use” and the one pertinent to our ultimate aim concerning objective knowledge of ideal things, is expressed early in the *Investigations*. A paraphrase of part of §10 is

*What a word signifies is shown by the kind of use it has.*

From the context in §10, it is clear that “signifies” here has or at least includes the sense of “refers to”, in which case what is signified is an object. In the context of Wittgenstein’s philosophy in *Investigations*, this corollary is an enrichment of Frege’s so-called ‘context principle’

...we ought always to keep before our eyes a complete sentence. Only in a sentence have the words really a meaning. It is enough if the sentence taken as a whole has a sense; it is this that confers on its parts also their content.

...

The self-subsistence that I am claiming for number is not to be taken to mean that a number word signifies something when removed from the context of a sentence, but only to preclude the use of such words as predicates or attributes, which appreciatively alters their meaning. [Frege, 1884, §60]

Of course the meaning of the second paragraph is not that number words have no meaning when standing alone, but rather that their meaning is derived entirely from the meaning of the sentences containing them. That much follows from the first paragraph, at least if one assumes that Frege believes that the ‘content’ of a term determines its reference. The reference to ‘self-subsistence’ is occasioned by an unfortunate feature of Frege’s ontology—his distinction between objects, i.e. self-subsistent objects, and
functions, which include concepts as truth-valued functions, and which are ‘incomplete objects’. Frege introduced the idea of incomplete objects presumably in the service of a compositional semantics, according to which the truth-value of $\forall x P(x)$, for example, is to be understood as obtained from the reference of $\forall$ and the reference of $P(x)$. However, bound variables, which motivated this idea, are in principle eliminable in such a manner that compositional semantics is preserved.\footnote{See the discussion and further references in W.W. Tait, “First-order Logic without bound variables: Compositional Semantics”, to appear in \textit{Dag Prawitz on Proofs and Meaning}, edited by Heinrich Wansing, \textit{Studia Logica}. The mode of composition in this ‘compositional semantics’ is application $Fa$ of a function $F$ to an argument $a$, where here “function” is used in its proper sense—not referring to Frege’s ‘incomplete objects’, but rather (essentially) to their courses-of-values in his sense. Frege’s problem with this seems to be a version of the third man argument: the notation $Fa$ involves not only the names “$F$” and “$a$”, but also the concatenation $xy$ representing the application of a function to an argument, i.e. denoting a doubly incomplete object. We may represent that incomplete object by a complete one $\Phi$, so that $Fa$ is $\Phi Fa$, but then we have a triply incomplete object: we are in infinite regress. The short answer to this argument is that, yes, $Fa$ implies $\Phi Fa$, but it doesn’t refer to $\Phi$ explicitly. Whatever infinite regress there is, it is not vicious.}

§10 is an enrichment of Frege’s §60 because, as Michael Dummett has pointed out in \textit{Frege: Philosophy of Mathematics} [1991], in the context of Wittgenstein’s account of meaning and understanding meaning it fills in “two lacunas in Frege’s account, at opposite ends: one at the end of sense, and the other at that of a speaker’s grasp of sense.” (p. 16.) Although Frege’s context principle refers to the sense of sentences and he speaks of the ‘grasping of sense’, he gives no real account of either the nature of the sense or of what it means to grasp sense. Frege’s position was further obscured by his own later contributions to semantics and in particular by his compositional semantics of what he referred to as ‘sense’ and ‘reference’: the sense or reference of a sentence is a function of the sense or reference, resp., of its components. If one fails to heed David Lewis’ warning (see footnote 6) above) and identifies this semantical notion of sense with the notion of sense of a sentence that occurs in his statement of the context principle, then there is an obvious conflict: the sense the name $t$ derives from that of sentences $P(t)$ containing it, but on the other hand the sense of $P(t)$ derives from that of $t$ and the ‘incomplete object’ $P(x)$. This conflict undoubtedly explains the difference between Dummett’s sympathetic treatment of the context principle in 1991 and his reference to it in his paper “Platonism” [Dummett, 1978a], which contains what may be a classic contemporary statement of the skepticism I am attacking, he writes:
When we scrutinize the doctrines of the arch-Platonist Frege, the substance of the existential affirmation finally appears to dissolve. For him mathematical objects are as genuine objects as the sun and moon: but when we ask what these objects are, we are told that they are the references of mathematical terms, and ‘only in the context of a sentence does a name have a reference.’ . . . So, then, to assert that there are, e.g., natural numbers turns out to be to assert no more than that we have correctly supplied the sentences of number theory with determinate truth conditions; and now the bold thesis that there are abstract objects as good as concrete ones appears to evaporate to a tame assertion that few would want to dispute. [pp.213-14]

