

# Liquidity and Monetary Policy

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## Abstract

This paper builds a variant of Kiyotaki-Moore (2008) to address the importance of liquidity frictions and their fluctuations on the allocation of resources, and how these outcomes can be influenced through the injection of liquidity, e.g. by means of lump sum transfers of cash or asset market trades by the central bank. The model features agents with linear utility in consumption, endowed with labor and capital, and randomly allowed to create new capital from consumable goods. Liquidity constraints are imposed upon the transactions of capital between investing and non-investing agents. I analyze steady states and dynamics with a government having an array of instruments: transfers, money creation, and direct trades of capital. The steady state displays a sub-optimal, monetary equilibrium only when liquidity frictions are substantially large. Money neutrality is found but not super-neutrality, the Friedman rule attains optimality irrespective of the instruments used to finance the deflation. As for dynamics, negative liquidity shocks can generate a contraction and a temporary drop in the asset prices. Moreover, the policy of purchasing capital has differential effects in magnitude, persistence and even direction depending on how it is financed.

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# 1 Introduction

How much do liquidity frictions and their fluctuations matter for the allocation of resources? How can these be influenced through the injection of liquidity, e.g. by means of lump sum transfers of cash or asset market trades by the central bank? The most recent example of the concerns about these issues is originated by the sub-prime mortgage crisis whose effects are still subject of study and evaluation. The following table shows succinctly the composition of the Fed's assets, before and during the crisis:

Federal Reserve Assets			
	Aug-07	Oct-08	Oct-09
New (TSLF,etc.) (in %)	0.0	32.1	39
Total (levels, billions)	903	1840	2198

Source: Fed data.

The table shows the emergence of what has been called "new instruments" of monetary policy. Different from conventional instruments, they allow agents to obtain liquid assets, such as loans or treasury bills using as collateral private commercial paper whose circulation were hampered by the economy's financial distress. What is the channel of transmission and the costs and benefits of the implementation of these policies?

I address these questions in a variant of a Kiyotaki-Moore (2008) model. To clearly spell out in which dimensions my analysis differ from theirs, let me synthesize their model environment. They construct a model with two kinds of agents: entrepreneurs, who posses log utility in consumption, produce output and are randomly allowed to create new capital from output; and workers, who just provide labor to entrepreneurs and consume their entire labor income. Liquidity constraints are imposed upon the transactions of capital between investing and non-investing entrepreneurs. These assumptions enables them to obtain and solve an aggregate version of the economy without dealing with the complications that heterogeneity creates. In

contrast to these authors, I assume that a single group of agents, own both factors of production and their services are rented to a representative firm that produces the output in the economy. I also assume that agents have preferences that are linear in consumption. The linearity of preferences assumption, although not suitable for analyzing issues of risk aversion, has the great advantage of facilitating aggregation (while retaining an otherwise standard ownership and factor endowment environment) and make transparent the mechanics through which liquidity frictions affect the economy. With these modifications, I also seek to obtain a better characterization of the role of money, which is treated as an asset free of those frictions. The resulting model allows to analyze a variety of government policies, which in spite of its highly stylized features, might serve to think about its potential effectiveness and its channels of transmission. Moreover, the methodology used for the dynamic analysis in this paper, permits a more flexible and complete evaluation of the effects of both the liquidity frictions and the government policy.

In the model, there are two assets: capital and money. Given agent's heterogeneity, in addition to the rental market for inputs, there is a market where agents can trade their post-production and newly created units of capital and money. Post-production capital refers to the un-depreciated capital that agents hold after renting their services to the firms. The distinction between these two kinds of capital matters as agents might face different degrees of liquidity frictions for them. These liquidity frictions are exogenously imposed on this market as agents cannot sell more than a given fraction of the capital they hold in a given period. When these constraints are tight enough, the market price of capital is higher than its frictionless counterpart. In these circumstances, agents with the investment technology have a private incentive to sell large amounts of capital and in this way obtain more consumable goods as an input for continuing producing capital. Were the liquidity constraints be relaxed, enough new units of capital would be created with the resulting equilibrium being Pareto optimal. When the liquidity constraints are tight enough, agents turn to the use of money which only then is valued. Therefore, money

serves the function of facilitating the flow of additional resources among agents.

