Compensation for Quality Difference in a Search Model of Money*

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Abstract

In this paper, we demonstrate a new and intuitive welfare-improving role for fiat money: compensating for quality differences between traded commodities. We construct a model with the following features; (i) there is no double-coincidence-of-wants problem, (ii) agents have perfect information about the qualities of goods, and (iii) there are no transaction costs nor any form of imperfect utility transfer. We identify situations where a market without money fails to motivate agents to produce goods of high quality. The introduction of money, however, allows agents with high quality goods to benefit from trade with agents who hold low quality goods and money. This can provide sufficient incentive for almost all agents to produce a high quality good.

1 Introduction

When we use a one-hundred dollar bill to purchase a commodity that is worth sixty dollars, we expect to get forty dollars in change. Such compensation for the difference in value might not exist in a barter economy. Consider a group of agents. Each specializes in producing one perishable good but wants to consume only others' goods. Every day, each agent decides to produce the good of either mediocre or superior quality. (The superior quality production requires extra effort.) Imagine that an agent goes to the marketplace with a good of superior quality looking for traders to exchange with her. Suppose the

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market is about to close and she ends up meeting someone carrying some mediocre good. She has no choice but to trade with the person because otherwise her own good would be rotten and she would not get anything to consume for that day. Knowing this before going to the market, agents have a natural incentive to produce only the mediocre good which involves less effort. Each individual must bear extra costs to produce the superior good, while the higher quality only directly benefits the buyer of the good. This generates inefficiency in the form of underproduction of the superior good.

Fiat money can serve as compensation for the quality difference between different traded commodities of different qualities. It turns out that compensating traders with money for the higher quality is a proper way to motivate the production of superior goods when the barter economy fails to do so.

In the earlier stages of the literature on search-theoretical models of money initiated by Kiyotaki and Wright (1989, 1991 and 1993), fiat money improves welfare by eliminating the double-coincidence-of-wants problem. In a later model in which goods are of different qualities and qualities are not always observable by others, Williamson and Wright (1994) show that fiat money can improve welfare by alleviating the problem of asymmetric information.[1] More recently, Engineer and Shi (1998) introduce imperfect transfer of utility into the model with two qualities of commodities. They show that introduction of fiat money can reduce the frequency of inefficient utility transfer and thus improve social welfare.

In this paper, we construct a model which abstracts from all three forms of imperfection in barter economies mentioned in the previous paragraph. In other words, (i) there is no double-coincidence-of-wants problem, (ii) agents have perfect information about the qualities of goods, and (iii) there are no transaction costs nor any form of imperfect utility transfer. We demonstrate an intuitive, yet overlooked, welfare improving role of fiat money– namely, compensation for quality difference between traded commodities.

In our model, the agents can choose to produce either a mediocre good or a higher cost superior good. The trading process is frictionless. There is no transaction cost and each trader wants to trade every day. Fiat money neither increases frequency of

[1] In contrast to Williamson and Wright's model in which qualities are endogenously chosen by agents, Haegler (1997) considers a similar setting in which qualities are exogenously determined. He further confirms this welfare improving role of money.

trade nor reduces transaction cost. We characterize those situations where the market without money fails to motivate people to produce the superior good. Then we show that introducing fiat money as compensation for quality difference between traded goods in this case provides agents enough incentives to do so.

The intuition that money improves welfare is as follows: If everybody accepts money as the standard compensation for quality difference, traders carrying the superior good meeting traders carrying the mediocre good with money can get money in addition to the mediocre good. On the other side, with money in hand, a trader can choose to produce the mediocre good in hopes of getting the superior good through trade. We show that compensation in the form of expected gain in the future can motivate almost all agents to produce the superior good.

In Section 2, we consider an economy without money and identify situations in which everybody chooses to produce the mediocre goods. Then in Section 3, we illustrate how the introduction of fiat money can generate efficiency gains. Section 4 concludes.

2 The Economy without Money

2.1 Production, Preferences, and Trading Technology

There is a continuum of agents with a unit mass who live forever. Everyday, each agent chooses effort levels (high or low) to produce one unit of perishable goods of different qualities.[2] The production cost of one unit of the mediocre good is normalized to zero. It costs c to produce a unit of a superior good. We consider the case in which people never consume their own production but equally prefer goods of the same quality produced by others. In consuming other agents' goods, one unit of superior good gives an instantaneous utility u_H and one unit of mediocre good gives instantaneous utility u_L. Consuming one's own good or consuming nothing gives an instantaneous utility $-\infty$. This both induces a natural incentive to trade and gets rid of the double-coincidence-of-wants problem. We also assume that agents do not discount the utility from consumption in the future. All of our results hold for any discount rate, but we want to demonstrate that it is not the discount factor that is driving our results[3]. Formally, we assume that the agents' preference relation

[2]It is the usual practice to assume goods are perishable. See for example Shi (1997).

