

Chapter 2

Cost Minimization

2.1 Introduction

In this chapter we begin the study of the firm's market behavior. We will begin by studying how a firm behaves in **competitive factor markets**. We will derive the set of **conditional or derived factor demand functions**. Conditional demand functions are functions of factor prices and a target level of output. Notice that regardless of the behavior of the firm in the goods market, it will always strive to minimize its cost in the factor market (question: are monopolies more or less efficient than competitive firms). Therefore, the set of conditional demands does not depend on the price of the output. The **cost function** is the maximum value function we derive from the cost minimization problem. It is also a function of factor prices and target output. In this chapter we will assume that the firm produces its output with two **variable factors**: labor and capital. That is, the firm can select the amounts of both capital and labor that it uses in production. We will see later on that we can interpret all the results of this chapter as the long run behavior of the firm.

The cost minimization problem is, mathematically speaking, a problem in constrained optimization. The firm wishes to minimize the cost of producing a certain level of output, but it is constrained by its technological possibilities, as summarized by the production function. Consequently, we will obtain a Lagrange multiplier as part of the answer. The Lagrange multiplier, μ^* , is the shadow value of the constraint. In this case, μ^* tells us the increase in the firm's cost that would arise if we expand output just a

little bit. This quantity is known as **marginal cost**.

Finally we will study the properties of the cost function and of conditional factor demands.

2.2 Optimal Choice

2.2.1 The Firm's Cost Minimization Problem

As we stated in the introduction, firms seek to minimize cost subject to the constraint that they produce y units of output. That is, the firm's cost minimization problem, $[CMP]$, is given by:

$$\begin{aligned} \min_{k,l} \quad & wl + rk \\ \text{s.t.} \quad & f(k, l) = y \\ & l \geq 0 \quad k \geq 0 \end{aligned}$$

In general we will ignore the non-negativity constraints on factor inputs by arguing that the firm is not endowed with any factors and so must be a net buyer in factor markets. We will rule out corner solutions by placing sufficient structure on the production function (basically that marginal products of both factors are always strictly positive. These conditions are due to Inada.).

Notice that in the $[CMP]$ parameters are given by factor prices, w and r , and the target level of output, y . If we were given values for these parameters we could obtain the actual quantities demanded. Since we are using arbitrary quantities, our solution will be quantities of l and k as functions of factor prices and target output levels:

$$l^*(r, w, y) \text{ and } k^*(r, w, y).$$

This set of demand functions is known as **conditional or derived demand functions**. Notice that conditional demand functions are functions of target output y . This means that as we vary the relative factor cost, the firm's optimal quantities will always be on the same isoquant.

2.2.2 Solving the Firm's Problem

As we saw in Chapter xxxx, we can solve a constrained optimization problem either via the substitution method or the method of Lagrange multi-

pliers. We will use the latter because it gives us insight into the behavioral content of the problem. The Lagrangian for this problem is given by:

$$L = rk + wl + \mu (y - f(k, l)).$$

Now, we minimize the Lagrangian with respect to the two choice variables and the Lagrange multiplier η :

$$\min_{k, l, \eta} rk + wl + \mu (y - f(k, l)).$$

We will ignore the non-negativity constraint on factor inputs and so our cost minimizing bundle will be characterized by the first order conditions that we present below:

$$\begin{aligned} [l] : w &= \mu^* f_l(k^*, l^*) \\ [k] : r &= \eta^* f_k(k^*, l^*) \\ [\mu] : f(k^*, l^*) &= y \end{aligned}$$

Notice that the choice variables and the Lagrange multiplier are starred. As we saw in Chapter xxx, this occurs because once we set the first order conditions equal to zero, we have three conditions that determine the solution to the consumer's problem. We no longer have arbitrary quantities; we have optimal quantities. Also, notice that we have three equations and three choice variables so the problem has a solution. After we solve the three equations we will obtain optimal quantities as functions of the parameters of the problem. Recall also, that the production function is a cardinal representation of production possibilities. Therefore, we can no longer use the monotonic transformation trick to simplify the problem. As we saw in Chapter xxx, scale matters in production.

