# Constructing Cardinals from Below

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The totality $\Omega$ of transfinite numbers was first introduced in [Cantor, 1883] by means of the principle

> If the initial segment $\Sigma$ of $\Omega$ is a set, then it has a least strict upper bound $S(\Sigma) \in \Omega$.

Thus, for numbers $\alpha = S(\Sigma)$ and $\beta = S(\Sigma')$, $\alpha < \beta$ iff $\alpha \in \Sigma'$; $\alpha = \beta$ iff $\Sigma = \Sigma'$; $S(\emptyset)$ is the least number 0 (although Cantor himself took the least number to be 1); if $\Sigma$ has a greatest element $\gamma$, then $\alpha$ is its successor $\gamma + 1$; and if $\Sigma$ is non-null and has no greatest element, then $\alpha$ is the least upper bound of $\Sigma$. The problem with the definition, of course, is in determining what it means for an initial segment to be a set. Obviously, not all of them are: for the totality $\Omega$ of all numbers is an initial segment, but to admit it as a set would yield $S(\Omega) < S(\Omega)$, contradicting the assumption that $\Omega$ is well-ordered by $<$. Cantor himself understood this already in 1883. In his earlier writings, e.g. [1882], he had essentially defined a set 'in some conceptual sphere' such as arithmetic or geometry, to be the extension of a well-defined property. But in these cases, he was considering sets of objects of some type $A$, where being an object of type $A$ is itself is not defined in terms of the notion of a set of objects of type $A$. But with his definition of the transfinite numbers, an entirely novel situation arises: the definition of $\Omega$ depends on the notion of a subset of $\Omega$. Accordingly, he abandoned his earlier definition of set in [1883] and, in his later writings, he distinguished between those initial segments which are sets and those which are not in terms of his concept of 'consistent multiplicity'; but that is just naming the problem, not solving it.

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However, we can replace the condition on $\Sigma$ of being a set or consistent multiplicity by certain other conditions $C$ which have a precise meaning, yielding the principle

> If the initial segment $\Sigma$ of $\Omega_C$ satisfies the condition $C$, then it has a least strict upper bound $S(\Sigma) \in \Omega_C$.

We will call such conditions $C$ *existence conditions*. Unlike the case of $\Omega$, no obvious contradiction arises in general from the assumption that we can take the least upper bound $S(\Omega_C)$. We can merely conclude that $\Omega_C$ does not satisfy the condition $C$ (since, otherwise, $S(\Omega_C) \in \Omega_C$ and so $S(\Omega_C) < S(\Omega_C)$); so that, in proceeding to construct $S(\Omega_C)$ and larger numbers, we must replace $C$ by some other existence condition which they satisfy. In this way, we are led to a hierarchy of more and more inclusive existence conditions, each of which can replace the condition "*is a set*" in Cantor's definition and so yields an initial segment of the transfinite numbers, but none of which yields 'all the numbers'.

In formulating existence conditions, we can make use of the fact that each transfinite number $\alpha$ can be armed with the structure $\langle R(\alpha), \in_\alpha, ran_\alpha\rangle$, where $R(\alpha)$ is obtained by starting with the null set and iterating the operation $D \mapsto \mathcal{P}(D)$ of taking power sets $\alpha$ times. ($R(S(\Sigma)) = \bigcup\{R(\alpha) \mid \alpha \in \Sigma\}$.) $\in_\alpha = \in \cap (R(\alpha) \times R(\alpha))$. The *rank* of a set is the least strict upper bound of the ranks of its elements, so that $R(\alpha)$ is the set of all sets of rank $< \alpha$. *ran* is the rank function, assigning to each set its rank, and $ran_\alpha$ is its restriction to $R(\alpha)$. Thus $ran_\alpha$ is a function with domain $R(\alpha)$ and range of values the set of ordinals $< \alpha$. We avoid taking the transfinite numbers to be external to set theory by identifying them with the von Neumann ordinals, i.e. with the transitive sets of transitive sets. (The set $x$ is transitive iff $y \in x \longrightarrow y \subset x$.) $\alpha$ then is the subset of $R(\alpha)$ consisting of all ordinals in $R(\alpha)$ and $\beta < \alpha$ simply means $\beta \in \alpha$. In terms of von Neumann ordinals, $S(\Sigma)$ is just $\Sigma$ and an existence condition is precisely a condition under which we admit the existence of $\Sigma$ as a set.

Thus, we will be discussing a particular conception of set theory which is explicitly formulated in [Gödel, *1933o] and [Gödel, 1947] but is implicit in [Zermelo, 1930]. In part, it is the conception according to which sets are the objects in any member of the hierarchy of *domains* obtained from the null domain by iterating the power set operation. Thus, this idea presupposes as well-understood the more primitive, logical notion of set, the one which is formalized in second-order predicate logic, namely that of forming from a given totality $D$ of objects the totality of all sets of objects in $D$. The other ingredient of this conception is that the process of iterating the power set operation does not presuppose a given external system of ordinals or 'stages' along which the iteration takes place, but is *autonomous*. The ordinals we consider are only the internal von Neumann ordinals, and the question, given some initial segment $\Sigma$ of ordinals, whether it has an upper bound, so that we may continue the iteration of taking power sets beyond the ordinals in $\Sigma$, should depend only on properties of $\Sigma$ or better of the universe of sets obtained by iterating the power set operation through $\Sigma$.

One should note that the 'bottom-up' conception described here is not the one motivating most contemporary work in set theory, where the choice of axioms is guided more by global considerations of the properties of the models of the axioms as a whole. Also, the cardinal numbers so far obtained on this conception are all relatively small: their existence is consistent with $V = L$. But, as we will explain, there seem to be resources in the conception that have not yet been investigated and that might lead beyond that barrier. In any case, it seems of interest to see what can be developed on the basis of our more 'constructive' conception—what large cardinal axioms can be founded on it.

# 1 Basic Set Theory

The $R(\alpha)$ with $\alpha > 0$ are, to within isomorphism, the models of a certain second-order theory $\mathcal{T}_0$, whose second-order quantifiers range over classes of individuals. Its language is that of second-order set theory with the unary function constant *ran* (for the rank function) added. $X, Y, Z, \ldots$ are the second-order variables, ranging over classes, $x, y, z, \ldots$ are the first-order variables ranging over sets and lower case Greek letters are to be understood as relativized variables ranging over the class $On$ of von Neumann ordinals. I.e.

$$\exists\alpha[\cdots\alpha\cdots] := \exists x(x \in On \wedge [\cdots x \cdots])$$

For any formula $\phi(x)$ of $\mathcal{T}_0$, the comprehension axiom of second-order logic yields a class (depending on the free variables in $\phi(x)$ other than $x$) consisting of just those sets $x$ such that $\phi(x)$. We denote such a class by

$$\{x \mid \phi(x)\}$$

We assume the axiom of extensionality for classes (as an axiom of second-order logic) and so there is exactly one such class. For example

$$On = \{x \mid x \text{ is transitive } \wedge \text{ every element of } x \text{ is transitive}\}$$

If $s$ is a set term (i.e. first-order) and $S$ a class term, $s = S$ will mean $\forall x[x \in s \longleftrightarrow x \in S]$ and '$S$ is a set' will mean $\exists x[x = S]$. $S \in s$ means of course that $\exists x[x = S \wedge x \in s]$. (As usual, we are using the symbol "$\in$" ambiguously, to denote both a relation between sets and sets and a heterogenious relation between sets and classes; but no confusion will result.)

The axioms of $\mathcal{T}_0$ are:

*Extensionality*

$$\forall xy(\forall z[z \in x \longleftrightarrow z \in y] \longrightarrow x = y)$$

*Foundation*

$$\forall X[\exists y(y \in X) \longrightarrow \exists y \in X \forall z(z \in X \longrightarrow z \notin y)]$$

It follows from this that the class $On$ of (von Neumann) ordinals is well-ordered by $\in$, which (as we noted) among ordinals we often denote by $<$. Foundation also enables us to prove that all sets have a certain property by showing that, if all subsets of a given set have the property, then so does the given set—the principle of $\in$-Induction.