[3]For further discussion see Section 4.

$\succ$ is defined by the overtaking criterion;[4] that is, $\{u_t - c_t\}_{t=0,1,\ldots} \succ \{u'_t - c'_t\}_{t=0,1,\ldots}$ if and only if $\liminf \sum_{t=0}^{T} (u_t - c_t - (u'_t - c'_t)) > 0$, where $u_t = u_H$ or u_L and $c_t = c$ or 0.

Assume that there is social gain in producing the superior good, i.e. $u_H - u_L > c$. Since in our model everybody trades every day, this assumption implies that it is always socially more efficient to have more agents produce the superior good. Finally, agents have perfect information regarding the quality of each agent's good. That is when they meet each other they can observe the quality of each other's good before deciding whether to accept the trade.

The timing of operation in the economy is specified as follows: There are infinitely many days. Every day, after production, agents go to the marketplace to trade with others. The market place opens in the morning. There are two rounds of trade in each day: the morning and the afternoon rounds. In each round the participants in the market are matched randomly, such that the agents face no aggregate uncertainty in their matches[5]. The trading process is modeled as follows: First, both traders simultaneously display what they have to offer for exchange. After seeing each other's offer, both traders announce simultaneously whether to accept the trade or not. The transaction is carried out if and only if both traders announce "trade". Everybody who has traded leaves the marketplace for that day.[6] Those who fail to trade in the morning get into the afternoon round of trade, which is identical to the morning round. There is no cost of waiting to get into the afternoon round of trade. However, people prefer getting the good in the morning to getting it in the afternoon, as long as the quality is the same. After consumption and before the marketplace opens again the next morning, agents produce another unit of the good of either quality, according to their choices.

[4]See for example Osborne and Rubinstein (1994) pp. 139.

[5]Suppose that the agents are indexed by $[0, 1]$. Then a possible matching process is described as follows. Pick a random number x from $[0, 1]$ according to the uniform distribution and divide this interval into two subsets: $[x, x + 1/2]$ modulo 1 and its complement in $[0, 1]$. Since the length of the two sets is the same the matching beween the two sets can be the natural bijection. For further discussion of the matching process see Appendix B.

[6]In other words, we exclude commodity money. We can imagine that traders only have the technology to preserve their own goods. Once goods change hands, they have to be consumed immediately.

2.2 Nonmonetary Equilibria

Suppose in the production stage each agent produces the superior good with probability λ and the mediocre good with probability $(1-\lambda)$. Then by the Law of Large Numbers[7] there is a proportion λ of agents producing the superior good and a proportion $1-\lambda$ producing the mediocre good. Consider the case in which $\lambda > 0$. We call traders with superior goods traders H and traders with mediocre goods traders L. In the morning and afternoon rounds of trade, both types of traders decide whether to trade with the traders they encounter. In the afternoon round, it is always optimal to trade, regardless of the quality of the good the other trader is offering. In the morning, it is optimal to trade if and only if the other trader holds the superior good unless the trader surely gets mediocre good in the afternoon round. The reason is that if a trader chooses not to trade with one who holds the superior good, then she might meet another trader with the mediocre good in the afternoon. On the other hand, if a trader chooses not to trade with a trader with the mediocre good, such patience always admits to possibility of meeting a trader with the superior good in the afternoon who has no better option other than trading with her. It is also optimal to any strategy to go to the morning round since it is costless and can not decrease the probability of meeting a superior good producer in the afternoon. From now onwards, we assume that all the agents behave this way.