2.2.3 Optimality

The first point to notice is that the optimality condition of [CMP] is a tangency condition:

$$\frac{f_l}{f_k} = \frac{w}{r},$$

where the marginal products are evaluated at the optimal quantities. Optimal firm behavior requires equality between the **technical rate of substitution** and **relative factor prices**. Thus, if the firm behaves optimally and the solution is interior it will use the input combination which

equalizes the slopes of the isocost curve and the isoquant. Therefore, once again, relative prices are the fundamental allocational tools used by the firm. Graphically, we know that the optimum quantities will be determined by the point of tangency of the budget set and the isoquant as we can see in Figure xxxx. The isoquant is fixed and we shift the isocost line in. Optimality tells us that we will stop shifting the isocost curve in when we have tangency between it and the indifference curve.

Insert Figure xxxx.1

Example 4 *Let us consider why*

$$\frac{f_l}{f_k} > \frac{w}{r}.$$

can not constitute an equilibrium. Consider Figure xxxx below and suppose that we are point A. At this point, the input bundle is given by

$$x_A = (k_A, l_A)$$

and its associated total cost is C_A dollars. Suppose, further, that we wish to increase consumption of l by one unit to $l+1$. Since we must remain on isoquant y , the change in demand for capital k , dk , must satisfy:

$$dy = 0 = f_l * 1 + f_k dk,$$

or

$$dk = -\frac{f_l}{f_k}.$$

This means that if we are on isoquant y and we increase labor input by one unit, capital input must decrease by exactly $-\frac{f_l}{f_k}$ units. Therefore, at point B we are on still on isoquant y and our input bundle is given by:

$$x_B = \left(k_A - \frac{f_l}{f_k}, l_A + 1 \right).$$

The question is whether at this new point we can achieve the same output level, but at a lower cost than at our original point A. That is, we wish to determine whether $C_B < C_A$, at the given prices. To determine this we look at point C. At point C, we have determined the amount of capital input k

that the firm would need to give up in order to keep the cost of the input bundle constant. That is, at point C , dk satisfies:

$$dy = 0 = w * 1 + rdk$$

which implies

$$dk = -\frac{w}{r}.$$

So the firm's input bundle at point C is:

$$x_C = \left(k_A - \frac{w}{r}, l_A + 1 \right).$$

Notice that it is on the same isocost line as input bundle A . It is clear that

$$\frac{f_l}{f_k} > \frac{w}{r}.$$

Since bundle A and bundle C are on the same isocost line, they both require an outlay of C_A dollars. Bundle B , on the other hand, guarantees the same output level as bundle A , but it is less costly. Notice that bundle A and bundle C have the same amount of l , but bundle B has less labor. The implication is that bundle B must be cheaper than bundle C , which costs the same as bundle A . Since bundle A and bundle B , both yield y units of output and bundle B is cheaper, bundle B is preferred.

Insert Figure xxxx.2

In the [CMP], we are using the market to construct trades that leave our target output level, y , unchanged but which decrease our cost. Consider Figure xxxx. In the above example we saw isocost lines such as C_1 and C_2 can not characterize the solution to the [CMP]. We can continue to push our isocost line closer to the origin, attain y and do so spending less money. This process must stop when the indifference curve and the isocost line are tangent to each other. An isocost line, such as the one labelled C_4 denotes lower cost than C_3 , but fails to achieve y units of output.

Insert Figure xxxx.3

2.2.4 Output Constraint

While the optimality condition tells us the proportions in which the firm optimally chooses to utilize inputs, the production function restricts input use levels to those which are consistent with achieving the target output level y . The feasibility condition is given by the firm's production constraint:

$$f(k, l) = y.$$

Notice that there are many possible tangency points, given prices.

Insert Figure xxxx.4

In fact, there is one tangency point for each target output level. The production constraint, then, pins down the factor input by indicating which point of tangency is the one in which we are interested. The production function tells us on which isoquant the firm must be producing. Figure xxxx, characterizes the optimum graphically.

