

# Chapter 6

## Profit Maximization

### 6.1 Introduction

In this section we begin our analysis of the firm's behavior in **product markets**. The behavioral assumption that underlies the analysis in this chapter is that of **profit maximization**, but we will see that there are two ways to study profit maximization. We will study the firm's **scale** decision first. That is, once the firm has selected its cost minimizing factor inputs, it must look to the goods market and decide what quantity to supply. The two stage problem uses the cost function to summarize conditions in the factor markets, and in the second stage assumes firms maximize profits by selecting the optimal quantity given factor market prices and its technology –as summarized by the cost function– and the goods price. In this chapter we will consider **perfectly competitive goods markets** and we will analyze the firm's price taking assumption carefully. Then, we will study the one shot profit maximization problem directly. In this alternative formulation, firms select quantities of factor inputs to maximize the difference between total revenues and total costs. These demand functions are known as **factor demand functions**. A key result is that under the price taking assumption the answer to both formulations of the profit maximization problem coincide. However, the two stage cost-scale decision is more general and applies even when the goods market is not competitive. The one-shot profit maximization problem, on the other hand, can only be used when both factor and goods markets are competitive. After establishing the equivalence of both problems we will study the properties of the

firm's factor demand functions. Then we will derive the **profit function** which tells us the maximum profit the firm can obtain, given factor market prices, and goods market prices.

Don't forget to address payments to factors of production Euler's theorem etc. Use Euler's theorem to explain quasi-rents as being inexistent in the long run, rents however may persist in the long run like entrepreneurial ability, or uncertainty as a payment for risk.

In the applications section we will study two cases in which profit maximization may not be a good behavioral assumption. One instance is that of **discrimination**, in which firms are interested in maximizing some function not only of profits but of other variables –such as race or gender– as well. A second instance is that of the **non-profit sector**.

## 6.2 The Price Taking Assumption

In this chapter we study the behavior of a profit maximizing firm that operates in a competitive market. The price taking assumption requires firms to take price as given when they make their output decisions. One way of formalizing this idea is to consider the elasticity of the market demand curve facing the firm, which is defined as:

$$\epsilon^d = \frac{dy}{dp} \frac{p}{y}.$$

A firm that operates in a perfectly competitive market assumes that the market demand curve it faces is infinitely elastic. Essentially, this assumption means that the firm can not influence price by its quantity choice. A firm that operates in a perfectly competitive market assumes that the demand curve it faces is flat. Therefore, we require that  $\frac{dp}{dy} = 0$ , or equivalently,  $\frac{dp}{dy} \frac{y}{p} = 0$ . If this assumption holds, then  $\epsilon^d$  must be infinite since both quantities are inversely related. This implies that the firm can sell any amount it wishes at the given market price, but nothing at a higher price. The firm can not take advantage of the elasticity of market demand when it makes its output decision. It is in this sense that the firm is a price taker.

The assumption of perfect competition is a theoretical construct designed to approximate the behavior of firms under certain conditions and may be inappropriate in certain industries. Therefore, it is desirable to

determine the characteristics of markets in which the assumption of perfect competition is reasonable. That is, we would like to determine the conditions under which we can assume that  $\epsilon^d$  is very large.

Intuitively speaking, a firm with small **market share** should not be able to influence price greatly.

Also, a firm is less likely to influence price in an industry in which market demand is very elastic. The **price elasticity of demand**, then, is also an important factor. Notice that there is a difference between market demand and the market demand curve facing the firm.

Finally, if other firms in the industry are very responsive to changes in market prices, then, it is unlikely that a firm can influence price through its output decision. The **price elasticity of supply of other firms in the industry** is the third important factor.

As we show in the appendix, we can decompose the elasticity of demand facing a particular firm into two effects as follows:

$$\epsilon^d = \frac{1}{ms} E_D - \left( \frac{1}{ms} - 1 \right) E_S,$$

where  $ms$  denotes the firm's market share,  $E_D$  denotes the price elasticity of demand and  $E_S$  denotes the price elasticity of supply of all other firms in the industry.

Notice that  $\epsilon^d$  tends to be large when  $ms$  is small, when  $E_D$  is large, and when  $E_S$  is large.

## 6.3 Profit Maximization

Consider the problem of a firm that has already chosen its optimal combination of factors to produce an arbitrary amount of output  $y$  in competitive factor markets. Now, the firm looks to goods markets and decides the quantity of the final good to supply for every possible price. That is, given its costs and market price of the good the firm makes its scale decision. The solution to the firm's profit maximization problem is its **supply function**. Notice that the supply function will be a function of the goods market price,  $p$ , since the firm will compare revenues to costs to decide the scale of its operation.