The steady state is solved analytically. There, I analyze how the equilibrium differs depending on the parameters of the economy. The key parameters that determine if the economy displays a suboptimal equilibrium are the liquidity parameters, the depreciation rate, the discount factor and technological parameters of the production function. I show that if the liquidity parameters are tight enough then it is not possible for the economy to finance the replenish of the depreciation of the optimal capital stock of the economy. Also, even when the liquidity constraint parameters are tight enough such that the economy display a suboptimal allocation, money may not be valued. That is, even when the economy is liquidity constrained, the resulting equilibrium might be non-monetary. It is required tighter liquidity frictions for money to become valued. Once the economy is in a liquidity constrained "region" in terms of the liquidity constraint parameters, I show that the negative effects are increasing in the degree of tightness of liquidity, but money prevents the economy to attain highly undesirable allocations. Money in this model is neutral, but not super-neutral. The steady state rate of growth of money have real effects, different from money-in-the-utility models such as Sidrauski (1967). The nature of the non-super-neutrality is different from models with this feature such as Cooley and Hansen (1989), because it is not based on consumption-leisure substitution due to inflation. Money is not super-neutral because anticipated inflation reduces the value of money which is the asset that enables the transfer of goods towards the production of investment.

Government has an array of instruments that include money creation, taxes and capital holdings. I show that the Friedman rule attains the Pareto optimal allocation, and that there exist several ways to finance the implied deflation that are equivalent in terms of efficiency, these include government purchasing capital in the market and or taxing the economy. When the government do not implement the Friedman rule and hold a positive constant amount of government capital, I show that in order for money to be valued, the economy must be even

more constrained than without any government capital holdings, suggesting a minimal long-run government intervention in the private assets market.

In a plausible numerical parametrization of the model I find that liquidity constraints need to prevent the selling of approximately 80 percent of the holdings of capital per quarter, in order for the steady state equilibrium to be suboptimal. With the liquidity constraints of this magnitude the resulting allocations are close to their frictionless counterpart<sup>1</sup>. I assess then if this feature is also present in the transition to the steady state. Implementing the parametric path method of Judd (2002), I show that the dynamic effects can be highly detrimental relative to a frictionless benchmark if the economy is relatively far away from the steady state. I also show the importance of money to narrow the gap between the time path of the frictionless economy's variables and the constrained ones.

How effective is the model in order to account for some features of the last crisis? To answer this question is necessary to conduct a short run analysis with stochastic fluctuations. In order to build some confidence on the appropriateness of the model as a dynamic description of the economy, I analyze several possible sources of fluctuations. Productivity shocks, money growth shocks and liquidity shocks are introduced. The stochastic dynamic system that describes the solution of the model is approximated by a log-linear system and solved near the liquidity constrained steady state economy.

The effects of productivity shocks are similar qualitatively to standard models in the neoclassical growth literature. The main difference with those models being that the effects of productivity shocks are relatively small in magnitude.

Positive and persistent money growth shocks are detrimental for investment and output. Different from models that get this result from the substitution between leisure and consumption,

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<sup>1</sup>Although further increases in the frictions have incremental detrimental effects.

my model shows that it is devalued money that function less effectively as a vehicle to transfer resources among individuals the channel for this result. As with productivity shocks, the magnitude of the effects are small. It seems that the liquidity constraint parameters that were set at low values to get a sub-optimal monetary equilibrium are also responsible for the low response of the system to general aggregate shocks.

Aggregate liquidity shocks are also introduced as fluctuations in the liquidity constraint parameters. In particular, following Kiyotaki and Moore, liquidity shocks are introduced as stochastic fluctuations in the parameter that determines the maximum fraction per unit of time that can be sold of the post production capital agents hold. This device meant to be the model analog of the last financial crisis, as the proceeds from the post production capital that can be sold are used by agents with the investment technology as input for capital creation.

Fluctuations in the liquidity parameter affect both the supply and demand of capital. It affects supply directly as a tighter constraint reduces the amount of new units of capital created and offered in the market. It affects demand because negative fluctuations that are expected to last for several periods reduce demand of capital as it becomes a less attractive asset as mean to save because future impediments on its selling are expected. I found that when the parameter falls transitorily, demand falls less than supply given that buyers of capital expect an eventual recovery. In this case the price of capital temporarily increases. When the liquidity parameter drops permanently to a new level, no recovery is expected and the fall in demand outweighs the fall in supply, reducing the price of capital. The model suggest then, that agents in the last crisis were expecting no recovery from the liquidity crunch.