*Second-order Separation*

$$\forall x Y \exists z[\forall u(u \in z \longleftrightarrow u \in x \wedge u \in Y)]$$

and, finally, two axioms concerning the rank function:

*Rank 1*

$$\forall x \forall y[y \in ran(x) \longleftrightarrow \exists z \in x \forall u(u \in y \longleftrightarrow u \in ran(z) \vee u = ran(z))]$$

In other words, if $z \in x$, then $ran(z) + 1 = ran(z) \cup \{ran(z)\}$ exists and moreover, $\{ran(z) + 1 \mid z \in x\}$ is a set and is by definition $ran(x)$. Writing

$$R(\alpha) = \{x \mid ran(x) \in \alpha\}$$

the second axiom concerning rank is

*Rank 2*

$$\forall \alpha[R(\alpha) \text{ is a set}].$$

From Foundation and Rank 1, it follows that $ran(x)$ is an ordinal and is the least ordinal $> ran(y)$ for each $y \in x$.

Note that, since

$$\bigcup x = \{y \mid \exists z[y \in z \wedge z \in x]\}$$

is a subclass of the set $R(ran(x))$ and so is a set, the axiom of *Union* is derivable in $\mathcal{T}_0$. In Zermelo-Fraenkel set theory, in particular in the presence of the axiom of Replacement, the rank function *ran* can be defined; but to begin with we want to consider arbitrary domains $R(\alpha)$ $(\alpha > 0)$, and these in general will not satisfy Replacement.

It is clear that the $R(\alpha)$ for $\alpha > 0$ are models of $\mathcal{T}_0$. Of course, when we speak of models of $\mathcal{T}_0$, we really should be speaking of the structure $\langle R(\alpha), \in_\alpha, ran_\alpha \rangle$ and not simply of its domain $R(\alpha)$; but, unless otherwise specified, when we say that a formula is satisfied in $R(\alpha)$, we will mean that it is satisfied in the corresponding structure. We prove now that the $R(\alpha)$ are, to within isomorphism, the models of $\mathcal{T}_0$.

**Theorem 1** *Every model of $\mathcal{T}_0$ is isomorphic to $R(\gamma)$ for exactly one $\gamma > 0$, and the isomorphism is unique.*

Let $\mathcal{M}$ be a model of $\mathcal{T}_0$. Then the class $On_{\mathcal{M}}$ of ordinals in $\mathcal{M}$ is well-ordered by $\in_{\mathcal{M}}$. Let $\gamma$ be the ordinal of this well-ordering. Since the domain of $\mathcal{M}$ is non-empty, $On_{\mathcal{M}}$ is non-empty and so $\gamma > 0$. There is exactly one isomorphism from $\gamma = On_{R(\gamma)}$ well-ordered by $\in_\gamma$ onto $\gamma' = On_{\mathcal{M}}$ well-ordered by $\in_{\mathcal{M}}$. Let $\alpha'$ be the image of $\alpha < \gamma$ under this isomorphism. There is exactly one isomorphism $f$ from $R(\gamma)$ onto the domain $M$ of $\mathcal{M}$, which is defined by $\in$-recursion: suppose that $f(y)$ is defined for all $y \in x$ and that $ran_{\mathcal{M}}(f(y)) = ran(y)'$. Then $\{f(y) \mid y \in x\}$ is included in $R_{\mathcal{M}}(ran(x)')$ and so is a set $\{f(y) \mid y \in x\}_M$ in $\mathcal{M}$. Define

$$f(x) = \{f(y) \mid y \in x\}_M$$

Clearly $ran_{\mathcal{M}}(f(x)) = ran(x)'$. By $\in$-induction, one easily proves that $f$ is injective (one-to-one). To show that every $y \in M$ is in the range of $f$, assume that every $z \in y$ is in the range of $f$—say $z = f(z*)$. Let $ran_{\mathcal{M}}(y) = \alpha'$. Then $ran(z*) < \alpha$ and so $\{z* \mid z \in_{\mathcal{M}} y\}$ is a set $x \in R(\alpha)$ and $f(x) = y$. Clearly, for $x$ and $y$ in $R(\gamma)$, $x \in y$ iff $f(x) \in_{\mathcal{M}} f(y)$ and we have shown that $ran_{\mathcal{M}}(f(x)) = ran(x)'$. So $f$ is indeed the required isomorphism. The uniqueness of $f$ is immediate using $\in$-induction. $\square$

## 2 Morse-Kelley Set Theory

In this section, we will discuss the existence conditions which lead to the (impredicative) second-order theory of sets first introduced in [Morse, 1965].

Given an existence condition $C$, $\Omega_C(\alpha)$ will denote the least initial segment of ordinals containing $\alpha+1$ which does not possess $C$. In what follows, $\Sigma$ will be an initial segment of ordinals. If $\mathcal{T}$ is an extension of $\mathcal{T}_0$, then by an *ordinal* of $\mathcal{T}$ we shall mean an $\alpha$ such that $R(\alpha)$ is a model of $\mathcal{T}$. In the following, we shall consider extensions $\mathcal{T}$ of $\mathcal{T}_0$ obtained by adding axioms which are formal expressions of existence conditions. Thus, the least ordinal of $\mathcal{T} > \alpha$ is $\Omega_C(\alpha)$, where $C$ is the existence condition in question.

The existence conditions that we shall consider in this paper are all formulated in terms of a formula $\phi(X)$, with only $X$ free. For now, the formula is one in the language of basic set theory and $X$ is a second-order variable. The corresponding condition is that $\phi(A)$ is true in $R(\Sigma)$ for some $A \subseteq R(\Sigma)$ and, for no $\alpha \in \Sigma$ is $\phi(A \cap R(\alpha))$ true in $R(\alpha)$. The formal expression that this condition is an existence condition is the axiom

$$\forall X[\phi(X) \longrightarrow \exists\beta\phi^{\beta}(X \cap R(\beta))] \tag{1}$$

where $\phi^{\beta}(X)$ is the result of restricting the first- and second-order bound variables in $\phi(X)$ to $R(\alpha)$ and $R(\alpha+1)$, respectively. Axioms of this form have been called *reflection principles*, because they express the fact that $R(\Sigma)$'s possession of a certain property is reflected by $R(\alpha)$'s possession of it for some $\alpha \in \Sigma$.

Since predicate logic assumes that all models are non-null, $\mathcal{T}_0$ already expresses that the condition given by $\phi(X) = \forall y(y \neq y)$ is an existence condition.

If we take

$$\phi(X) = \exists\alpha(X = \{\alpha\})$$

then (1) is equivalent to the axiom of

*Successor*

$$\forall\alpha[\alpha \cup \{\alpha\} \text{ is a set}].$$

With this axiom, we can derive the axioms of *Unordered Pairs* and *Powerset*, since if $x$ and $y$ are in $R(\alpha)$, then the classes $\{x, y\}$ and $\{z \mid z \subseteq x\}$ are included in $R(\alpha+1)$ and so are sets. Since we have arbitrary unordered pairs, we can now introduce the usual coding

$$(x, y) = \{\{x\}, \{x, y\}\}$$

of ordered pairs. By means of this coding, functions $F$ may be represented by means of their graphs $\{(x, F(x)) \mid x$ is in the domain of definition of $F\}$. Let "$X$ is a function" mean that $X$ is a class of pairs such that $(x, y) \in X$ and $(x, z) \in X$ implies that $y = z$. $dom(X)$ $(rang(X))$ is the set of $x$ $(y)$ such that for some $y$ $(x)$, $(x, y) \in X$.

We are now able to formulate the axiom of

*Choice*

$$\forall x \exists F[F \text{ is a choice function for } x]$$

where a choice function for $x$ is a set consisting of exactly one pair $(y, z)$ for each non-empty element $y$ of $x$, where $z \in y$. The axiom of Choice is a consequence of the Global Choice principle,

$$\exists F[F \text{ is a choice function for } V]$$

which, like [Zermelo, 1930], I take to be a principle of second-order logic, following (at least as I see it) from the meaning of $\forall\exists$. $\mathcal{T}_1$ will denote $\mathcal{T}_0$ together with the axioms of Successor and Choice. Its ordinals are precisely the limit ordinals. In $\mathcal{T}_1$ we can prove that every set can be well-ordered ([Zermelo, 1908]).