Insert Figure xxxx.5

2.3 Conditional or Derived Demand

As we discussed previously, conditional demand functions are the solution to the firm's cost minimization problem:

$$l^*(w, r, y) \text{ and } k^*(w, r, y).$$

In this section we will discuss two properties of conditional demand functions.

Property I: Conditional demands are homogeneous of degree 0 in prices.

Notice that in the cost minimization problem, the isoquant curve is fixed. The optimality point is determined by the slope of the isocost line. This is precisely the price ratio. Therefore, if we double prices, the price ratio is unchanged and since the indifference curve is fixed the optimal point, given by the set of conditional demands, does not change.

Property II: The solution is characterized by no excess production.

Once more, the consumer will choose its optimality point to satisfy the restriction with equality.

2.4 Cost Function

If we take the conditional demand functions and plug them into the objective function we obtain the firm's cost function. This function tells us the minimum money outlay necessary to achieve production y , given factor prices.

$$C(w, r, y) = wl^*(w, r, y) + rk^*(w, r, y).$$

In this section we will discuss the properties of the cost function.

Property I: The cost function is homogeneous of degree 1 in prices.

This property states that if we double prices, we will double the cost necessary to achieve the same target level of output y . This property follows from the homogeneity of degree 0 of conditional demands:

$$\begin{aligned} C(\lambda w, \lambda r, y) &= \lambda wl^*(\lambda w, \lambda r, y) + \lambda rk^*(\lambda w, \lambda r, y) \\ &= \lambda wl^*(w, r, y) + \lambda rk^*(w, r, y) \\ &= \lambda [wl^*(w, r, y) + rk^*(w, r, y)] \\ &= \lambda C(w, r, y) \end{aligned}$$

where the second equality follows from homogeneity of degree 0 of Hicksian demands.

Property II: The cost function is increasing in y .

This property states that the cost of achieving a higher target level of output is strictly positive:

$$\frac{dC(w, r, y)}{dy} > 0.$$

Insert Figure xxxx.6.

Property III: The cost function is non decreasing in w and r .

This property states that if a factor price increases then the necessary expenditure to achieve the optimal bundle can not decrease:

$$\frac{dC(w, r, y)}{dp} \geq 0,$$

where $p = w, r$. If the firm doesn't purchase the good whose price increased then minimal expenditure would be unaffected; otherwise it would have to increase.

Property IV: The cost function is concave in w and r .

This is perhaps the most interesting of all the properties of the expenditure function. It states that the shape of the expenditure is as displayed in the Figure xxxx.

Insert Figure xxxx.7.

Notice that if factor prices doubled, and the firm did not alter its spending pattern at all, then cost would grow linearly with factor prices. However, the firm will respond to a change in relative factor prices. The optimality condition, which pins down the tangency condition will vary if we change the relative price. Since the firm can always choose not to alter its behavior, its costs can not grow faster than linearly. Therefore, we rule out cost function $C'(w, r, y)$. In general, the consumer will take advantage of substitution opportunities and his costs will grow slower than linearly as a result of a relative factor price change. Therefore, $C(w, r, y)$ is the proper shape of the cost function. It is concave in prices.

2.5 The Lagrange Multiplier

Let us now consider the interpretation of the Lagrange multiplier, μ^* . Consider the cost function

$$C(w, r, y)$$

and let us find the extra cost that the firm must pay if it wishes to increase its target output, y by a tiny amount. Mathematically speaking, we are looking for

$$\frac{dC(w, r, y)}{dy}.$$

If we use the definition of the cost function and compute its derivative with respect to target output, we obtain:

$$\frac{dC(y)}{dy} = w \frac{dl^*(y)}{dy} + r \frac{dk^*(y)}{dy},$$

where we have suppressed w and r as arguments of the cost function and

of the conditional demand functions. If we recall the first order conditions with respect to l and k :

$$\begin{aligned} [l] & : w = \mu^* f_l \\ [k] & : r = \mu^* f_k \end{aligned}$$

and use them to substitute the for prices we obtain:

$$\frac{dC(y)}{dy} = \mu^* \left(f_l \frac{dl^*(y)}{dy} + f_k \frac{dk^*(y)}{dy} \right).$$