Consider, then, the objective function of a profit maximizing firm that

operates in a competitive goods market:

$$\max_y py - C(w, r, y).$$

This is a very simple one variable maximization problem with first order condition:

$$[y]: p = \frac{dC(w, r, y)}{dy}.$$

The first order condition implies that the firms optimal scale decision is a function of factor prices,  $r$  and  $w$ , and output price,  $p$ :

$$y^* = y^*(w, r, p)$$

Notice the interpretation of the firms first order condition. It tells us that the profit maximizing competitive firm will choose its output to equate marginal revenue and marginal cost. Since the firm is assumed to be a price taker in goods markets, marginal revenue is constant and equal to  $p$ . That is, the firm does not consider the impact that its scale decision has on market price. Marginal revenue, the addition to revenue from the last unit firm sold by the firm, is constant and equal to  $p$  dollars.

Suppose, on the other hand, that the firm chose an output that was not consistent with the first order condition. Let us assume, for example, that price exceeds marginal cost:

$$p > \frac{dC(w, r, y)}{dy}.$$

A profit maximizing firm would not be satisfied with this outcome because they could increase profits by expanding output. One more unit of output would increase revenues by  $p$  and would increase cost by only  $\frac{dC(w, r, y)}{dy}$ , which is less than  $p$ . Therefore, the profit maximizing firm would exhaust these gains and would produce where price was exactly equal to marginal cost. The case in which price is strictly less than marginal cost is left as an exercise.

Notice that the second order condition for the profit maximization problem is very important, since it is telling us that in order to have a maximum the cost function must be convex in output:

$$-\frac{d^2C(w, r, y)}{dy^2} < 0.$$

One other important property of this formulation of the profit maximization problem is that it holds for any goods market structure. We solved the profit maximization problem of a competitive firm and saw that the optimal quantity can be found where marginal cost is equal to (a constant) marginal revenue. However, if we wanted to consider a non-competitive market for goods we would need to consider the effect that the firm's quantity choice has on market price. We will study Monopoly in Chapter xxxx.

### 6.3.1 Maximum Profits and Returns to Scale

The relationship between returns to scale and profit maximization is seen by looking at the second order condition which requires that costs be convex in output. As we saw in Chapter xxxx, convexity of the cost function essentially requires a decreasing returns to scale technology. Notice that the second order is a sufficient condition, and therefore....

To be completed.

We know from the duality results that if the cost function is convex in output, then technology must exhibit decreasing returns.

To be completed.

If the firm possessed a constant returns to scale technology, for example, the cost function would be linear in output:

$$C(w, r, y) = c(w, r, 1) y$$

In this case the second order condition would be violated. and the only possible solution involves zero profits and an indeterminate amount of output. In this case, we know that doubling input use doubles output and doubles cost so the only possible solution is zero profits and there is an infinite number input bundles that are optimal. The firm is indifferent between them since they all give zero profit and hence, the supply function is perfectly elastic: the firm will supply any output as long as it cover its costs. When firms possess a constant returns to scale technology, the solution to the profit maximization problem is not unique. Note that we can rewrite the profit maximization problem as:

$$\max_y (p - c(w, r, 1)) y$$

and any value of  $y$  is maximal, as long as:

$$p = c(w, r, 1).$$

When there are increasing returns to scale, no profit maximizing output plan exists.

To be completed.

## 6.4 The Profit Function

Suppose that a firm faces competitive factor and goods markets. Then, an alternative formulation of the firm's profit maximization problem is given by:

$$\max_{k,l} pf(k, l) - rk - wl.$$

Notice that in this case we assume the firm has a given technology and faces given factor and goods prices. The firm's economic problem is to find the capital-labor combination that maximizes the difference between **revenues**, given by  $pf(k, l)$ , and **total costs**, given by  $rk + wl$ .