I use the model to analyze the unorthodox policy of direct government purchases of capital in the market. Two exercises are performed, first a given transitory purchase of capital financed with taxes and second, the same transitory purchase financed with money creation. While both

policies succeed in increasing output and investment, they differ in magnitude and persistence, and more importantly, inflation increases under the first policy, while it decreases under the second. Interestingly, when the government temporarily purchases capital and has only money creation as an additional instrument, then following the initial purchase, it must retire money from the economy. The anticipation of this action by agents lead to a higher demand of money at the time of the implementation which produce a deflation in spite of the cash injection. Furthermore, as agents expect money to gain value following the policy, investment and output remain for a longer time above their steady state values compared to the first policy.

When the purchase is permanent, there is a stark difference between the effects of both policies. When the purchase is financed with taxes, then results are similar to the transitory purchase, while financing the purchase with money creation have permanent contractive effects on capital and output. Since no withdrawal of money is possible after the initial purchase, then the rate of growth of money remains at a permanent higher level, and this devalues money with its consequent negative effects on investment.

**Related Literature** There exists several papers that introduce heterogeneity as a source of liquidity needs. At least since the papers of Lucas (1980) and Diamond and Dybvig (1983), heterogeneity in the marginal utility of consumption was considered important to analyze the role of money and assets in the economy. It is interesting that there are recent papers that take the view of liquidity as it was first conceptualized by these highly stylized seminal papers on the topic. A couple of such papers are Curdia and Woodford (2009) and Yi Wen (2009). The progress achieved by the models developed in these papers is that such ideas are embedded in models that are useful for monetary policy analysis.

In another strand of the literature, papers such as De Fiore and Tristani (2007) introduce heterogeneity in investment opportunities as opposed to heterogeneity in preferences following

the seminal contribution by Bernanke, Gertler and Gilchrist (1999) and Carlstrom and Fuerst (1997). These papers also assume the existence of a financial intermediary. Internal funds of the firms are not sufficient to finance the desired amount of investment, so entrepreneurs need to raise external finance. It is assumed that, since investors might default upon facing a negative shock, households are not willing to lend to them directly. The financial contract between the financial intermediaries and the investors, solve explicitly an agency problem. The way this is modeled goes back to Carlstrom and Fuerst (1997), and represents an inter-period solution of a moral hazard problem between investors and intermediaries. The virtue of such a modeling strategy is that it is possible to incorporate the financial friction in an inter-temporal model without analyzing dynamic contracting. The i.i.d. assumption and the absence of inter-period links between investors' decisions enable to solve the agency problem period by period which greatly simplifies the analysis. Kiyotaki and Moore (2008) and my own work that is based on theirs, share the same feature with these papers in incorporating heterogeneity in investment opportunities, although no agency problem is formally modeled. Instead, it is exogenously assumed a friction that prevents agents to trade as much as they desire. The restriction has a temporal nature, since agents have to hold for the next period at least a fraction of initial period capital. This feature can be viewed as a shortcut to get the approximate result that a formal informational problem solving would deliver.

In all the papers mentioned above, the complications that arise from solving the model with heterogeneity were in part avoided by clever assumptions that facilitated aggregation. In Curdia and Woodford, the assumption of a financial contract with sporadic encounters among individuals was useful. In De Fiore and Tristan, the assumption of a financial intermediary that diversified risk and the linearity of the investment technology helped to obtain neat results. Also, the special preference structure and timing of Wen, allowed to obtain closed form solutions through which aggregation were possible. It seems that methodologically, the main obstacle when deal-

ing with models that address liquidity issues, is precisely challenges that are present due to heterogeneity. Kiyotaki and Moore assume two measures of agents, members of one of them differing intrinsically in several aspects from members of the other. I assume a linear structure of preferences for the agents which given their implied corner solutions facilitates aggregation.

The rest of the document is organized as follows. Section 2 presents the main model. Section 3 analyzes the steady state of the economy. Section 4 focus on the transitional dynamics of the model without aggregate uncertainty. Section 5 presents an analysis of aggregate uncertainty. Section 6 presents an exploration in the model to understand current crisis and Section 7 concludes.