Consider the condition corresponding to

$$\phi(X) = dom(X) \text{ is a function } \wedge \exists y[rang(X) = \{y\} \wedge dom(dom(X)) = y]$$

Here we are coding a pair $(F, y)$, consisting of a class $F$ and a set $y$ by $X = F \times \{y\}$, consisting of the pairs $(x, y)$ with $x \in F$. From now on, we shall use $F$ as a bound variable ranging over functions. So the formal expression of the condition corresponding to $\phi(X)$ as an existence condition is expressed by the axiom of

*Replacement*

$$\forall F[dom(F) \text{ is a set } \longrightarrow rang(F) \text{ is a set}]$$

Adding this axiom to $\mathcal{T}_1$, we obtain the usual system $\mathcal{T}_2$ of (impredicative) second-order set theory—so-called *Morse-Kelley set theory*—without the axiom of Infinity. From this axiom, it follows that a choice function for a set is itself (coextensive with) a set. Also, with this axiom, and providing that we replace the axiom of Successor by the axioms of Unordered Pairs and Powerset, we can drop the primitive constant *ran* and its axioms *Rank*1 and

$Rank2$, since $ran$ can now be defined by $\in$-induction and these two axioms can be deduced from the remaining axioms [Zermelo, 1930].

As usual, we identify cardinal numbers with initial ordinals, i.e. with ordinal numbers which are not in one-to-one correspondence with smaller ordinals. In $\mathcal{T}_2$ we can prove that every set $x$ has (i.e. is in one-to-one correspondence with) a cardinal number $|x|$.

Let $2^\alpha = |\mathcal{P}(\alpha)|$. A cardinal $\kappa$ is called a *strong limit cardinal* if $\alpha < \kappa$ always implies that $2^\alpha < \kappa$. So, every ordinal of $\mathcal{T}_2$ must be a strong limit cardinal. An ordinal $\gamma$ is called *singular* iff it is of the form $\bigcup_{\alpha<\beta} F(\alpha)$, where $dom(F) = \beta < \gamma$; otherwise, $\gamma$ is called *regular*. Note that 0 and $\omega$ are both regular strong limit cardinals. Clearly a regular ordinal must be a cardinal and, in view of the axiom of Replacement, the ordinals of $\mathcal{T}_2$ must be regular. Conversely, if $\kappa$ is a regular strong limit cardinal $> 0$, then it is an ordinal of $\mathcal{T}_2$. First, note that we can define the ordinal function $\Phi$ (i.e. ordinal-valued function on ordinals) in $R(\kappa)$ by

$$\Phi(0) = 0$$
$$\Phi(\alpha + 1) = 2^{\Phi(\alpha)}$$

and for limit ordinals $\gamma$

$$\Phi(\gamma) = \bigcup_{\alpha<\gamma} \Phi(\alpha)$$

For $\alpha < \kappa$, $\Phi(\alpha) = |R(\alpha)|$ and so $x \in R(\kappa) = \bigcup_{\alpha<\kappa} R(\alpha)$ implies $|x| \in R(\kappa)$. To see that $R(\kappa)$ satisfies the axiom of Regularity, let $F$ be a function with $dom(F) \in R(\kappa)$ and $rang(F) \subseteq R(\kappa)$. Let $g$ be a one-to-one correspondence from $|dom(F)|$ to $dom(F)$. Then $G = ran \circ F \circ g$ is a function from $|dom(F)| < \kappa$ and so, since $\kappa$ is regular, $\bigcup rang(G) = \beta < \kappa$. But $rang(F) \subseteq R(\beta)$ and so $rang(F) \in R(\kappa)$. So, as was first proved in [Zermelo, 1930], the ordinals of $\mathcal{T}_2$ are precisely the regular strong limit cardinals $> 0$.

$\Phi$ is an example of a *normal function*, i.e. an ordinal function $f$ which is *order-preserving*, i.e. $\alpha < \beta$ implies $f(\alpha) < f(\beta)$, and *continuous*, i.e., when $\gamma$ is a limit ordinal, then $f(\gamma) = \bigcup_{\alpha<\gamma} f(\alpha)$. When $f$ is an order-preserving ordinal function, then $\alpha \leq f(\alpha)$ for all $\alpha$. For, $f(\alpha) < \alpha$ implies $f(f(\alpha)) < f(\alpha)$; and so there can be no $\alpha$ such that $f(\alpha) < \alpha$ because there can be no least such $\alpha$. When $f$ is a normal function, then it has fixed-points $f(\beta) = \beta \geq$ any given $\alpha$. Either $\alpha$ is already a fixed point or else, using the fact that $f$ is order-preserving,

$$\alpha < f(\alpha) < f(f(\alpha)) < \cdots < f^n(\alpha) < \cdots .$$

In the second case, let $\beta = \bigcup_{n<\omega} f^n(\alpha)$. Then $\alpha < \beta$ and

$$f(\beta) = \bigcup_{n<\omega} f^{n+1}(\alpha) = \beta.$$

Of course, in the case in which the fixed point $\beta$ is $> \alpha \geq \omega$, $\beta$ is singular, since it is $> \omega$ and cofinal with the range of the function $F$ defined on $\omega$ by $F(n) = f^n(\alpha)$.

The strong limit cardinals $\delta$ are precisely the fixed-points of $\Phi$. For $\delta < \Phi(\delta)$ would imply by continuity that there is a $\beta < \delta$ with $\delta < \Phi(\beta)$. The least such $\beta$ would be a successor ordinal $\gamma + 1$. So $\Phi(\gamma) < \delta \leq 2^{\Phi(\gamma)}$. Conversely, if $\delta$ is a fixed point, $\delta \neq 0$ , then $\alpha < \delta$ implies $2^\alpha = \Phi(\alpha + 1) < \Phi(\delta) = \delta$. So $\delta$ is a strong limit.

If we apply the axiom (1) to

$$\phi(X) = [0 \in On \wedge \forall\alpha\exists\gamma(\alpha < \gamma)]$$

(in which $X$ does not occur), then we obtain the existence of a limit ordinal and so of the least one $\omega$. This is the axiom of Infinity. Because this axiom is commonly just assumed, it is usual to speak only of the regular strong limit cardinals $> \omega$, which are called the *(strongly) inaccessible* cardinals. Thus, the ordinals of models of Morse- Kelley set theory $\mathcal{T}_3 = \mathcal{T}_2+$ *Infinity* are precisely the inaccessible cardinals, i.e. the regular fixed points of the normal function $\alpha \mapsto \beth_\alpha$, defined by

$$\beth_\alpha = \Phi(\omega + \alpha)$$

For $\Phi(\omega + \alpha) = \omega + \alpha$ for $\alpha > 0$ implies that $\alpha$ has power $> \omega$ and so $\omega + \alpha = \alpha$.

So far, we have applied the reflection principle only to first-order formulas $\phi(X)$, i.e. formulas in which all the bound variables are first-order. It turns out that, not only are the axioms of Successor, Replacement and Infinity consequences in $\mathcal{T}_0$ of reflection for such formulas, but reflection for first-order $\phi(X)$ is deducible in $\mathcal{T}_3$.

# 3 Zermelo's conception of set theory

It is an embarassment in set theory, as it is often understood, that an absolute distinction must be drawn between totalities such as the totality of 'all ordinals' or 'all cardinals' or 'all sets'—the totalities which Cantor called 'inconsistent manifolds' and we call *proper classes*—on the one hand, and those totalities which form sets. For when we take the former totalities to be well-defined objects, then we must make this absolute distinction: the two kinds of objects must be treated quite differently. But why, if the totality of all sets has a well-defined extension, is it not a set in an more extensive totality? The only grounds for the distinction is the negative one that, if we treat proper classes like sets, we are led to inconsistency—in the examples cited, to the familiar 'paradoxes of set theory'. Thus, on this understanding of set theory, these paradoxes truly are paradoxical: there is no accounting for them. We can, as Cantor did, only introduce the distinction between those totalities which are 'consistent' and those which are not. (Of course, relative to any domain $R(\alpha)$ there are proper classes, i.e. subclasses of the domain which are not coextensive with elements of it; but these are proper classes *only* in this relative sense: each subclass of $R(\alpha)$ will be coextensive with a set in $R(\alpha + 1)$ for example.)