Of course, a fairer formulation of Frege’s response to the question of what the numbers are is that they are the numbers and the fact that that is what we are referring to is determined by what we say about them.12 We find the root of Dummett’s criticism in the first sentence of the paper:

Platonism, as a philosophy of mathematics, is founded on a simile: the comparison between the apprehension of mathematical truth to the perception of physical objects, and thus of mathematical reality to the physical universe. (p. 202)

In his later writings Dummett presumably came to understand Lewis’s distinction between semantical structure, concerning which Frege was a pioneer, and meaning in the sense that is in play for a correct understanding of the context principle, concerning which Frege was silent; and that the Investigations can be read as filling the gap in Frege’s argument that the natural

---

12It is reasonable to suppose that Tyler Burge had at least this passage in mind when he wrote in “Frege on knowing the third realm” [1992]

Some philosophers have suggested that Frege’s use of the context principle somehow suggests a qualification on his Platonism. Issues surrounding Frege’s context principle(s) are, of course, extremely subtle and complex. But it seems to me that the suggestion must involve some confusion. The context principles govern relations between linguistic expressions and their senses or referents. They do not bear directly on the nature of the senses or referents themselves at all. (p. 640 n. 11)

Burge’s use of the plural seems to be at least partially motivated by a possible ambiguity concerning whether context in propositions determines the sense of a word or simply its reference. It seems clear though that, although Frege had not yet stated his principle that sense determines reference, he had to be referring to sense—and therefore to reference as well.
numbers are a part of our ontology.\textsuperscript{13}

If one replaces Dummett’s reference to Platonism as a philosophy of mathematics with reference to the actual development of mathematics as a science of ideal things, he is certainly right to the extent that it developed as a class of idealized theories of physical phenomena. But, as we have already pointed out, this is a fact about priority, not about validity.

10. Overpopulation and Non-Uniqueness. One challenge that has been many times raised to the present point of view and, in particular, to the context principle is that it would seem to leave us harnessed with all kinds of ridiculous ontologies. It is a bit ironical that Frege, whose views about ontology were really quite conservative, should be associated with this supposed profligacy. The objection might seem to make sense against the background of Wittgenstein’s fluid conception of language, comparing our language to an ancient city: a maze of little streets and squares, of old and new houses, of houses with extensions from various periods, and all this surrounded by a multitude of new suburbs with straight and regular streets and uniform houses. §18

What are the criteria for when a ‘genuine’ suburb of our language, in which we are able to speak of new kinds of objects, has been established? The protest is against the idea that new objects are unveiled just because a new language in which we seem to speak about them evolves or is established.

Georg Cantor addressed this objection in the case of mathematics in his Foundations of a General Theory of Manifolds [Cantor, 1883a], in which he first introduced his transfinite numbers. He first of all suggests that the meaning of the usual term pure mathematics is better expressed by the term free mathematics and goes on to write

Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must be both consistent with each other and also stand in exact relationship[s, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. . . .

