THE UNIVERSITY OF CHICAGO
DEPARTMENT OF ECONOMICS
Economics 209: Honors Econometrics
Problem Set #2

This problem set contains 7 questions. They cover asymptotics, maximum likelihood, and OLS.

Asymptotics

Question A1: MLE revisited.

In Question MLE4 from Problem Set #1 you found that the MLE estimator for a random sample from the pdf given by:

$$f(x;\theta) = \frac{1}{\theta}$$

for $x \in (0, \theta)$ and $\theta \in (0, \infty)$ was given by $\widehat{\theta}^{MLE} = \max_i \{x_i\}$. In this question we explore further properties of this estimator.

a. Find the cdf for the random variable $z = \max_i \{x_i\}$. (Hint: Let $F(z)$ denote the probability that a given observation is less than z. If an observation is the maximum of a sample then all the x_i must be less than z. Determine the probability of this event and you will have the cdf for the maximum of a sample.)

b. What is the pdf of z?

c. Compute $E[z]$ using the density from b. Is the estimator unbiased?

d. Find the variance of the estimator.

f. Prove that the estimator is consistent.

Question A2: More asymptotics.

Let $\overline{X}$ denote the sample mean in random sampling, sample size n, from a population in which the random sample $X\tilde{}\exp(\lambda)$. That is, the density of X is given by

$$f(X;\theta) = \lambda e^{-\lambda x}$$

for $x > 0$. For convenience let $\theta = E[X]$.

a. Find $E[X]$.

b. Find $Var[X]$.

c. Find $E\left[\overline{X}\right]$.

d. Find $Var\left[\overline{X}\right]$.

e. Show that $p\lim \overline{X} = \theta$.

f. Find the limiting distribution of $\sqrt{n}\left(\overline{X} - \theta\right)$.

Question A3: Asymptotics of a function.

Consider the setup from question A2. Let $U = \frac{1}{\overline{X}}$.

a. Find $p\lim U$. You must be explicit about the use of any theorems.

b. Find the limiting distribution of $\sqrt{n}(U - \lambda)$.

c. Use your result to approximate $\Pr\left(U \leq \frac{5}{2}\right)$ in random sampling, size 16, from an exponential distribution with $\lambda = 2$.

Regression

Question R1: Variances.

Compute the variance of the OLS estimator using the definition of variance. That is, use the fact that:

$$Var[\beta] = E\left[\left(\widehat{\beta} - \beta\right)\left(\widehat{\beta} - \beta\right)'\right].$$

Justify every step of the proof.

Question R2: Relationship of MLE to OLS.

Consider the following simple regression model:

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

and assume that the X_i are predetermined. Let Y_i come from a normal distribution with expectation $E[Y_i] = \beta_0 + \beta_1 X_i$ and variance σ^2.

a. What is the likelihood function of the sample? Explain clearly the steps in you derivation.

b. What are the first order conditions?

c. Find $\widehat{\beta}_0^{MLE}$ and $\widehat{\beta}_1^{MLE}$.

d. Explain clearly why these estimators are identical to the OLS estimators.

e. Find the information matrix.

f. Find $\widehat{\sigma}^2_{MLE}$. Is it equal to $\widehat{\sigma}^2_{OLS}$?

Question R3: The covariance of the OLS estimators.

Let the simple regression model be given by:

$$Y_i = \beta_0 + \beta_1 X_i + u_i.$$

a. Compute $Cov\left(\widehat{\beta}_0, \widehat{\beta}_1\right)$.

b.What is its sign?

c. Explain why you answer makes sense. (Hint: use a graph).

Question R4: The Least Absolute Deviations estimator.

Consider the following simple regression model:

$$Y_i = \beta_0 + u_i.$$

a. What is the criterion function that OLS instructs us to minimize?

b. Show that the OLS estimator is given by:

$$\widehat{\alpha}^{OLS} = \overline{Y}.$$

c. Suppose that instead, you want to minimize the sum of the absolute deviations by choice of α :

$$\min_{\alpha} \sum_{i=1}^{n} |Y_i - \alpha|.$$

d. Show that $\widehat{\alpha}^{LAD} = median\{Y_i\}$. Hint: Start with $n = 3$ and, without loss of generality, assume that $Y_1 < Y_2 < Y_3$. Then, consider $n = 4$. Then, state the general case and prove it.