## 2 The Model

### Environment

The economy is populated by a measure one of infinitely lived households or agents which are indexed by  $j \in [0, 1]$ . At time 0, each agent has a given amount of homogeneous capital and a given amount of intrinsically useless pieces of paper, money. Each period they have a labor endowment normalized to unity, to which no disutility is attached.

At the beginning of a given period, an agent has carried over from previous period  $k_t^j$  units of capital and  $m_{t-1}^j$  units of money. Both physical capital and labor is rented to a constant returns to scale firm which pays the rental rate  $r_t$  and wage rate  $w_t$  for each unit of capital and labor respectively, in units of the consumable good. The firm also returns the un-depreciated amount of capital to each agent. Government gives to each agent a transfer  $\tau_t^j$  of goods<sup>2</sup>. After

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<sup>2</sup>Given that the government can buy capital in the market, as it will be specified shortly, the transfers might be financed with the proceeds from its capital holdings, or financed with money creation.

all these payments are made, a fraction  $\pi$  of the agents receive exogenously and for free, a constant returns to scale technology of investment, by which they can transform one unit of the consumable good to one unit of capital<sup>3</sup>. Following Kiyotaki and Moore (2008), henceforth KM, I will call an agent with the investment technology an investor and an agent without the investment technology a saver<sup>4</sup>. The probability of an agent acquiring the investor status is  $\pi$  and is independent in time and across individuals. The status of the agent  $j$  in period  $t$  is denoted by  $z_t^j \in Z = \{1, 0\}$ , where 1 denotes an agent with investor status and 0 denotes an agent with saver status.

After knowing their status, agents decide on how much to consume and how much of capital and money to sell and buy. Investors decide on how much of new units of capital to sell. Capital is traded at price  $q_t$ <sup>5</sup>, and money at price  $\mu_t$ , both in units of consumable goods. After these transactions are made agents end up with capital holdings of  $k_{t+1}^j$  and money holdings of  $m_t^j$ , and the process starts again next period.

Government may buy or sell capital in the market, create new units of money, and transfer resource to agents, which might be negative. I now turn to state more formally the optimization problem of the agents and the behavior of the government.

### The Agent $j$ optimization problem

Each agent has preferences over consumption given by a utility function:

$$E \left[ \sum_{t=0}^{\infty} \beta^t c_t^j \right]$$

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<sup>3</sup>This technology is reversible in the sense that it can be used to transform one unit of capital good into a unit of consumable good.

<sup>4</sup>Both agents can in principle save, the investor directly by creating new units of capital or buying them in the market and the saver by buying capital in the market.

<sup>5</sup>Later, we will specify some frictions, the liquidity constraints, that are exogenously imposed upon these transactions.

where  $\beta \in (0, 1)$ . The expectation operator refers both to the aggregate uncertainty introduced later and the idiosyncratic uncertainty of the changing status of agents.

The flow constraint for an investor is given by:

$$c_t^j + i_t^j + q_t [k_{t+1}^j - i_t^j - (1 - \delta)k_t^j] + \mu_t [m_t^j - m_{t-1}^j] \leq w_t + r_t k_t^j + \tau_t^j, \quad z_t^j = 1. \quad (1)$$

To understand this flow constraint, let us consider the case where investment is positive. The second term on the left hand side is the cost of investment. The investor might decide, for example, to maintain for the next period the depreciated amount of capital he received from the firm. Hence  $k_{t+1}^j = (1 - \delta)k_t^j$ , in this case he is selling all new units of capital created, for which he receives  $q_t i_t$ . On the other hand he might hold for himself all new units of capital created, and even buy more from the market, in this case  $k_{t+1}^j > i_t^j + (1 - \delta)k_t^j$ . What the investor actually does will depend of course on the market conditions and the liquidity constraints he faces. Investors can only sell up to a fraction  $\theta_s$  of the new units of capital created, also they can only sell up to a fraction  $\theta_r$  of the capital they hold from previous period<sup>6</sup>. It is assumed that this parameter follow an exogenous stochastic process:

$$\ln \theta_{r,t} = (1 - \rho_\theta) \ln \bar{\theta}_r + \rho_\theta \ln \theta_{r,t-1} + \epsilon_{\theta,t}, \quad (2)$$

where  $\bar{\theta}_r$  is the steady state value of  $\theta_{r,t}$ ,  $0 \leq \rho_\theta \leq 1$ , and  $E[\epsilon_{\theta,t}] = 0$ . In this way fluctuations or "shocks" of liquidity are introduced.