At the end of [1930], Zermelo sketched a quite different approach to understanding set theory and the so-called paradoxes, one which precludes this embarassment. He begins with the Hilbertian thesis that we may speak about the existence of this or that object in mathematics only when we have specified a consistent and categorical theory in which we can speak of such objects. More generally, we may assert mathematical propositions (including existence propositions) only within such a theory. The background of this view is that there are no mathematical phenomena (such as Kantian pure intuitions) on the basis of which mathematical propositions are meaningful: however it is that we come to accept a body of propositions in some mathematical sphere, whether derived from empirical experience or by analogy or whatever, there can be no definitive criterion for existence or truth until we have specified an axiomatic basis for these propositions. The requirement of consistency aims at making sure that the distinctions between existence and non-existence and, more generally, between truth and falsety do not collapse. The requirement of categoricity, for Hilbert and Zermelo, probably aimed at making sure that questions of truth had determinate answers. Since the non-trivial categorical axiomatic theories are all second-order and we now know that second-oder logic is incomplete, we also know that categoricity does not yield determinate answers in all cases. But for us, categoricity still seems a reasonable requirement: it means that the reference of the theory—the structure to which it refers—is fixed (to within isomorphism): no new axioms about the specific mathematical concepts involved are needed. Consistency is, of course, on the grounds stated above, an indispensible requirement; but as we now also know (and Zermelo didn't in 1930), there is no reasonable sense in which we can establish it conclusively even for $\mathcal{T}_1$. We must live with the possibility that an inconsistency may be discovered in even elementary parts of mathematics (or that there is one, but we will never be able to discover it because the proof is too long).

Zermelo takes as his starting point $\mathcal{T}_2$.[1] He notes that it is not a categorical theory and so does not itself determine a structure. Rather, it is its various categorical (and consistent) extensions that determine structures. For example, if we add the axiom that there are no infinite ordinals, then we obtain a categorical extension whose model is $R(\omega)$. But, contrary to a common understanding of set theory as a theory about 'the' universe of sets, there is no such universe: the notion of 'all ordinals' or 'all sets' lacks rigorous mathematical sense. Or rather it has only a relative sense: all sets or ordinals in this or that categorically determined domain. More generally, there is no absolute notion of proper class; there are only proper classes relative to a particular model $R(\alpha)$. He concludes from this analysis that the paradoxes of set theory (*die "ultrafiniten Antinomien der Mengenlehre"*) are illusory and arise from the confusion between the (non-categorical) theory $\mathcal{T}_2$ and its models, by which he has to mean its categorical extensions. For if we take set theory itself to have a model which includes all sets, then its universe is indeed the 'class of all sets' in the absolute sense and paradoxes arise if we do not make the absolute distinction, which otherwise has no foundation, between sets and proper classes.

By adopting Zermelo's point of view, we see that there is no conceptual difficulty with admitting second-order quantifiers, i.e. over classes. The models that we consider are the $R(\alpha)$'s, and the second-order quantifiers relative to this simply range over the elements of $R(\alpha + 1)$, *but now regarded as classes*. Of course, we must distinguish the elements of $R(\alpha)$ *qua* first-order objects from their occurrence in $R(\alpha + 1)$ as second-order. To make

[1]Presumably, he included the axioms of Unordered Couples and Replacement in order not to have to take the rank function as primitive.

this precise, we may take the range of the second-order variables to be the isomorphic replica $R(\alpha+1)^*$ of $R(\alpha+1)$ which consists in replacing each element $x$ of the latter by the pair $(\alpha, x)$. For $y \in R(\alpha)$ and $(\alpha, x)$ a class, $y \in (\alpha, x)$ means simply that $y \in x$. In the same way, we can equally well consider quantifiers of order higher than 2 as well.

# 4 Formulas of Finite Order

In fact, in what follows, we are going to want to consider not only second-order formulas, but formulas of arbitrary finite order, where the quantifiers of order $n+1$ ($n > 0$) range over classes of objects of order $n$. So we will begin with some details that will apply generally and not just to the formulas of second-order set theory.

The *order* of a formula is the maximum of the orders of the bound variables in it. Note that the formulas of order 0 are those which contain no quantifiers. The relativization $\phi^\beta$ of the formula $\phi$ to $R(\beta)$ is extended from formulas of second-order set theory to those of arbitrary finite order in the obvious way: all the quantifiers of order $n+1$ are restricted to $R(\beta+n)$. When talking about sets and classes, it is more natural to speak of their *types* rather than their orders. The type of an object of order $n+1$ is $n$.

Let $A$ be of type 1. From the point of view of $R(\beta)$, $A$ is $A^\beta = A \cap R(\beta)$. For, when $A$ and $B$ are type 1, then $A$ and $B$ are equal relative to $R(\beta)$ just in case $\forall z \in R(\beta)[z \in A \longleftrightarrow z \in B]$ iff $A^\beta = B^\beta$. It follows that, from the point of view of $R(\beta)$, $B$ is

$$B^\beta = \{A^\beta \mid A \in B\}$$

for $B$ of arbitrary type $> 1$.

So, for $A$ of type $> 0$, $\phi^\beta(A^\beta)$ expresses the truth of $\phi(A)$ in $R(\beta)$. When $\phi = \psi(A, \ldots, B)$ is a formula of finite order containing no parameters other than $A, \ldots, B$, each of which is of type $> 0$, $\phi^\beta$ will denote $\psi^\beta(A^\beta, \ldots, B^\beta)$.

In order to code higher order and possibly heterogeneous relations as objects of finite type, we introduce the following operations: We define $A\uparrow$ of type $n+1$ for $A$ of any type $n$, and $A\downarrow$ of type $n+1$ for A of any type $n+2$, such that $A\uparrow\downarrow = A$.

$$A\uparrow = \{x \mid x = A\}.$$

If $A$ is type $n+2$, define the set $A\downarrow$ of type $n+1$ by

$$A\downarrow = \begin{cases} B & \text{if } A = \{B\} \\ \emptyset & \text{if } A \text{ is not a singleton} \end{cases}$$

Using these operations, we can raise the order of quantifiers in formulas. E.g., if $X$ is of type $n+1$ and $Y$ and $Z$ are of type $n+k+1$, then

$$\forall X \forall Y \phi(X, Y) \longleftrightarrow \forall Z \forall Y \phi(Z \downarrow \cdots \downarrow, Y)$$

where there are $k$ occurences of $\downarrow$ . In this way, every formula of order $n > 1$ is equivalent in $T_1$ (regarded now as being embedded in predicate logic of finite order rather than just second-order) to a formula

$$(2) \qquad Q_1 X_1 \cdots Q_m X_m \phi(X_1, \ldots, X_m)$$

where $\phi$ is of order $< n$, the $Q_i$ are quantifiers $\forall$ or $\exists$ and the $X_i$ are all of order $n$.

Again, in $\mathcal{T}_1$ we can define $A_{n\times}$ and $A_{/n}$ for $n < \omega$ and $A$ of any type as follows. Let $A$ be of type $\leq 1$.

$$A_{n\times} = \{\langle n, x\rangle \mid x \in A\}$$

$$A_{/n} = \{x \mid \langle n, x\rangle \in A\}.$$

If $B$ is of type $> 1$, then

$$B_{n\times} = \{X_{n\times} \mid X \in B\}$$

and

$$B_{/n} = \{X_{/n} \mid X \in B\}.$$

Let $A$ and $B$ be of the same type. Define

$$A + B = A_{0\times} \cup B_{1\times}.$$

Then $(A+B)_{/0} = A$ and $(A+B)_{/1} = B$.