Notice that trader L never gets to trade in the morning; therefore both in the morning and in the afternoon there is measure $1-\lambda$ of L traders. Let us look at the expected instantaneous payoff of different traders. First, we consider an H trader. An H trader has a probability λ of meeting another trader H and ending up getting utility u_H in the morning. If she meets an L trader instead, which is with probability $(1-\lambda)$, she will not accept the trade and instead will choose to get into the afternoon round of trade. By the afternoon, a proportion λ^2 of traders have left the market. Among the remaining $\left(1-\lambda^2\right)$ proportion of traders, $(1-\lambda)$ of them are L traders and the remaining $\left(\lambda-\lambda^2\right)$ are H traders. In this round, regardless of the quality of the good in hand, each trader has a

[7] We are aware of the technical inadaquacy of applying the Law of Large Numbers to continuum–many iid random variables; see for example Judd (1985). However the reader should keep in mind that we are really interested in the case of large but finitely many paleyrs, in which case we have in mind an appropriate version of the Law for which the technical problem does not arise. In any case, in Appendix B we provide a solution for the problem.

$\left(\lambda-\lambda^2\right)/\left(1-\lambda^2\right)$ probability of trading for the superior good and getting utility u_H, and a $(1-\lambda)/\left(1-\lambda^2\right)$ probability of trading for the mediocre good and getting utility u_L. Therefore, the expected instantaneous payoff of an H trader is

$$\begin{aligned} V_H(\lambda) &= \lambda u_H + (1-\lambda)\left[\frac{\lambda-\lambda^2}{1-\lambda^2}u_H + \frac{1-\lambda}{1-\lambda^2}u_L\right] - c \qquad (1) \\ &= \lambda u_H + (1-\lambda)\left[\frac{\lambda}{1+\lambda}u_H + \frac{1}{1+\lambda}u_L\right] - c. \end{aligned}$$

An L trader trades only in the afternoon, but in the afternoon every trader will be treated identically. Therefore, the expected instantaneous payoff of a trader with the mediocre good is

$$\begin{aligned} V_L(\lambda) &= \frac{\lambda-\lambda^2}{1-\lambda^2}u_H + \frac{1-\lambda}{1-\lambda^2}u_L \qquad (2) \\ &= \frac{\lambda}{1+\lambda}u_H + \frac{1}{1+\lambda}u_L. \end{aligned}$$

We claim the following

Proposition 1 *(i) There always exists an equilibrium in which every agent produces the mediocre good everyday. (ii) The outcome in which every agent produces the superior good everyday is never an equilibrium outcome. (iii) In particular, when $c < u_H - u_L < 2c$, the only equilibrium outcome is that everybody chooses to produce the mediocre good everyday.*

Proof. See Appendix A. ■

In this nonmonetary economy, by modelling two rounds of trade in a day instead of one, we let H traders have some bargaining power over L traders: they can refuse to trade with L traders in the morning. When all H traders do that, they have a better chance of getting the superior good than L traders , and this potentially provides incentive for production of the superior good. The proposition above suggests that superior good production is equilibrium only if the efficiency gain from producing the superior good is at least double the cost of quality improvement. And even in this case, there is also an equilibrium with no superior good production.

When $c < u_H - u_L < 2c$, the only equilibrium outcome is that everybody produces the mediocre good. The reason is that no matter what proportion of the agents produce a superior good in the economy, producing a mediocre good has a higher expected payoff than producing a superior good. The natural question is: Can the introduction of fiat money into the economy make a difference? We provide a positive answer to this question in the next section.

3 Economy with Money

We now consider another economy that is similar to the previous one in every respect except that now, in the initial period, a proportion μ of agents are provided with a unit of fiat money before they make their production decision. Because of Proposition 1 part (iii), we focus on the case $c < u_H - u_L < 2c$. As in other search-theoretical models of money, there always exists one equilibrium in which money has no value. Our main concern here is whether there also exists a monetary equilibrium that Pareto dominates the nonmonetary equilibrium in which the market fails to motivate agents to produce the superior good even though it is socially optimal to do so.

We show that there exist stationary monetary equilibria in which a positive proportion of the agents produce a superior good every day. The strategies of the players are described as follows.

(A_1) A money holder always produces a mediocre good. She offers both the mediocre good and money in the morning round and trades if and only if she is offered the superior good. If she fails to trade in the morning, she hides and stores the money, shows the mediocre good in the afternoon, and agrees to trade for a good of any quality.

(A_2) A non-money holder produces the superior good with probability $\lambda/(1-\mu)$. Suppose she produces the superior good. In the morning, she offers the superior good and trades if and only if she is either offered the superior good or the mediocre good accompanied by money. In the afternoon, she offers the superior good and trades for a good of any quality. Suppose she produces the mediocre good instead. Then in the morning, she offers the mediocre good and trades if and only if she is either offered the superior good or the mediocre good accompanied by money. In the afternoon, she offers the mediocre good and agrees to trade for any quality.