The liquidity constraint is then given by:

$$k_{t+1}^j \geq (1 - \theta_s) i_t^j + (1 - \theta_{r,t})(1 - \delta) k_t^j, \quad z_t^j = 1.$$

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<sup>6</sup>In specifying the liquidity constraints I follow KM entirely. Several informational or agency problems can be thought of as justifications for these constraints.

Consumption, investment and money holdings are also restricted to be non-negative:

$$c_t^j \geq 0, \quad i_t^j \geq 0, \quad m_t^j \geq 0, \quad z_t^j = 1.$$

The flow constraint for savers is similar to that of the investors, the only difference is that they lack the investment technology, hence any accumulation of capital must be done by buying it in the market:

$$c_t^j + q_t [k_{t+1}^j - (1 - \delta)k_t^j] + \mu_t(m_t^j - m_{t-1}^j) \leq w_t + r_t k_t^j + \tau_t^j, \quad z_t^j = 0. \quad (3)$$

savers are also subject to a liquidity constraint:

$$k_{t+1}^j \geq (1 - \theta_{r,t})(1 - \delta)k_t^j, \quad z_t^j = 0.$$

In this case the liquidity constraint is reflected only in the difficulty of selling old units of capital. Finally, savers are also restricted to have non-negative consumption and money holdings:

$$c_t^j \geq 0, \quad m_t^j \geq 0, \quad z_t^j = 0.$$

### The Firms optimization problem

Firms optimization problem is standard and simple. CRS firms rent individual capital and labor from each agent at the start of each period. This is equivalent to a single firm renting the aggregate amount of capital  $K_t$ , and aggregate labor  $L_t$ . The representative firm's problem is:

$$\max_{K_t, L_t} [F(K_t, L_t) - r_t L_t - w_t L_t], \quad t = 0, 1, 2, \dots \quad (4)$$

It then follows that the optimality conditions are:

$$\begin{aligned} r_t &= F_K(K_t, L_t), \quad t = 0, 1, 2, \\ w_t &= F_L(K_t, L_t), \quad t = 0, 1, 2, \dots \end{aligned} \tag{5}$$

I will use throughout the Cobb-Douglas production function:  $Y_t = A_t K_t^\alpha L_t^{1-\alpha}$ ,  $0 < \alpha < 1$ . The parameter  $A_t$  is assumed to follow an exogenous stochastic process:

$$\ln A_t = (1 - \rho_A) \ln \bar{A} + \rho_A \ln A_{t-1} + \epsilon_{A,t}, \tag{6}$$

where  $0 \leq \rho_A \leq 1$ , and  $E[\epsilon_{A,t}] = 0$ .  $\bar{A}$  is normalized such that the steady state value of capital of a representative agent economy equals one:  $\bar{A} = \frac{1-(1-\delta)\beta}{\alpha\beta}$ .

## The Government

It is assumed that at time zero the central bank has provided in lump sum fashion a given amount of money to the economy  $M_0$ . Recall that it has not been stated conditions such that  $\mu_t > 0$  yet. As it will be shown, there exist some conditions on the parameters such that money will not be valued in the economy, and hence monetary policy is vacuous. The objective of introducing government is simply to model the effects of its using of different instruments. I do not attempt to derive optimal government policy but just understand the mechanisms through which government policy might affect the performance of the economy. It is assumed that the government must respect the following budget constraint:

$$T_t + q_t [H_{t+1} - (1 - \delta)H_t] = r_t H_t + \mu_t (M_t - M_{t-1}), \tag{7}$$

where  $T_t$  is the aggregate amount of transfers,  $H_t$  is the amount of capital holdings, and  $M_t$  is

the stock of money at time  $t$ . When holding capital, government acts as another private agent in the economy, in the sense that it rents capital services to the firms<sup>7</sup>. Since  $H_{t+1}$  is capital acquired from the private sector is reasonable to assume that liquidity constraints affect also any transactions that the central bank conducts, hence a liquidity constraint is given by<sup>8</sup>:

$$H_{t+1} \geq (1 - \theta_{r,t})(1 - \delta)H_t.$$

It is evident that different policies that the government implements are linked by its budget. If for example the government decides to not buy any private capital, then any money creation must be transferred to agents. To be concrete, initially, it is specified rules for money creation and capital holdings, leaving transfers as a variable to "adjust". Later in the paper it will be considered a different policy where transfers are set to a constant value.