Using compositions of these operations, which compositions we shall call the *contracting operations*, we can reduce the formula (2) to one in which $Q_i \neq Q_{i+1}$ for $i < m$. For example, if $X$ and $Y$ are of the same type

$$\forall X \forall Y \phi(X, Y) \longleftrightarrow \forall X \phi(X_{/0}, X_{/1})$$

in $\mathcal{T}_1$. In particular, every formula of order $n+2$ is equivalent to a $\Pi^{n+1}_m$ formula for some $m$ in the sense of the following definition.

**Definition 1** *Let $n \geq 0$.*

- *A formula of order $\leq n$ is called a $\Pi^n_0$ formula and a $\Sigma^n_0$ formula.*
- *A $\Pi^n_{m+1}$ is one of the form $\forall Y \psi(Y)$ where $\psi$ is a $\Sigma^n_m$ formula and $Y$ is a variable of type $n$.*
- *A $\Sigma^n_{m+1}$ formula is the negation of a $\Pi^n_m$ formula.*

If $\beta$ is a limit ordinal, then $R(\beta)$ is closed under all the operations $X \uparrow$, $X \downarrow$, $X_{n\times}$ and $X_{/n}$. We note also, for later use

**Lemma 1** *For $\beta$ a limit ordinal, each of the following hold when both sides of the equation are defined:*

- $(A \uparrow)^\beta = A^\beta \uparrow$
- $(A \downarrow)^\beta = A^\beta \downarrow$
- $A^\beta_{n\times} = (A^\beta)_{n\times}$
- $A^\beta_{/n} = (A^\beta)_{/n}$
- $(A + B)^\beta = A^\beta + B^\beta$.

□

# 5 Reflection on a Second-order Parameter

When $\phi$ is restricted to $\Pi^n_m$ and the type of $X$ to 1, then we denote the axiom schema (1) by

$$RF(n, m).$$

**Definition 2**

- *An ordinal $\gamma$ is said to be $\phi$-*indescribable *if (1) holds in $R(\gamma)$.*
- *If $\Theta$ is a class of formulas, then $\gamma$ is called $\Theta$-*indescribable *if it is $\phi$-indescribable for each $\phi \in \Theta$ containing only the free variable $X$ of order 2.*

- $\gamma$ *is* totally indescribable *if it is* $\Theta$*-indescribable where* $\Theta$ *is the class of all formulas of set theory of finite type.*

So $\gamma$ is $\Pi^n_m$-indescribable if $RF(n,m)$ holds in $R(\gamma)$.

In the remainder of this section, we discuss the strength of $RF(n+1,m)$. Let $\psi$ denote the conjunction of the axioms of Morse-Kelley set theory. Applying (1) to the $\Pi^1_1$ formula $\phi(X) = \psi \wedge \exists\alpha(\alpha \in X)$ (with $X = \{\alpha\}$), we obtain the axiom that, for any ordinal $\alpha$, there is as inaccessible cardinal $> \alpha$, i.e. that there is an unbounded sequence of inaccessible cardinals. In fact, we can strengthen this.

A class $C$ of ordinals is *unbounded* iff $C \not\subset \alpha$ for all $\alpha$. Similarly, $D \subseteq \beta$ is *unbounded in* $\beta$ if $D \not\subseteq \alpha$ for any $\alpha < \beta$. $C$ is called *closed* if, for every ordinal $\beta$, if $C \cap \beta$ is unbounded in $\beta$, then $\beta \in C$. It is easy to see that the closed unbounded classes, called $CLUB$ classes, are precisely the range of values of normal functions. Namely, corresponding to the $CLUB$ class $C$ is the normal function which enumerates it. A class $S$ of ordinals is called *stationary* if, for every $CLUB$ class $C$, $S \cap C \neq \emptyset$. For any limit ordinal $\beta$, we can also speak of closed unbounded subsets of $\beta$ and of stationary subsets of $\beta$: simply relativize the definitions to $R(\beta)$.

**Lemma 2** *Let* $m,n > 0$ *and* $\phi(X) \in \Pi^n_m$*, where* $X$ *is of type* 1*. Then* $RF(n,m)$ *implies*

$$\forall X[\phi(X) \longrightarrow \{\beta \mid \phi^\beta(X^\beta)\} \text{ is stationary}].$$

For assume $\phi(A)$ and let $C$ be $CLUB$. Apply $RF(n,m)$ to $[\phi(A) \wedge C$ is unbounded$]$ (coding the pair $(A,C)$ by an object of type n) to obtain a $\beta$ such that $\phi(A)^\beta$ (i.e. $\phi^\beta(A^\beta)$) and $C \cap \beta = C^\beta$ is unbounded in $\beta$, so that $\beta \in C$.

So, in particular, it follows from $RF(1,1)$ that the class $A$ of inaccessible cardinals is stationary. Applying $RF(1,1)$ to the assertion that $A$ is stationary, which is $\Pi^1_1$, we obtain a cardinal $\kappa$ such that $A \cap \kappa$ is stationary in $\kappa$, i.e. $\kappa$ is a so-called *Mahlo* cardinal. Using Lemma 2, we can iterate this procedure and obtain a stationary class $B$ of Mahlo cardinals, and so cardinals $\kappa$ such that $B \cap \kappa$ is stationary in $\kappa$, i.e. *hyper-Mahlo* cardinals; and so on.

However, $RF(1,1)$ yields something more than the existence of Mahlo cardinals, hyper-Mahlo cardinals and the like:

**Definition 3**

- *A* binary tree *is a class* $\boldsymbol{T}$ *of functions* $f$ *such that, for some ordinal* $\beta$, $f : \beta \longrightarrow 2$ *and such that, if* $f \in \boldsymbol{T}$ *and* $f$ *has domain* $\beta$*, then* $f$ *restricted to any ordinal less than* $\beta$ *is in* $\boldsymbol{T}$.

- *A binary tree* $\boldsymbol{T}$ *is* path-bounded *iff for every function* $F : \Omega \longrightarrow 2$*, there is an* $\alpha$ *such that* $F$ *restricted to* $\alpha$ *is not in* $\boldsymbol{T}$.

- $\boldsymbol{T}$ *is* bounded *iff there is an* $\alpha$ *such that for all* $F : \Omega \longrightarrow 2$*,* $F$ *restricted to* $\alpha$ *is not in* $\boldsymbol{T}$.

- *The* binary tree property *is that every path-bounded binary tree is bounded.*

The instance

$$\mathbf{T} \text{ is bounded} \longrightarrow \exists\beta[\mathbf{T}^\beta \text{ is bounded}]^\beta$$

of $RF(1,1)$ implies the binary tree property. So, since a cardinal $\kappa$ is weakly compact just in case it is inaccessible and $R(\kappa)$ has the binary tree property, $RF(1,2)$ implies the existence of a stationary class of weakly compact cardinals. We can of course go on to construct a stationary class of *hyper-weakly compact* cardinals; and so on.

We noted that $RF(1,0)$ not only implies, but is equivalent in $\mathcal{T}_0$ to the axioms of Successor, Replacement and Infinity. Likewise, $RF(1,1)$ not only implies, but is equivalent in $\mathcal{T}_3$ to the binary tree property. In other words, in $\mathcal{T}_0$, inaccessibility is equivalent to $\Pi^1_0$-indescribability and weak compactness is equivalent to $\Pi^1_1$-indescribability ([Hanf and Scott, 1961]).

For $m, n > 1$, there is a single $\Pi^n_m$ formula $\phi(X, y)$ such that every $\Pi^n_m$ formula containing at most the second-order variable $X$ free is equivalent in $\mathcal{T}_1$ to $\phi(X, e)$ for some finite ordinal $e$. It easily follows that, not just for $m, n = 1$, but for all $n, m > 0$, $RF(n, m)$ is equivalent to the single $\Pi^n_{m+1}$ formula

$$\psi_{n,m} = \forall X \forall n \in \omega[\phi(X, n) \longrightarrow \exists\beta\phi^\beta(X^\beta, n)]$$

in $T_1$. So $\psi_{n,m}$ expresses $\Pi^n_m$-indescribability. [Kanamori, 1994, Section 6.9] So by Lemma 2, $RF(n, m + 1)$ implies that the class of $\Pi^n_m$-indescribable cardinals is stationary. In other words, the class of $\Pi^n_m$-indescribable cardinals less than a given $\Pi^n_{m+1}$-indescribable cardinal $\kappa$ is stationary in $\kappa$. So the principle $RF(n, m)$ strictly increases in strength as $m$ increases.