Now, we totally differentiate the feasibility condition with respect to y :

$$f_l \frac{dl^*(y)}{dy} + f_k \frac{dk^*(y)}{dy} = 1,$$

which establishes that

$$\frac{dC(w, r, y)}{dy} = \mu^*.$$

Consequently, we can interpret μ^* as the firm's marginal cost. This is the dollar price of one additional unit of output. Notice that we have essentially proven the envelope theorem while carrying out the above discussion (see the discussion in chapter 2).

2.6 The Cost Function and Returns to Scale

We noted in Chapter xxxx that a very important difference between consumer theory and production is that the utility function is an ordinal concept while the production function is a cardinal concept. What this means is that the output level attached to an isoquant is a meaningful quantity, but the utility index assigned to an indifference curve is arbitrary. Since the scale on which we measure production is meaningfully defined we can study the effect of returns to scale on the shape of the firm's cost function.

To be completed.

2.7 Applications

2.7.1 Learning-By-Doing

Consider a cost minimizing firm.

To be completed.

2.8 Exercises

Exercise 2.8.1 Consider a firm that has a Cobb-Douglas technology. The firm wishes to minimize cost of producing y units of output and has access to perfectly competitive factor markets. The firm's cost minimization problem is given by:

$$\begin{array}{ll} \min_{k,l} & wl + rk \\ \text{s.t.} & k^\alpha l^\beta = y \end{array}$$

Let μ denote the Lagrange multiplier on the output constraint.

- What are the parameters of the problem?
- Find the conditional demand functions. Label them $l^*(w, r, y)$ and $k^*(w, r, y)$.
- Find the cost function: $C(w, r, y)$. What is its interpretation?
- Find μ^* . What is its interpretation?
- Find $\frac{dC}{dy}$ and show that it is equal to μ^* .
- How does $\frac{dC}{dy}$ vary with $(\alpha + \beta)$?

Exercise 2.8.2 Consider a firm that produces holes with shovels and people. Assume that people and shovels are perfect complements so that technology is given by:

$$f(k, l) = \min\{k, l\}$$

The firm wishes to minimize cost of producing y units of output and has access to perfectly competitive factor markets. Let μ denote the Lagrange multiplier on the output constraint.

- What is the firm's cost minimization problem?
- What is the optimality condition?

- b. Find the conditional demand functions. Label them $l^*(w, r, y)$ and $k^*(w, r, y)$.
- c. Find the cost function: $C(w, r, y)$. What is particular about the expression you obtained. That is discuss the economic intuition of your result.
- d. Find μ^* . What is its interpretation?
- e. Find $\frac{dC}{dy}$ and show that it is equal to μ^* .
- f. Explain why the expression that you obtain makes intuitive sense.

Exercise 2.8.3 Suppose that the firm's utility function is given by:

$$f(k, l) = k^\alpha l^{1-\alpha}.$$

Also suppose that factor prices are given by w and r and that the target output parameter is given by y . Denote the Lagrange multiplier by μ .

- a. Write down the agent's cost minimization problem. Be very precise.
- b. Find the conditional demand curves, $l^*(w, r, y)$ and $r^*(w, r, y)$ and show that they are homogeneous of degree 0 in w and r .
- c. Find μ^* .
- d. Find the cost function, $C(w, r, y)$.
- e. Show that the cost function is homogeneous of degree 1 in w and r . What is the intuition?
- f. Show that $\frac{dC}{dw} \geq 0$ and $\frac{dC}{dr} \geq 0$. What is the intuition?
- g. Find $\frac{dC}{dy}$.
- h. Show that $\frac{dC}{dy} = \mu^* > 0$. What is the intuition?
- i. Show that the expenditure function is concave in w . Do this graphically and mathematically. What is the intuition?