This is an unconstrained multivariate optimization problem. Therefore we solve it by standard optimization techniques. The first order conditions of the problem are given by:

$$\begin{aligned} [l] : \quad pf_l(k^*, l^*) &= w \\ [w] : \quad pf_k(k^*, l^*) &= r \end{aligned}$$

and the solution is the set of **factor demand functions**:

$$\begin{aligned} l^* &= l^*(w, r, p) \\ k^* &= k^*(w, r, p) \end{aligned}$$

Unlike the set of conditional factor demand functions, demand functions depend only on factor prices  $w$  and  $r$ , and the goods price,  $p$ . In this formulation of the profit maximization problem, the firm selects the best combination of inputs to maximize the difference between revenues and costs. The scale decision is implicit in the above formulation of the problem since optimal output is just equal to:

$$y^*(w, r, p) = f(k^*(w, r, p), l^*(w, r, p)).$$

Profit maximization instructs the firm to equate **value marginal product** of each factor to its market price. The reason is simple. Value marginal product represents the addition to revenue of one more unit of the factor, while its market price represents the marginal cost of using one more unit of that factor. Suppose that the firm chose an input bundle that did not satisfy the first order condition. In particular, suppose that

$$pf_l(k^*, l^*) > w.$$

Then, the firm could increase its profits by increasing the amount of labor that it uses. The **marginal unit of labor** (the last unit used by the firm) contributes  $pf_l(k^*, l^*)$  to revenue, while it costs the firm only  $w$ . Therefore, using a little bit more labor in production would generate more revenue than cost. This is a net addition to profit. The firm will continue to do this until it exhausts the profits it can earn from labor. This requires that the first order condition hold with equality.

### 6.4.1 Equivalence of both Problems

If we study a competitive firm, then both statements of the profit maximization problem give the same solution. The justification is simple, under competition both problems have the same first order condition. The key step is that in a competitive market price is equal to marginal cost.

Let us consider the scale decision formulation of the problem first. In this setting, the first order conditions require that price equal marginal cost, which is the Lagrange multiplier of the cost minimization problem:

$$p = \frac{dC(w, r, y)}{dy} = \mu^*.$$

The last equality follows because we are using the cost function in the scale optimization problem. In addition, we know that the cost function is a minimum value function and is derived from the first order conditions of the cost minimization problem:

$$\mu^* = \frac{w}{f(k^*, l^*)}$$

and

$$\mu^* = \frac{r}{f(k^*, l^*)}.$$

Combining these three results, we obtain:

$$pf(k^*, l^*) = w$$

and

$$pf(k^*, l^*) = r$$

which are precisely the first order conditions of the profit maximization problem. Notice that the key step is that in a competitive goods market price must equal marginal cost. If the goods market is not competitive there would be a wedge between price and marginal cost and we would be unable to substitute price for  $\mu^*$ . The equivalence between both problems would break down.

Suppose, then, that we solve the one shot profit maximization problem, but that price exceed marginal cost.

To be completed.

## 6.4.2 Properties of Factor Demands

**Property I:** Homogeneous of degree 0 in  $w$ ,  $r$ , and  $p$ .

$$k^*(cw, cr, cp) = k^*(w, r, p).$$

## 6.4.3 Properties of the Profit Function

If we evaluate the objective function at the profit maximizing level of output, we obtain the maximum value function known as the profit function:

$$\begin{aligned} \pi^*(w, r, p) &= py^*(w, r, p) - wl^*(w, r, p) - rk^*(w, r, p) \\ &= pf(k^*, l^*) - wl^* - rk^*. \end{aligned}$$

The profit function tells us the maximum profit a firm can obtain given factor market and goods market conditions. It has several very interesting properties that do not depend on the properties of the technology, they follow simply from the assumption of maximization.

**Property I:** The profit function is homogeneous of degree 1 in  $w$ ,  $r$ , and  $p$ :

$$\pi^*(cw, cr, cp) = c\pi^*(w, r, p).$$



This property makes intuitive sense.

**Property II:** The derivative of the profit function with respect to factor prices gives the factor demand functions:

$$\frac{d\pi^*}{dw} = -l^*(w, r, p)$$

and

$$\frac{d\pi^*}{dr} = -k^*(w, r, p).$$

**Property III:** The derivative of the profit function with respect to  $p$  is the firm's supply function. This is known as Hotelling's Lemma:

$$\frac{d\pi^*(w, r, p)}{dp} = y^*(w, r, p).$$

This is an application of the envelope theorem, which we just studied.

**Property IV:** The profit function is convex in

To be completed.

**Property V:** The profit function is non-decreasing in output prices:

$$\frac{d\pi^*}{dp} \geq 0.$$

**Property VI:** The profit function is non-increasing in input prices:

$$\frac{d\pi^*}{dw} \leq 0$$

and

$$\frac{d\pi^*}{dr} \leq 0.$$

## 6.5 Rents and Euler's Theorem

One aspect that economists are concerned about has to do with the distribution of income between the different factors. In order to study this we rely on Euler's theorem, which gives us a very nice decomposition theorem for homogeneous functions. Suppose that the technology is constant returns to scale.