Nevertheless, there is in any case a relatively low limit to the cardinals obtained by $RF(n,m)$ for arbitrary n and m. Let $[C]^n$ denote the set of all $n$-element subsets of $C$. $[C]^{<\omega}$ denotes the set of all finite subsets of C.

**Definition 4** *Let $D \subset \kappa$.*

- *$D \longrightarrow (stationary)^n_\lambda$ denotes the following partition property of $D$: for any function $f : [D]^n \longrightarrow \lambda$, there is a stationary subset $S$ of $\kappa$ such that $S \subseteq D$ and $f$ is constant on $[S]^n$.*
- *$D \longrightarrow (stationary)^{<\omega}_\lambda$ denotes the following partition property of $D$: for any function $f : [D]^{<\omega} \longrightarrow \lambda$, there is a stationary subset $S$ of $\kappa$ such that $S \subseteq D$ and, for each $n < \omega$, $f$ is constant on $[S]^n$.*
- *We may also write $D \longrightarrow (\alpha)^n_\lambda$ or $D \longrightarrow (\alpha)^{<\omega}_\lambda$ meaning that the set $S$ is to be of order type $\alpha$.*

Notice that the notation suppresses the cardinal $\kappa$; but, in each case that we don't explicitly mention it, will be determined by the context.

Now, the relative weakness of $R(n,m)$ from the point of view of large cardinal axioms is measured by the fact that, even when $\kappa$ satisfies the relatively weak principle $\kappa \longrightarrow (stationary)^2_2$, the set of totally indescribable cardinals $< \kappa$ is stationary in $\kappa$.[2]

# 6 Reflection on a Higher-order Parameter

From now on, we will assume the background theory $\mathcal{T}_3$ together with the axiom that there is an unbounded sequence of inaccessible cardinals, and $\kappa$ will denote an ordinal of that theory, i.e. an inaccessible limit of inaccessible cardinals.

[2] In fact, this holds when $\kappa$ satisfies the weaker condition of being *subtle*. A. Kanamori has pointed out to me that this follows from an easy modification of Theorem 4.1 and its proof in [Baumgartner, 1973]. Indeed, the restriction to finite types can also be removed: call $\kappa$ *absolutely indescribable* if (1) holds in $R(\kappa)$ for every formula $\phi(X)$ of order $< \kappa$ with $X$ of type 1. (The relativization $\phi^\beta(X)$ is defined in this case in the obvious way.) Then, when $\kappa$ is subtle and, in particular, when $\kappa \longrightarrow (stationary)^2_2$, the set of absolutely indescribable cardinals $< \kappa$ is stationary in $\kappa$.

As we have noted, relative to $R(\beta)$, the second-order class $A$ is $A^\beta = A \cap R(\beta)$ and so the relativization $B^\beta$ of a third- or higher-order class $B$ is

$$B^\beta = \{A^\beta \mid A \in B\}.$$

With this definition, $\phi(A)^\beta (= \phi^\beta(A^\beta))$, where $A$ is of arbitrary finite type $> 0$, expresses that $\phi(A)$ is true in $R(\beta)$ and (1) has meaning for $X$ of arbitrary finite type. But there is a problem with the generalized (1), even for $X$ of third-order: when $U$ is the class of all bounded (or, alternatively, of all unbounded) second-order classes, for example, we have the true sentence $\phi(U)$ that every class in $U$ is bounded (or unbounded); whereas for every $\beta$, $\phi^\beta(U^\beta)$ is false since $U^\beta$ is just $R(\beta+1)$ and, in particular, contains both $R(\beta)$ and the null set. Therefore, we must restrict (1) to special classes of formulas.

One plausible way to think about the difference between reflecting $\phi(A)$ when $A$ is second-order and when it is of higher-order is that, in the former case, reflection is asserting that, if $\phi(A)$ holds in the structure $\langle R(\kappa), \in, A\rangle$, then it holds in the substructure $\langle R(\beta), \in, A^\beta\rangle$ for some $\beta < \kappa$. (We are no longer considering the rank function *ran* as part of the structure, since it is definable in $\mathcal{T}_3$.) But, when $A$ is higher-order, say of third-order, this is no longer so. Now we are considering the structure $\langle R(\kappa), R(\kappa+1), \in, A\rangle$ and $\langle R(\beta), R(\beta+1), \in, A^\beta\rangle$. But the latter is not a substructure of the former, i.e. the'inclusion map' of the latter structure into the former is no longer single valued: for subclasses $X$ and $Y$ of $R(\kappa)$, $X \neq Y$ does not imply $X^\beta \neq Y^\beta$. Likewise, for $X \in R(\beta+1)$, $X \notin A$ does not imply $X^\beta \notin A^\beta$. For this reason, the formulas that we can expect to be preserved in passing from the former structure to the latter must be suitably restricted and in particular should not contain the relation $\notin$ between second- and third-order objects nor the relation $\neq$ between second-order objects. In the general case of reflecting $\phi(A)$ when $A$ is of order $n \geq 2$, one should not admit in $\phi(X)$ the relation $\notin$ between $k$th-order and $k+1$th-order objects or the relation $\neq$ between $k$th-order objects, for $1 < k < n$.

However, we will not pursue the question of the most general general formulation of the reflection principle with higher-order parameters here. Rather, we shall consider (1) for a very special class of formulas $\phi(X)$. Let $\psi$ be a formula. A *$\kappa$-instance* of $\psi$ is the result of substituting for each free variable of order $n$ (the name of) an object of order $n$ over $R(\kappa)$, for each $n$.

**Definition 5**

- *A formula $\phi$ is called* almost downward absolute (ada) *for $\kappa$ iff for each $\kappa$-instance $\psi$ of it, either $\psi$ is false in $R(\kappa)$ or else the set of $\beta < \kappa$ such that it is true in $R(\beta)$ includes a $CLUB$ set in $\kappa$.*

- *A formula is called* reflectable *for $\kappa$ if it is obtained from an ada formula for $\kappa$ by prefixing a string of zero or more quantifiers.*

- *A formula is* positive *iff it is built up by means of the operations $\wedge, \vee, \forall$ and $\exists$ from atoms of the form $x = y$, $x \neq y$, $x \in y$, $x \notin y$, $x \in Y$, $x \notin Y$, $X = Y$ and $X \in Y$.*

- *A formula is called* positive in the extended sense *iff it is obtained from a positive formula $\phi$ by the substitution of zero or more terms $S(X, \ldots, Y)$, expressing contraction operations, for free variables. When $\phi$ is first-order, we call the resulting formula first-order positive in the extended sense.*

**Lemma 3** *Let $\lambda$ be a regular uncountable cardinal. Then every first-order formula which is positive in the extended sense is ada for $\lambda$.*

First, let $\phi(X, \ldots, Y)$ be first order positive and suppose that $\psi = \phi(A, \ldots, B)$ is true in $R(\lambda)$. For some complete set of Skolem functions for $\psi$, let $C$ be the class of ordinals $\beta < \lambda$ such that $R(\beta)$ is closed under them. To see that $\beta \in C$ implies that $\psi$ is true in $R(\beta)$, just note that if $D$ is of type $> 0$ and $D \in E$ occurs in $\psi$ then it occurs positively; and so if it is false in $R(\lambda)$ it contributes nothing to the truth of $\psi$ in any $R(\beta)$. On the other hand, if it is true in $R(\lambda)$, then it is true in $R(\beta)$ for all $\beta < \lambda$.