Given that players use actions (A_1) and (A_2), there are three types of traders in the morning. We call the traders with money and the mediocre good by M traders, those with the mediocre good alone L traders and those with the superior good H traders. By the Law of Large Numbers[8], the proportions of these traders in the economy are stationary at μ, $1-\lambda-\mu$ and λ respectively. In the morning round, a proportion λ^2 of traders H meet each other and trade. Furthermore $\lambda\mu$ of traders M pair up with another $\lambda\mu$ of H traders and trade. All the other traders (a proportion $1-\lambda^2-2\lambda\mu$) get into the afternoon round.

Given these strategies, the expected instantaneous payoff plus the expected future gain from keeping money of a money holder with the mediocre good is

$$V_M(\lambda,\mu) = \lambda u_H + (1-\lambda)\left[\frac{\lambda(1-\lambda-\mu)}{1-\lambda^2-2\lambda\mu}u_H + \frac{1-\lambda-\lambda\mu}{1-\lambda^2-2\lambda\mu}u_L + v\right], \tag{3}$$

where v denotes the expected gain from holding money in the future. If a non-money holder chooses to produce the superior good, the expected instantaneous payoff net the cost of production plus the expected future gain from receiving money is

$$\begin{aligned} V_H(\lambda,\mu) &= \lambda u_H + \mu(u_L+v) + \\ &\quad (1-\lambda-\mu)\left[\frac{\lambda(1-\lambda-\mu)}{1-\lambda^2-2\lambda\mu}u_H + \frac{1-\lambda-\lambda\mu}{1-\lambda^2-2\lambda\mu}u_L\right] - c. \end{aligned} \tag{4}$$

If the non-money holder chooses to produce the mediocre good instead, her expected instantaneous payoff is

$$V_L(\lambda,\mu) = \frac{\lambda(1-\lambda-\mu)}{1-\lambda^2-2\lambda\mu}u_H + \frac{1-\lambda-\lambda\mu}{1-\lambda^2-2\lambda\mu}u_L. \tag{5}$$

If the strategy profile described above is a stationary equilibrium, every non-money holder must be indifferent between producing the superior good and the mediocre good every day as long as she has no money, i.e. $V_H(\lambda,\mu) = V_L(\lambda,\mu)$ must hold. Therefore, it must also be optimal for a non-money holder to commit to producing the mediocre good forever instead of randomizing. By doing so, she obtains $V_L(\lambda,\mu)$ everyday. A money holder always produces the mediocre good until she gets rid of the money. Afterward, deciding to produce the mediocre good forever again provides the same lifetime utility as any other alternative decision. The gain from holding money is realized on the day when a money holder meets a trader H in the morning. By the Law of Large Numbers, this happens eventually with probability one. The payoff of the money holder on that day is u_H, while the expected payoff of the mediocre good producer without money is $V_L(\lambda,\mu)$.

[8] See Footnote 7.

Therefore we can conclude that

$$v = u_H - V_L(\lambda, \mu) > 0. \tag{6}$$

This argument also shows that at the beginning of the first period the expected payoff of money holders is strictly higher than the expected payoff of the other agents.

We claim the following

Proposition 2 *Suppose $c < u_H - u_L < 2c$. For any $\lambda \in (0,1)$, there is a unique $\mu \in (0, 1-\lambda)$, such that there exists a monetary equilibrium, in which proportion λ of agents produces the superior good, and proportion μ receives fiat money. Money is generally accepted as compensation for the quality difference between the superior and the mediocre good and players follow the strategy profile (A_1, A_2) described above. Furthermore μ is decreasing in λ. All such monetary equilibria outcome Pareto dominate the nonmonetary equilibrium outcomes, and the Pareto Optimal outcome can be approximated arbitrarily closely.*

In order to prove this proposition, we need to verify that there is no incentive for the following deviations: (i) a money holder producing a superior good, (ii) an M trader hiding money in the morning, (iii) an M trader accepting a trade with an L or M trader or refusing to trade with an H trader, (iv) an M trader showing money in the afternoon, (v) a non-money holder not playing a mixed strategy in the production stage, (vi) an H trader accepting a trade with an L trader or refusing to trade with an H or M trader. The proof is provided in Appendix A.