To be completed.

## 6.6 Quasi-Rents

Fixed in the short term but not in the long run.

## 6.7 Applications

### 6.7.1 Discrimination

To be completed.

### 6.7.2 Marshall's Other Laws

To be completed.

### 6.7.3 Non-Profits

To be completed.

## 6.8 Exercises

## 6.9 Appendix

Let  $Y^S(p)$  denote aggregate supply in the industry. Let  $q(p)$  denote supply by other firms in the industry and let  $y(p)$  denote a particular firm's supply. Clearly, then,

$$Y^S(p) = q(p) + y(p)$$

Also, let  $p = P(Q^D)$  denote inverse market demand. Notice that lower case  $p$  denotes a particular price, while upper case  $P$  denotes the inverse demand function. In equilibrium quantity demanded equals quantity supplied, therefore,

$$Y^S(p) = Q^D.$$

The equilibrium condition of the market is given by:

$$p = P(q(p) + y(p)).$$

If we totally differentiate the equilibrium condition with respect to  $y$ , we obtain:

$$\frac{dp}{dy} = \frac{dP}{dQ^D} \left( \frac{dq}{dp} \frac{dp}{dy} + \frac{dy}{dp} \frac{dp}{dy} \right).$$

which we can simplify to obtain:

$$\frac{dp}{dy} = \frac{dP}{dQ^D} \left( \frac{dq}{dp} \frac{dp}{dy} + 1 \right).$$

Now, solving for  $\frac{dp}{dy}$ , we obtain:

$$\frac{dp}{dy} = \frac{\frac{dP}{dQ^D}}{1 - \frac{dP}{dQ^D} \frac{dq}{dp}}.$$

Under suitable mathematical assumptions we obtain:

$$\frac{dy}{dp} = \frac{1 - \frac{dP}{dQ^D} \frac{dq}{dp}}{\frac{dP}{dQ^D}},$$

which simplifies to:

$$\frac{dy}{dp} = \frac{dQ^D}{dP} - \frac{dq}{dp}.$$

Notice that the left hand side is the derivative of the demand function facing the firm, while on the right hand side we have the derivative of the demand function with respect to price, and the derivative of the supply of all other firms in the industry with respect to price.

Now, we must convert the above equation into elasticity. Recall that in the text we defined

$$\epsilon^d = \frac{dy}{dp} \frac{p}{y},$$

therefore, multiplying both sides by  $\frac{p}{y}$  we obtain:

$$\epsilon^d = \frac{p}{y} \frac{dQ^D}{dP} - \frac{p}{y} \frac{dq}{dp}.$$

Notice that,

$$\frac{p}{y} \frac{dQ^D}{dP} = \frac{p}{y} \frac{dQ^D}{dP} \frac{Q^D}{Q^D}$$

$$\begin{aligned}
&= \frac{dQ^D}{dp} \frac{p}{Q^D} \frac{Q^D}{y} \\
&= E_D \frac{Y^S}{y} \\
&= E_D \frac{1}{ms}.
\end{aligned}$$

where the third equality follows because in equilibrium market demand,  $Q^D$ , equals, aggregate supply,  $Y^S$  and we define market share as  $ms = \frac{y}{Y^S}$ .

Also, note that  $\frac{dQ^D}{dP} = \frac{dQ^D}{dp}$ , since its just a notation change.

Similarly,

$$\begin{aligned}
\frac{p}{y} \frac{dq}{dp} &= \frac{p}{y} \frac{dq}{dp} \frac{q}{q} \\
&= \frac{p}{q} \frac{dq}{dp} \frac{q}{y} \\
&= E_S \frac{q}{y} \frac{Y^S}{Y^S} \\
&= E_S \frac{\frac{q}{Y^S}}{\frac{y}{Y^S}} \\
&= E_S \left( \frac{1 - ms}{ms} \right).
\end{aligned}$$

Combining both results we obtain:

$$\epsilon^d = E_D \frac{1}{ms} - E_S \left( \frac{1 - ms}{ms} \right).$$

One final simplification can be obtained if we assume that all firms in the industry are identical. Then,  $ms = \frac{1}{n}$ , and the above equation simplifies to:

$$\epsilon^d = nE_D - (n - 1)E_S.$$