\textsuperscript{13}In his later writings Dummett adopted the formula ‘meaning is use’ in general terms, but he rejected Wittgenstein’s account of understanding or grasping meaning, the account I have outlined here. But I think his argument for the rejection is weak and I won’t discuss it now. The argument can be found in Chapter 4 of his book The Logical Basis of Metaphysics [1991b].
It is not necessary, I believe, to fear, as many do, that these principles present any danger to science. ... every mathematical concept carries within itself the necessary corrective: if it is fruitless or unsuited to its purpose, then that appears very soon through its uselessness, and it will be abandoned for lack of success. [Cantor, 1883a, §8, paragraphs 4-5]

Cantor's point can be made more general: no truth- or existence-police are needed. If the contractor begins to put up new structures in the outskirts of old town but they bear no relation to the needs or desires of people, no suburb will grow there and he will go broke. If someone down the hall begins developing a weird new 'mathematics', with no interesting relation to what has gone before, worry not! He won't get tenure.

Another objection that has been voiced arises from the fact that a structure like the system \( \mathcal{N} \) of natural numbers can be given to us purely conceptually, i.e. by a propositional characterization. For many different structures may satisfy the characterization, even in the case such as that of \( \mathcal{N} \) in which all the structures satisfying the characterization are isomorphic to one another. Thus, the characterization of \( \mathcal{N} \) can be satisfied if we take the natural numbers 0, 1, 2, ... to be

\[
\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots
\]

or

\[
\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots
\]

Which of these two or indeed which of the infinitely many distinct simply infinite systems is \( \mathcal{N} \)? This is the question posed by Paul Benacerraf in [1974] The answer, given in effect by Dedekind almost a century before in a letter to Weber [1932, pp.488-90], is that \( \mathcal{N} \) is none of these: In the two examples above, the numbers are represented by sets. But numbers are not sets: it is ungrammatical to ask whether one number is an element of another. On the contrary, the numbers are sui generis and the question posed by both Frege and Dedekind, “What are the numbers?”, should not be answered, as it was by Frege, by defining them to be something else, but rather by determining their basic properties, as it was by Dedekind.

There is, however, a somewhat different take on the uniqueness problem: why do the formal properties that we have accepted pick out just one sui generis system of natural numbers? The prima facie more sensible version of this question depends on the fact that formal mathematical theories such as second order arithmetic and set theory are incomplete with respect to
number theoretic propositions. So how is it that a particular object, the simply infinite system of natural numbers, is picked out by our mathematics? This version of the problem was raised by Dummett in his paper “The philosophical significance of Gödel’s theorem” [1963]. But our knowledge of the natural world is also incomplete and we are not tempted on that account to think that there are more than one natural world. One might wish to separate the two cases by pointing out that there may be grounds, namely empirical evidence, for deciding the truth-value of a given undecided empirical sentence; but there are grounds, too, in the case of mathematics. For example, the undecided sentences constructed by Gödel for a given theory are decided in their very construction, on grounds that can be formalized in a slightly stronger theory; and certain statements (viz. large cardinal axioms) are reasonably adopted as true because they, and not their negations, lead to the solution of problems in classical descriptive set theory. It is not clear that there are or will be grounds for deciding truth in all cases (e.g. the case of the continuum hypothesis); but that too is true in the case of our knowledge of the natural world. (For example: is the universe infinite?) Moreover—and here the comparison with propositions about natural objects ends—there is the fact that we can prove that all simply infinite systems are isomorphic and so have the same true sentences, even if we do not always know which are true and which are false. Of course, the notion of a simply infinite system \( \langle \mathbb{N}, o, ' \rangle \) involves the condition that every set of numbers containing 0 and closed under ' contains all the numbers, where the notion of a set of numbers must be taken to be open-ended. But nevertheless, if our beliefs about arithmetic truth are founded on proof, we will agree about arithmetic truth so long as we agree about what constitutes a valid definition of a set of numbers.