In the model without money, when $c < u_H - u_L < 2c$, agents always prefer to produce the mediocre good. In other words, the market cannot provide enough incentive for agents to produce the superior good. In the proof of Proposition 1 we show that the difference between the instantaneous payoff of an H trader and an L trader is:

$$V_H(\lambda) - V_L(\lambda) = \frac{\lambda}{1+\lambda}(u_H - u_L) - c. \tag{7}$$

Notice that $\lambda/(1+\lambda)$ is less than half on $[0,1]$, and since $u_H - u_L < 2c$ this expression is always negative. However $\lambda/(1+\lambda)$ is strictly increasing on $[0,1]$. Hence, the more agents that are producing a superior good, the less attractive it is to produce the mediocre good relative to the superior good. The introduction of money can decrease the gap between the payoffs of L traders and H traders. In order to understand this, it is worth expressing the excess instantaneous payoff of an H trader over an L trader in the monetary economy:

$$V_H(\lambda,\mu) - V_L(\lambda,\mu) = (u_H - u_L)\frac{\lambda(1-2\mu) - \lambda^2 + \mu}{1-\lambda^2-2\lambda\mu} - c. \tag{8}$$

Observe that if $\mu = 0$ this equation is equivalent with (7). From (7), in a nonmonetary economy (where $\mu = 0$), the coefficient of $u_H - u_L$ is $\lambda/(1+\lambda)$. Since $\lambda \leq 1$ this coefficient is weakly smaller than $1/2$, hence $V_H(\lambda,0) - V_L(\lambda,0)$ is always negative. However if μ is positive and λ is converging to $1-\mu$, the coefficient of $(u_H - u_L)$ is converging to one. That is, the upper bound of this coefficient is one instead of one-half. Hence, by making λ dependent on μ, it is possible to achieve the result that agents without money are indifferent between producing superior and mediocre goods.

The intuition is the following: Consider a situation where there is a certain positive proportion of H traders and the rest are L traders. Recall that it cannot be an equilibrium because mediocre good production is too attractive. Suppose we inject money in order to replace some of the L traders by M traders. Like in the nonmonetary economy, L traders obtain the superior good in the afternoon. Also notice that the proportion of H traders in the afternoon never exceeds the proportion of L traders. Those M traders who do not trade in the morning will hide their money and go to the afternoon round of trade. This will decrease the chance that an L trader gets a superior good. As we replace sufficiently many L traders by M traders, the remaining L traders' probability of getting the superior good could become arbitrarily small. Furthermore, replacing L traders by M traders increases the expected instantaneous payoff of H traders because their probability of getting the superior good is not affected, and they enjoy the additional opportunity to be compensated by getting money in addition to the mediocre good. This explains why any proportion of superior good production (which cannot be an equilibrium outcome in the nonmonetary economy) can be supported as an equilibrium by injecting an appropriate amount of money into the economy. The most desirable outcome is to have almost every agent producing the superior good. In order to achieve this, the policy maker should inject an infinitesimal amount of money into the economy.

4 Conclusion

When goods are of different qualities, one natural situation under which money improves welfare is when there is private information about quality. The homogeneity of money reduces the problem of private information and thus lubricates trade. However, when there is no private information about the qualities, what will be the most natural welfare improving role of money? Engineer and Shi (1998) reintroduce some transaction cost in the form of imperfect service sidepayments. They show that there exists a monetary equilibrium in which money can cut down the frequency of trade between the mediocre and superior goods and thus reduce efficiency loss caused by the service sidepayment.

In this paper, we suggest an alternative way that money may improve welfare without reintroducing any form of transaction cost. We find a monetary equilibrium in which money is generally accepted as compensation for the quality difference between traded commodities. This form of compensation provides enough incentive for agents to produce goods of superior quality when the barter economy fails to do so.

We have shown that the effect of money is tremendous; a very small amount of money can induce almost all agents to produce the superior good. The optimal monetary policy is to provide money to a positive proportion of agents, but this proportion should be as small as possible.

All of our results are valid if the agents discount the future. In a nonmonetary economy it is because the current decisions do not affect the future payoff. In the monetary equilibrium, the current decisions about whether to accept or use money indeed can effect the payoff in the future. We have shown that a money holder is indifferent between using the money today or in a future day. However, if they discount the future they are in fact more eager to use the money today since the future benefit is discounted. On the other hand, the value of the money is clearly less than if they had not discounted the future and (6) is modified, but the argument of the proof of Proposition 2 goes through without additional difficulties (other than that for each possible μ we will find different λs).

