

Philosophy 30200  
Historical and Philosophical Foundations of  
Set Theory  
Syllabus: Autumn:2005

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Meeting times: Wednesday 9:30-1200, starting Sept 28.

Meeting place: Classics 11.

I will be away October 12 and 19: We may arrange another lecture, whose title (subject-matter) will be announced (decided upon) later on.

There will be one lecture a week, with discussion. Following is a list of the lectures, with required readings and, in curly brackets, suggested readings. (Some of the latter may be too technical or in the wrong language for some members of the class.) All of the required readings (at least) will be found on e-reserve. Those taking the course for credit will be expected to produce a paper relating to the subject matter of the course. I will read and comment on drafts turned in by the end of eighth week, so that there will be an opportunity to revise.

Set theory in its lower reaches, however disguised, is part of the warp and woof of contemporary mathematics. But these lower reaches point irresistibly (at least for those unwilling to close their eyes) to something more—to higher infinities—that, however much of it one brings within the bounds of ordinary mathematics by means of axioms, always spills over the bounds and demands new axioms. Here is the essential incompleteness of our understanding that was suggested by Kant with his antinomies, if not quite where he located it.

On what grounds do we accept new axioms? The question invites some historical perspective on the mathematics of the infinite that we already

have, as well as philosophical perspective on the question of what constitutes “grounds”. And when we add to this question the reactionary challenges that essential incompleteness and the whiff of paradox associated with it engender, and the tendency to blend these with the historical resistance to the actual infinite, a need for both historical and conceptual clarity becomes even more apparent. The need was exacerbated by the wide misunderstanding, especially in the first half of the twentieth century (but spilling over into the present), of the essential incompleteness of set theory, leading to the so-called *paradoxes of set theory*.

This course will thus be a blend of philosophy, history (of the philosophy-through-history species) and a bit of mathematics. The main issues that we will discuss will not require much knowledge of the latter, but it will require a minimum skill in understanding elementary arguments. I blush somewhat at the broad strokes with which I will paint the history up to the nineteenth century in these lectures; but clear traces of influence go back at least to Greece in the fourth century B.C.: We have lots to cover.

Although we shall in time have discussed the axioms of set theory, this is not a course in axiomatic set theory. I list, in order of increasing demand on the reader, a few texts or treatises on that subject.: [Enderton, 1977] (quite elementary) [Kunen, 1980] (axiomatic set theory and independence proofs: beautifully written) [Jech, 1978] (a treatise) [Kanamori, 1994] (A very lively and attractive treatment, focusing on the investigation of axioms asserting the existence of large transfinite numbers).

**Lecture 1.** Introduction: Now and Then. (Sept. 28)

**Lecture 2.** Exact Science in Ancient Greece: Uncovering the Infinite. (Oct. 5)

**Readings:** Aristotle’s *Physics*, Book III, Ch. 4-8 and Book VI, Euclid’s *Elements*, Books I, V, X and XII. (Look at the definitions, postulates, common notions, and theorems at least), {[C.H. Edwards, 1979, pp. 10-19].}

**Lecture 3.** Sets, the Infinite, and Paradoxes in Late Medieval and Early Modern Times : Philosophy and Mathematics. Oct 26)

**Readings:** [Murdoch, 1982], {[Duhem, 1985, Ch. 1-2]}, [Mancosu, 1996, Ch. 3-4], [Grattan-Guinness, 1980, Ch 2], An excerpt from [Berkely, 1834], {[C.H. Edwards, 1979, Ch. 8 and 9]}. Primary sources concerning the origin and development of the calculus can be found in [Struik, 1969, Ch. 4-5].}

**Lecture 4.** Sets, Functions, and the Actual Infinite in Nineteenth Century Mathematics. (Nov. 2)

**Readings:** [Grattan-Guinness, 1980, Ch 3 and (optional) 4], an excerpt from [Bolzano, 1851], [Ferreirós, 1999, Ch 1].

**Lecture 5.** The Foundations of Arithmetic and Analysis. (Nov 9) **Readings:** [Dedekind, 1887; Dedekind, 1872], {[Frege, 1884]}.

**Lecture 6.** Cantor: Uncountable Sets, Well-orderings, Transfinite Numbers, and Powersets. (Nov. 16)

**Readings:** [Cantor, 1874], [Cantor, 1883a], [Zermelo, 1908], {[Cantor, 1891]}.

**Lecture 7.** The so-called ‘Paradoxes of Set Theory’ and Reactions to It. (Nov. 23)

**Readings:** An excerpt from [Hilbert, 1900], [Russell, 19308], [Baire, Borel, Hadamard and Lebesgue, 1905], [Weyl, 1921], [Bishop and Bridges, 1985, Ch. 1].

**Lecture 8.** Cumulative Hierarchies of Sets, Zermelo-Fraenkel Set Theory. (Nov. 30)

**Readings:** [Zermelo, 1930], [Gödel, 1964], [Scott, 1974].

**Lecture 9.** Essential Incompleteness [the Absolute Infinite] and the Search for New Axioms. (Whenever)

**Readings:** [Jensen, 1995], [Feferman, Friedman, Maddy and Steel, 2000, Especially the piece by J. Steel].

## References

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