An even less plausible take on the uniqueness problem is to accept the categoricity (in the above sense) of the notion of a simply infinite system and to agree that when we are speaking of the natural numbers we are speaking of a *sui generis* such system, but then ask how we know that there is just one such system and not two or even more of them? In this case, there is nothing that distinguishes between them; there is just, so to speak, a pure ambiguity. Rather than discussing this absurdity further, though, suffice it to say that the same pure ambiguity applies to language about the natural world: how do we know that there aren’t two or more such empirical worlds, indistinguishable, and generating for us the same patterns of sense data? If this in itself does not answer the challenge of non-uniqueness, it shows that

---

14 See the discussion in Charles Parsons' [2008, §49].
the challenge is not unique to language about ideal things.

9. **Platonism.** One might define Platonism in mathematics as the doctrine that mathematical objects—structures and their elements—exist and are ideal. In Øystein Linnebo’s article in the *Stanford Encyclopedia of Philosophy* it is defined to be the doctrine that mathematical objects exist and are both “abstract” and “independent of intelligent agents and their language, thoughts and practices”, but I think that this definition includes too much and too little. I have already explained why I avoid speaking of mathematical objects as ‘abstract’: they are not abstracted from anything; but leaving that aside, the class of abstract objects includes things such as the various species of organisms and sets of natural objects, which are indeed independent in Linnebo’s sense but whose properties are not independent of what is true of the natural world. Likewise, Linnebo includes Quine among the Platonists because he accepts the existence of certain mathematical objects on the grounds that reference to them is essential to natural science (the so-called ‘indispensability argument’). But according to this argument empirical evidence is relevant to the existence of these objects: they would constitute a radical kind of ‘theoretical entity’.

In his paper “Sur le platonisme dans les mathématiques” [1935], Paul Bernays contrasts the ‘platonistic tendency’ in mathematics to constructive mathematics, where the latter is less “cut off from all links with the reflecting subject”; but no matter what the special character and attractions of constructive mathematics are, its objects—e.g. numbers, functions, sets, proofs—are not natural objects whose properties can be established empirically. In particular, they are not ‘mental’ objects. If, for example, the natural numbers are understood to be built up from 0 by means of the successor operation, what mental object would be? Besides, whose mind does it inhabit? (The universal mind doesn’t exist in nature.) If the constructions referred to in ‘constructive mathematics’ are restricted to what we can physically or mentally construct, then constructive mathematics would be an impoverished thing. So in the above sense at least, constructive mathematics is also Platonistic.

Gödel [*1951, pp. 321-322*] argued that, if one accepts that the theorems of ordinary number theory, for example, are true and there is no alternative interpretation of them as true propositions, then one must admit that they

---

15 There is an accompanying supplement to definitions of Platonism in the literature which are more or less in accord with Linnebo’s definition.
are about ideal things. Although Gödel leaves somewhat open the question of alternative interpretations, I believe that we can safely say that there are no alternative interpretations in a non-ideal domain; so that the theorems of ordinary number theory are all true only if they are taken to be about ideal things, (and that this assurance also extends to set theory).

An argument against this conclusion has been offered by making a distinction between the notion of existence expressed within mathematics by the existential quantifier $\exists_M x$ which is proper to mathematics and an external notion of existence expressed by $\exists_P$. Thus, the theorems of mathematics are true when ‘existence’ is read as internal, but false when it is read as external. The quantifier $\exists^M x$ makes sense: there is a recognizable language of mathematics in the sense that there is agreement among mathematicians over the meaning of mathematical propositions. A less tolerant response is that there is just one genuine notion of existence, $\exists^P x$, which is overarching and, so understood, existential propositions in mathematics are just false. But one has to question whether any real meaning has ever been given to the external, philosophers’ quantifier $\exists^P x$ that is not question-begging. Substantive objections to the existence of ideal objects—and I am excluding question-begging ones—have come, not from an analysis of a general notion of existence and a demonstration that it doesn’t accommodate ideal objects; rather they have been based on the skepticism that I claim the remarks above about the nature of language and meaning refute. From this point of view, the proper content of Quine’s formula

$$To \ be \ is \ to \ be \ the \ value \ of \ a \ bound \ variable.$$  

is not, as it was for Quine, a criterion for the ontological commitment of a theory, in the sense that the theory could be entirely satisfactory in every way except that it is committed to objects that just don’t exist. The proper understanding of it is that, on the contrary, there is no external, transcendental notion of existence upon which to judge theory or language. What “existence” means is defined by the use of the existential quantifier within the language. To reject the objects is to reject the language which gives meaning to sentences referring to them: it is to reject a natural form of expression. There can certainly be grounds, such as inconsistency, for

---

16I referred to this in [2001] as a default argument for Platonism.