Obviously our model is abstract and restrictive. However, we believe that our result has captured the importance of having fiat money in different denominations so that products of different qualities can be appropriately compensated with different terms of trade. Agents are willing to make extra efforts to produce high quality goods only if the disutilities involved will be properly compensated.

Appendix A

Proof of Proposition 1. (i) If every other agent produces the mediocre good, then an agent always gets the mediocre good through trade, regardless of the quality of her own good. Since producing the superior good costs more, an agent's best response is to produce a mediocre good.

(ii) Suppose that every agent produces the superior good. We claim that if one of the agents decides to produce the mediocre good, she surely obtains the superior good by the end of the day. In the morning round she surely meets a trader with a superior good, who may refuse to trade with her. However this trader with the superior good will surely be in the second round, and only superior good holders can be there. Therefore the probability that the trader with the mediocre good meets somebody with a superior good in the afternoon round is one. Thus, each player has an incentive to produce the mediocre good.

(iii) In order to establish this point we have to consider two different cases: Case 1: the proportion of those who produce a mediocre good is zero. Case 2: the proportion of these traders is positive.

Case 1: Consider first the possibility of having everyone except a finite number of agents producing the mediocre good. Let n denote the number of agents producing the mediocre good. The probability that a superior good trader meets a mediocre good trader is zero. Therefore the payoff of a superior good trader is the same as if everybody else was producing the superior good, i.e. u_H. If a trader chooses to produce the mediocre good instead, then she has zero probability of trading in the first round. In the second round, with probability one there will be $2n$ agents left among which n agents carry the superior good. The probability for a mediocre good trader to meet a superior good trader will be $n/(2n-1)$. Therefore,

$$V_L = \frac{n}{2n-1}u_H + \frac{n-1}{2n-1}u_L.$$

For this profile to constitute an equilibrium, it is required that $V_H = V_L + c$ or

$$u_H = \frac{n}{2n-1}u_H + \frac{n-1}{2n-1}u_L + c.$$

Rewriting this equality we get:

$$u_H - u_L = \left(\frac{2n-1}{n-1}\right) c.$$

However, this is impossible since by assumption $u_H - u_L < 2c$ and $2 \leq (2n-1)/(n-1)$. If there were infinitely many people producing the superior good, but their proportion were still zero, the argument would be basically identical to the previous one, with $n/(2n-1)$ being replaced by $\frac{1}{2}$.

Case 2: Suppose $0 < \lambda < 1$. (Recall that λ is the proportion of people who choose to produce the superior good.) With some manipulation of (1) and (2), it can be shown that the difference in payoffs from producing the superior good and the mediocre good is

$$V_H - V_L = \frac{\lambda}{1+\lambda}(u_H - u_L) - c.$$

In order to guarantee that no agent has incentive to deviate, $V_H - V_L = 0$ must be satisfied. This implies that

$$\lambda = \frac{c}{u_H - u_L - c}.$$

Since $\lambda < 1$, it follows that $c/(u_H - u_L - c) < 1$ or, equivalently, $2c < u_H - u_L$, which again contradicts the assumption that $u_H - u_L < 2c$. ■

Proof of Proposition 2. (i) It is optimal for money holders to produce a mediocre good. If a money holder produces a superior good, it is easy to verify that she has no incentive to show the money in any round: If she postpones the opportunity to use the money for a finite number of days, then she still achieves the same benefit of v because there is no discounting. If she postpones forever, then it is a waste of the money.

(ii) It is optimal for an M trader to show the money in the morning: By hiding money, an M trader gets $V_L(\lambda, \mu)$ instead of $V_M(\lambda, \mu)$ in that period. Again, hiding the money in the morning for a finite number of days does not affect the utility, but hiding it forever wastes the money.

(iii) It is optimal for an M trader to trade only with an H trader in the morning. If an M trader trades with an L trader, then she loses the positive probability of meeting an H trader in the afternoon which provides her with higher instantaneous utility. If she trades with an M trader, then she keeps the money and gets u_L. It is always better not to trade but enter the afternoon round of trade with money hidden. By doing so, she still keeps the money but gets $V_L(\lambda, \mu) > u_L$ instead.

(iv) It is optimal for an M trader to hide the money in the afternoon. Since the other party always accepts trade as long as she receives an offer which provides positive utility, there is no need to show the money.