A first-order formula which is positive in the extended sense is of the form

$$\chi(U, \ldots, V) = \phi(S(U, \ldots, V), \ldots, T(U, \ldots, V))$$

where $\phi$ is positive and the terms $S(U, \ldots, V), \ldots, T(U, \ldots, V)$ are contraction operations. Let $\psi = \chi(D, \ldots, E) = \phi(A, \ldots, B)$ be a $\kappa$-instance which is true in $R(\kappa)$, where $A = S(D, \ldots, E), \ldots, B = T(D, \ldots, E)$. We have just shown that there is a $CLUB$ set $C$ of $\beta < \kappa$ such that $\phi^\beta(A^\beta, \ldots, B^\beta)$. Let $C'$ be the set of limit ordinals in $C$. $C'$ is also $CLUB$ and, according to Lemma 1, for $\beta \in C'$, $A^\beta = S(D^\beta, \ldots, E^\beta)$ and $B^\beta = T(D^\beta, \ldots, E^\beta)$. So for $\beta \in C'$, $\chi^\beta(D^\beta, \ldots, E^\beta$. $\square$

**Definition 6**

- $\Gamma$ *is the class of first-order formulas positive in the extended sense.*
- $\Gamma_n$ *is the class of formulas*

$$\forall Y_1 \exists Z_1 \cdots \forall Y_n \exists Z_n \psi$$

*where* $\psi \in \Gamma = \Gamma_0$ *and the* $Y_i$ *are second-order. (The* $Z_i$ *may be of any order.)*

So the $\Gamma_n$ formulas are all positive in the extended sense and so, by Lemma 3, reflectable for all regular cardinals.

We want to study the cardinals $\kappa$ for which $R(\kappa)$ satisfies (1) for all formulas in $\Gamma_n$. More generally:

**Definition 7** *$D \subseteq \kappa$ is $n$-*reflective *iff*

$$\forall X[\phi \longrightarrow \exists \beta \in D \phi^\beta]$$

*for all* $\phi \in \Gamma_n$ *containing just the free variable* $X$ *of arbitrary order* $> 1$*.*

We shall investigate the class of subsets of cardinals $\kappa$ which are $n$-reflective. In particular, we shall prove

**Theorem 2** *If* $D \subseteq \kappa$ *is n-reflective, then* $D \longrightarrow (\mathit{Stationary})^{n+1}$*.*

# 7 n-Stationarity

**Definition 8**

- *For* $n > 0$*, a function* $K$ *defined on* $[\kappa]^n$ *such that, for* $\kappa > \beta_1 > \ldots > \beta_n, K(\beta_1, \ldots \beta_n) \subseteq R(\beta_n)$*, is called an* n-sequence *(on* $\kappa$*).*
- *When, more strictly,* $K(\beta_1, \ldots \beta_n)$ *is always* $\subseteq \beta_n$*, then* $K$ *is called a* thin n-sequence *on* $\kappa$*.*
- *Let* $K$ *be an n-sequence on* $\kappa$*. A subset* $H$ *of* $\kappa$ *is called* homogeneous *for* $K$ *iff there is a* $B \subseteq R(\kappa)$ *such that* $K(\beta_1, \ldots \beta_n) = B \cap R(\beta_n)$ *for all* $(\beta_1, \ldots \beta_n) \in [H]^n$ *with* $\beta_1 > \ldots > \beta_n$*.*

If $K$ is a 1-sequence on $\kappa$, then the maximal homogeneous classes for $K$ are precisely those of the form

$$[K, B] = \{\alpha \in \kappa \mid K(\alpha) = B \cap R(\alpha)\}$$

for some $B \subseteq R(\kappa)$.

**Lemma 4** *For every 1-sequence $K$ there is a thin 1-sequence $\hat{K}$ such that, if $H$ is homogeneous for $\hat{K}$, then $H \cap C$ is homogeneous for $K$, where $C$ is the $CLUB$ class of inaccessibles or limits of inaccessibles less than $\kappa$.*

Choose a bijection

$$F \ : \ \kappa \longleftrightarrow R(\kappa)$$

such that its restriction $F_\alpha$ to $\alpha$ is a bijection

$$F_\alpha \ : \ \alpha \longleftrightarrow R(\alpha)$$

for each $\alpha \in C$. For $\alpha \in C$, set

$$\zeta \in \hat{K}(\alpha) \longleftrightarrow F(\zeta) \in K(\alpha)$$

For $\alpha \notin C$, set $\hat{K}(\alpha) = \emptyset$. $\square$

**Definition 9** *Let $D \subseteq \kappa$.*

- *$D$ is* 0-stationary *iff it is stationary.*
- *$D$ is $n+1$-stationary iff every 1-sequence $K$ has a homogeneous $n$-stationary set $\subseteq D$—i.e. there is an $X$ such that $[K, X] \cap D$ is $n$-stationary.*

If $D$ is n-stationary and $C$ is $CLUB$, then $D \cap C$ is n-stationary. So, by the lemma above, "1-sequence" can be equivalently replaced by "thin 1-sequence" in the definition of the notion of $n+1$-stationary.

We will prove that $D \subseteq \kappa$ is n-reflective iff it is n-stationary.

Using the contraction operations, we can code an $n$-tuple of objects by single object whose order is the maximum of the orders of the objects in the $n$-tuple.

**Definition 10** *Let $D \subseteq \kappa$.*

- *We define the notion of an $n$-*box *for $D$. An $n$-box is of order $n+2$.*
  - *A 0-box for $D$ is a CLUB class $C$ such that $C \cap D = \emptyset$.*
  - *An $n+1$-box for $D$ is a class $T$ of triples $(K, X, S)$ for some fixed 1-sequence $K$, called the* witness *for $T$, such that, for every second-order class $X$, there is an $S$ with $(K, X, S) \in T$ and $S$ is an $n$-box for $[K, X] \cap D$.*
- *Let $T$ be of order $n+2$. We define the $\Gamma_n$ formula $\theta_n(T)$ by induction on $n$:*
  - $\theta_0(T) \longleftrightarrow T$ *is an unbounded subset of* $\kappa$
  - $\theta_{n+1}(T) \longleftrightarrow \forall X \exists K \exists S[(K, X, S) \in T \wedge \theta_n(S)]$

The order of the bound variables $X$, $A$ and $S$ are, respectively, $2, 2$ and $n+2$. Note that $\theta_{n+1}(T)$ can be put in the form of a $\Gamma_{n+1}$ formula by means of contracting operations.

**Lemma 5**

*a) $D$ is not $n$-stationary iff it has an $n$-box.*

*b) If $T$ is an $n$-box for some $D$, then $\theta_n(T)$ is true (in $R(\kappa)$).*

*c) If $T$ is an $n$-box for $D$, then $\theta_n(T)^\beta$ is false for all $\beta \in D$.*

*d) If $D$ is $n$-reflective, then it is $n$-stationary.*

The proof of a) and b) is immediate by induction on $n$. d) follows immediately from a)-c). For c), let $T$ be an n-box for $D$, $\beta \in D$, and assume that $\theta_n(T)^\beta$ is true. We derive a contradiction, by induction on n.

$n = 0$. $\theta_0(T)^\beta$ asserts that $T^\beta = T \cap \beta$ is unbounded in $\beta$. Since $T$ is $CLUB$, $\beta \in T$, contradicting $D \cap T = \emptyset$.

$n = m+1$. Let $K$ be the witness for $T$. Then there are $K'$ and $S'$ such that $\langle K', K(\beta), S' \rangle \in T^\beta$ and $\theta_m^\beta(S')$. $K' = K^\beta$, $K(\beta) = A^\beta$ for some $A$, and $S' = S^\beta$, where $\langle K, A, S \rangle \in T$. So $S$ is an m-box for $D \cap [K, A]$, $\theta_m(S)^\beta$ is true, and $\beta \in D \cap [K, A]$—a contradiction. □

So we have proved half of

**Lemma 6** *$D$ is $n$-reflective iff it is $n$-stationary.*

To prove the other half, let $D$ be n-stationary and let $\phi \in \Gamma_n$ be true in $R(\kappa)$. We proceed by induction on n.

$n = 0$: By Lemma 3, there is a $CLUB$ class $C \subseteq \kappa$ such that $\phi^\beta$ for all $\beta \in C$, and $D$ is stationary.