17I am ignoring the difference between constructivist propositions and a classical (non-constructive) propositions. But I conjecture that the two traditions in mathematics are inter-interpretable in such a way that every constructive theorem interprets into a classical theorem and every classical theorem interprets into a constructive theorem conditional on the law of excluded middle.
rejecting a theory of ideal objects, but they are not the external and essentially meaningless grounds that the objects in question just don’t happen to exist.

Of course it is a consequence of our position that the sense of existence in which ideal objects exist is the humble sense, internal to mathematics, which we learn in the context of learning existence proofs. For our conclusion is not that there is a univocal across the board notion of existence according to which ideal objects exist. It is rather that there is no such notion and that the meaning of existence in mathematics is the well-established one within the practice of mathematics. Notions such as that of object, existence and identity take their meaning in the language, not as a part of a scaffolding on which language is built. All that transcends the particular language is the formal rules of identity and quantification.

One would expect full support of this point of view from Wittgenstein, who wrote

For this is what disputes between idealists, solipsists and realists look like. The one party attacks the normal form of expression as if they were attacking a statement; the others defend it, as if they were stating facts recognized by every reasonable human being. §402

Unfortunately, Wittgenstein’s conception of mathematics as constructing and computing belongs to the eighteenth century and does not match the state of the subject in his time; and so his criticism of what was part of the warp and woof of mathematics by the 1920’s (see §426, for example) is in conflict with his statement that philosophy “leaves mathematics as it is” (§124). In particular, the anti-Platonist should be listed with the idealist and the solipsist in the camp of the transcendental ‘they don’t exist’ folk in §402 and the correlative Platonist about mathematical objects should be included along with the realists about physical objects and about other minds in the camp of the transcendental “they do exist”. There is some irony here, though, in that my aim is to defend ordinary mathematical practice as it developed through the nineteenth and twentieth centuries against skepticism about ideal objects. The argument led me somewhat far a field to a position on the nature of meaningful discourse, before leading finally to §402. But there we find conspicuously absent from the catalogue of empty skepticisms the very one it is my concern to dissolve.\footnote{It should be noted that it is not just skepticism that Wittgenstein is addressing in §402; he is complaining, too, about the realist, who believes that there is a transcendental}
Leaving aside Wittgenstein’s own conservatism and, in particular, his peculiar time-warp in connection with mathematics, there is a wonderful liberation implicit in *Investigations* that fits with the bold approach of Dedekind, Cantor and Hilbert to ontology. For those of us philosophers who have found distasteful the constraint and demand for humility bound up with Kant’s response to skepticism—reason must be restricted to its ‘proper’ object, the natural world is given to us in intuition, and the basic laws of this world are hard-wired into its givenness—the vision is most welcome that, indeed with some prompting from nature, we are free to give full reign to our creative rational powers and get to decide without any but logical restraint what we want to talk about.

**References**


Cantor, G. [1883a]. *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*, Leipzig: Teubner. A separate printing of [Cantor, 1883b] with a subtitle, preface and some footnotes added. A translation *Foundations of a General Theory of Manifolds: A Mathamatico-Philosophical notion of existence according to which physical objects, other minds or [pace Wittgenstein] ideal objects, exist—just as the skeptic believes that there is a transcendental notion of existence according to which they don’t.
Investigation into the Theory of the Infinite by W. Ewald is in [Ewald, 1996, pp. 639-920].