(v) Non-money holders produce the superior goods with probability $\lambda/(1-\mu)$. We show that for each $\lambda \in (0,1)$ there exists a μ such that

$$V_H(\lambda, \mu) = V_L(\lambda, \mu). \tag{9}$$

Define $f(\lambda, \mu) \equiv V_H(\lambda, \mu) - V_L(\lambda, \mu) = \lambda(u_H - V_L) + \mu(u_L + u_H - 2V_L) - c$. We show that for all $\lambda \in (0,1)$, there exists a unique corresponding $\mu(\lambda)$ such that $f(\lambda, \mu(\lambda)) = 0$. We are going to argue that if μ is zero, $f(\lambda, \mu)$ is negative, and if μ is $1-\lambda$ then, $f(\lambda, \mu)$ is positive. Since f is continuous on $\{(y,x) \in \mathbb{R}^2 : y \in (0,1), x \in [0, 1-y]\}$ there must exist a $\mu \in (0, 1-\lambda)$ for which $f(\lambda, \mu) = 0$. In addition, we show that $f(\lambda, \mu)$ is strictly increasing in μ hence, $\mu(\lambda)$ must be unique. Using (5):

$$f(\lambda, \mu) = (u_H - u_L)\frac{\lambda(1-2\mu) - \lambda^2 + \mu}{1 - \lambda^2 - 2\lambda\mu} - c. \tag{10}$$

Therefore

$$f(\lambda, 0) = (u_H - u_L)\frac{\lambda}{1+\lambda} - c.$$

In the proof of Proposition 1, part (iii), Case 2, we have shown that this expression is indeed negative. Furthermore

$$f(\lambda, 1-\lambda) = (u_H - u_L) - c,$$

which is positive by assumption.[9]

We have shown that $f(\lambda, \mu) < 0$ when μ is zero and $f(\lambda, \mu) > 0$ when μ is $1-\lambda$. Since

$$\frac{\partial f(\lambda, \mu)}{\partial \mu} = (u_H - u_L)\frac{(\lambda+1)^2}{\left(-1 + \lambda^2 + 2\lambda\mu\right)^2} > 0 \text{ if } \mu < 1 - \lambda,$$

we can conclude that $f(\lambda, \mu)$ is strictly increasing in μ. Hence, for each $\lambda \in (0,1)$ there exists a unique $\mu \in (0, 1-\lambda)$ such that $f(\lambda, \mu) = 0$.

[9] Although $1 - \lambda^2 - 2\lambda\mu$ is zero at $\mu = 1 - \lambda$, $\lim_{\mu \to \lambda}\left(\lambda(1-2\mu) - \lambda^2 + \mu\right)/\left(1 - \lambda^2 - 2\lambda\mu\right) = \left(\lambda(1-2\mu) - \lambda^2 + \mu\right)/\left(1 - \lambda^2 - 2\lambda\mu\right)\rfloor_{\mu=1-\lambda} = 1$.

(vi) An H Trader accepts a trade with an M trader and with an H trader but refuses to trade with an L trader. To ensure that an H trader is willing to trade with an M trader, it must be that

$$u_L + v \geq V_L.$$

Plugging in v from (6) and V_L from (5), this inequality can be rewritten as

$$u_L + u_H \geq 2\frac{\lambda(1-\lambda-\mu)}{1-\lambda^2-2\lambda\mu}u_H + 2\frac{1-\lambda-\lambda\mu}{1-\lambda^2-2\lambda\mu}u_L.$$

This inequality holds because $\lambda(1-\lambda-\mu)$ is always less than $(1-\lambda-\lambda\mu)$. Since $u_H > V_L$, she is also willing to trade with an H trader. Since $u_L < V_L$, she prefers to go to the afternoon round instead of trading with an L trader.

(vii) Finally we show that $\mu(\lambda)$ is decreasing. Notice that it is enough to show that $f(\lambda,\mu)$ is strictly increasing in λ. But

$$\frac{\partial f(\lambda,\mu)}{\partial\lambda} = (u_H - u_L)\frac{1+\lambda^2+2\lambda\mu-2\mu-2\lambda+2\mu^2}{\left(-1+\lambda^2+2\lambda\mu\right)^2}.$$

It is easy to show that at $\lambda = 0$ this expression is positive and the numerator (as a function of λ) has only complex roots. Therefore $\partial f(\lambda,\mu)/\partial\lambda$ must be positive and $f(.,\mu)$ must be increasing on $(0, 1-\mu]$.