$n = m + 1$: Let $\phi = \forall Y \exists Z \psi(Y, Z)$, where $\psi \in \Gamma_m$. If $\phi^\beta$ is false for all $\beta \in D$, then we can define a 1-sequence $K$ such that $\exists Z \psi^\beta(K(\beta), Z)$ is false for all $\beta \in D$. Choose $A$ so that $E = D \cap [K, A]$ is m-stationary. For $\beta \in E$, $\exists Z \psi^\beta(A^\beta, Z)$ is false. But $\exists Z \psi(A, Z)$ is true. Choose $B$ so that $\psi(A, B)$. But this is a $\Gamma_n$ formula and $\psi^\beta(A^\beta, B^\beta)$ is false for all $\beta$ in the m-stationary class $E$, contrary to the induction hypothesis. □

# 8 Ineffability

**Definition 11** *For $n > 0$, $D \subseteq \kappa$ is* n-ineffable *iff every thin n-sequence $K$ has a stationary homogeneous class $\subseteq D$.*

**Theorem 3 ([Baumgartner, 1973])** *Let $\kappa$ be a regular uncountable cardinal and $n > 0$. $D \subseteq \kappa$ is n-ineffable iff it satisfies $D \longrightarrow (stationary)^{n+1}$.*

So, since stationary sets $D$ trivially satisfy $D \longrightarrow (stationary)^1$, we need only show that n-stationary classes are n-ineffable for $n > 0$.

Baumgartner's proof shows: For every thin 1-sequence $K$, there is a club class $C_K$ and an $f_K \mid [\kappa]^2 \longrightarrow 2$ such that, if $H$ is homogeneous for $f_K$, then $C_K \cap H$ is homogeneous for $K$.

**Lemma 7** *Let $K$ be an $n + 1$-sequence on $\kappa$, $n > 0$. Then there is a 1-sequence $K'$ on $\kappa$ and a function $K''$ defined on $R(\kappa+1)$ such that $K''(X)$ is an n-sequence on $\kappa$ for all $X \subseteq R(\kappa)$ and, if $H$ is homogeneous for $K''(X)$, then $H \cap [K', X]$ is homogeneous for $K$.*

Let $K'$ be defined by

$$G(\alpha) = \{\langle \beta_1, \ldots, \beta_n, K(\alpha, \beta_1, \ldots, \beta_n)\rangle \mid \beta_1 < \alpha\}.$$

For each $X \subseteq R(\kappa)$, define $K''(X)$ by

$$K''(X)(\beta_1, \ldots, \beta_n) = K(\alpha, \beta_1, \ldots, \beta_n)$$

for all (i.e. any) $\alpha \in [K', X]$ with $\alpha > \beta_1$. If there is no such $\alpha$, let $K''(X)(\beta_1, \ldots, \beta_n) = \emptyset$. □

Theorem 2 now follows from

**Lemma 8** *If $n > 0$ and $D \subseteq \kappa$ is n-stationary then $D$ is n-ineffable.*

(When $n = 1$, the converse also holds.) Let $n = m + 1$ with $m > 0$ and let $K$ be an n-sequence (thin or otherwise.) Choose $A$ so that $D \cap [K', A]$ is m-stationary and hence m-ineffable. So $K''(A)$ has a stationary homogeneous class $H \subseteq D \cap [K', A]$. So $H$ is homogeneous for $K$. □

# 9 Bounds on Higher-Order Reflection

A very generous upper bound for the least $n$-reflective cardinal is simply obtained:

**Theorem 4** *Let $\kappa$ be measurable and let $\mathbf{U}$ be a normal ultrafilter on $\kappa$. Then, for every thin 1-sequence $K$, there is a set $A \subseteq \kappa$ such that $[K, A] \in \mathbf{U}$. So every $D \in \mathbf{U}$ is n-stationary, i.e. n-reflective.*

The second assertion easily follows from the first: Every set in $\mathbf{U}$ is stationary, i.e. 0-stationary, since $\mathbf{U}$ is a normal ultrafilter. Assume that every set in $\mathbf{U}$ is n-stationary and let $K$ be a thin 1-sequence. There is an $A \subseteq \kappa$ such that $[K, A] \in \mathbf{U}$. So for $D \in \mathbf{U}$, $[K, A] \cap D$ is in $\mathbf{U}$ and so is n-stationary. Hence $D$ is $n + 1$-stationary.

To prove the first assertion, let $K$ be a thin 1-sequence and define $B_\alpha = \{\zeta < \kappa \mid \alpha \in K_\zeta\}$. $A \subseteq \kappa$ is defined by

$$\alpha \in A \;\Leftrightarrow\; B_\alpha \in \mathbf{U}$$

Let $C_\alpha$ be $B_\alpha$ if the latter set is in $\mathbf{U}$ and let it be $\kappa - B_\alpha$, otherwise. In any case, $C_\alpha \in \mathbf{U}$. Then $\alpha \in [K, A]$ iff $K_\alpha = A \cap \alpha$ iff

$$\forall \beta < \alpha[\alpha \in B_\beta \;\Leftrightarrow\; B_\beta \in \mathbf{U}$$

iff $\forall \beta < \alpha[\alpha \in C_\beta]$ iff $\alpha \in \Delta_\zeta C_\zeta$, where, since $\mathbf{U}$ is a normal ultrafilter containing each $C_\alpha$, $\Delta_\zeta C_\zeta = \{\alpha \mid \forall \beta < \alpha[\alpha \in C_\beta]\}$ is in $\mathbf{U}$. □

Peter Koellner has shown that there is a cardinal $\delta$ less than the Erdös cardinal $\kappa(\omega)$ ($\kappa(\alpha)$ being the least $\kappa$ such that $\kappa \longrightarrow (\alpha)^{<\omega}$) which is $n$-reflective for each n.

A simple extension of the notion of $n$-reflectiveness is obtained as follows:, let $\Gamma^m_n$ be the class of formulas

$$\forall Y_1 \exists Z_1 \cdots \forall Y_n \exists Z_n \psi$$

where $\psi \in \Gamma = \Gamma_0$ and the $Y_i$ are (m+1)-order. (The $Z_i$ may again be of any order.) So $\Gamma_n = \Gamma_n^1$. The $\Gamma_n^*$ formulas are all positive in the extended sense and so, by Lemma 3, reflectable for all regular cardinals. We now could study the cardinals $\kappa$ for which $R(\kappa)$ satisfies (1) for all formulas in $\Gamma_n^m$. More generally: call $D \subseteq \kappa$ *n-reflective*$^m$ iff

$$\forall X[\phi \longrightarrow \exists \beta \in D \phi^\beta]$$

for all $\phi \in \Gamma_n^m$ containing just the free variable $X$ of arbitrary order $> 1$. Generalizing the notion of an $n$-sequence: For $n > 0$, a function $K$ defined on $[\kappa]^n$ such that, for $\kappa > \beta_1 > \ldots > \beta_n, K(\beta_1, \ldots \beta_n) \subseteq R(\beta_n + m)$, is called an *n-sequence*$^m$ (on $\kappa$). Let $K$ be an $n$-sequence$^m$ on $\kappa$. A subset $H$ of $\kappa$ is called *homogeneous* for $K$ iff there is a $B \subseteq R(\kappa + m)$ such that $K(\beta_1, \ldots \beta_n) = B^{\beta_n}$ for all $(\beta_1, \ldots \beta_n) \in [H]^n$ with $\beta_1 > \ldots > \beta_n$. Finally, we may generalize the notion of $n$-stationarity:

**Definition 12** *Let $D \subseteq \kappa$.*

- *$D$ is* 0-stationary$^m$ *iff it is stationary.*
- *$D$ is $n+1$-stationary$^m$ iff every 1-sequence$^m$ $K$ has a homogeneous n-stationary$^m$ set $\subseteq D$.*

The proof of Lemma 6 extends straightforwardly to a proof that n-reflectiveness$^m$ is equivalent to n-stationarity$^m$. However, it is not yet known where in the chart of large cardinals this leads. It appears that Koelner's argument (for $m = 1$) extends to show that there are cardinals $< \kappa(\omega)$ which are $n$-reflective$^2$ for all $n$; but it is not clear at the moment whether the argument can be extended to $m > 2$.

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