Therefore, we have shown that there exist monetary equilibria in which a positive proportion of the agents produces superior goods. According to Proposition 1, in every nonmonetary equilibrium all agents produce mediocre goods when $c \leq u_H - u_L \leq 2c$. Therefore these monetary equilibria outcomes Pareto dominate the nonmonetary equilibrium outcome. Furthermore since λ can be arbitrarily close to one, the Pareto Optimal outcome in which every agent consumes a superior good, can be approximated arbitrarily closely. ■

Appendix B

The problem of having continuum-many agents randomizing independently is that the outcome of a randomization is generally not measurable according to the Kolmogorov construction (see for example Judd (1985)). Therefore the meaning of "proportion α of the agents produces the superior quality good" does not make sense. Even if the set of all possible outcomes is endowed with a new measure which assigns zero measure to the set of outcomes which were not measurable, another problem arises: The Law of Large Numbers may not be true (see for example Judd (1985)). For further discussion and possible solution concepts see Dubey and Shapley (1994), Al-Najjar (1995), and Khan and Sun (1997). Here we propose our own solution to this problem.

The concept we develop here clearly can be applied to a more general model, but here we only apply it to our model of the nonmonetary economy presented in Section 2. We mimic the (inadmissible) mixed-strategy profiles with (admissible) correlated pure strategy profiles. We want to accomplish two goals here. First, in an economy where every agent is producing the superior good with probability λ, the proportion of those agents who are actually producing the superior good is indeed λ (that is, we want to guarantee that the conclusion of the Law of Large Numbers remains true in our model). And second, we want satisfy the intuitive criterion of uniform random matching: if proportion λ of the agents are producing the superior good in the economy, then every agent meets a superior good holder with probability λ in the first round (and a straightforward analogue for the second round).

In what follows we show how to mimic the strategy profile described in Subsection 2.2 in which every day each agent produces the superior good with probability λ and trades in the first round if and only if she meets a superior good holder, but she trades with everybody in the second round.

Suppose there is a game master who recommends strategies to each player. Assume that the agents are indexed by the points of a unit circle. First, the game master picks a random point on this circle according to the uniform distribution and maps this point to zero and the circle to $[0, 1]$ according to the natural bijection.

1. The game master recommends a strategy to each player. For example he suggests that each agent in $[0, \lambda)$ produce the superior good and each agent in $[\lambda, 1)$ produce the mediocre good. Furthermore he recommends to trade in the morning only with traders holding superior good.

2. If each agent follows the recommendation of the game master, he matches them in the following way. First he divides the superior good producers into two subgroups: he picks a number x in $[0, \lambda]$ according to the uniform distribution and then the two subgroups

are defined as follows:

$$S_S = [x, x+\lambda^2] \text{ modulo } \lambda$$
$$S_M = [x+\lambda^2, x+\lambda] \text{ modulo } \lambda.$$

Similarly he divides the mediocre good producers into two subgroups:

$$M_M = \lambda + \{[x, x+(1-\lambda)^2] \text{ modulo } 1-\lambda\}$$
$$M_S = \lambda + \{[x+(1-\lambda)^2, x+(1-\lambda)] \text{ modulo } 1-\lambda\}.$$

Observe that the lengths of S_M and M_S are the same, so the game master uses the natural bijection to match every agent from S_M with an agent from M_S. Then he matches everybody from S_S (M_M) with another agent from S_S (M_M). Therefore if each agent follows the recommendation of the game master, the probability of meeting a superior good holder in the morning is indeed λ.

If each agents follows the recommendation of the game master, after the morning round only the agents from S_S will leave the market and clearly the game master can match up the remainder in a similar manner. If a single agent deviates, the game master puts him into the group to which he deviated and matches him accordingly. (Deviation of a single agent does not effect the matching of the others.) The recommended strategy profile is an equilibrium if no agent has any incentive not to follow the recommendation of the game master.

The game master only takes care of measurability problems. He does not induce any correlated equilibrium what cannot be justified as a mixed-strategy equilibrium on intuitive grounds. The game master in Dubey and Shapley (1994) forced the agents to "cooperate", and the agents must solve the measurability problems themselves . Our game master helps them to coordinate instead. The advantage of this approach is that it avoids the use of the Law of Large numbers in a model of continuum many players.

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