Let  $\gamma_t$  be the gross growth rate of money that the central bank creates:

$$M_t = \gamma_t M_{t-1}.$$

In order to introduce surprise changes in the money supplied by the central bank, an stochastic process for the rate of growth is postulated:

$$\ln \gamma_t = (1 - \rho_\gamma) \ln \bar{\gamma} + \rho_\gamma \ln \gamma_{t-1} + \epsilon_{\gamma,t}, \quad (8)$$

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<sup>7</sup>This budget constraint links for the government, money creation and capital transactions with transfers of money to agents. If for example, in a given period new money is not created:  $M_t = M_{t-1}$ , and  $q_t[H_{t+1} - (1 - \delta)H_t] < r_t H_t$ , then the real government income from past holdings of capital is strictly higher than the current expenditures in acquiring capital. Its budget constraint in this case implies that  $r_t H_t - q_t[H_{t+1} - (1 - \delta)H_t]$  goods ought to be given to agents as transfer goods. If  $M_t > M_{t-1}$  and  $H_t = H_{t+1} = 0$ , then all money creation is used to buy goods that are transferred to agents.

<sup>8</sup>Later in the paper, monetary policy buying capital in the market will be analyzed. In this case this restriction will not be binding. If selling of government capital is analyzed then the possibility that this constraint is binding needs to be considered.

where  $0 \leq \rho_\gamma \leq 1$  and  $E[\epsilon_{\gamma,t}] = 0$ .  $\bar{\gamma}$  is the steady state growth of money.

I assume that the government establish a stochastic rule for capital transactions as follows:

$$\ln H_t = (1 - \rho_H) \ln \bar{H} + \rho_H \ln H_{t-1} + \epsilon_{H,t}, \quad (9)$$

where  $0 \leq \rho_H \leq 1$  and  $E[\epsilon_{H,t}] = 0$ .  $\bar{H}$  is the steady state amount of government capital holdings.

This rudimentary way of introducing policy in capital holdings, serves the purpose of analyzing the effects of non-conventional policy such as transitory transactions of capital.

## 2.1 Recursive Representation and Definition of Equilibrium

The investor's Bellman equation ( $z = 1$ ) is<sup>9</sup>:

$$\begin{aligned} v_1(k, m) &= \max_{k', m', i} E \{ w + rn + \tau - i - q[k' - i - (1 - \delta)k] - \mu(m' - m) \\ &+ \beta\pi v_1(k', m') + \beta(1 - \pi)v_0(k', m') \} \end{aligned}$$

such that:

$$w + rn + \tau - i - q[k' - i - (1 - \delta)k] - \mu(m' - m) \geq 0$$

$$k' \geq (1 - \theta_s)i + (1 - \theta_r)(1 - \delta)k, \quad i \geq 0, \quad m' \geq 0.$$

The saver's Bellman equation ( $z^j = 0$ ) is:

$$\begin{aligned} v_0(k, m) &= \max_{k', m'} E \{ w + rn + \tau - q[k' - (1 - \delta)k] - \mu(m' - m) \\ &+ \beta\pi v_1(k', m') + \beta(1 - \pi)v_0(k', m') \} \end{aligned}$$

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<sup>9</sup>I omit the index  $j$  from the variables below to simplify notation. When some ambiguity is created because of this, it will be specified and clarified in the text.

such that:

$$w + rn + \tau - q[k' - (1 - \delta)k] - \mu(m' - m) \geq 0$$

$$k' \geq (1 - \theta_r)(1 - \delta)k, \quad m' \geq 0.$$

The expectation operator refers to the aggregate uncertainty reflected in the processes (2),(6),(8) and (9).

Government is represented by:

$$T + q[H' - (1 - \delta)H] = rH + \mu(M' - M)$$

$$H' \geq (1 - \theta_r)(1 - \delta)H, \quad M' = \gamma'M.$$

### Formal statement of the problem and definition of equilibrium

**Definition** *A recursive competitive equilibrium consists of factor prices  $(r,w)$ ; price of capital  $q$ , price of money  $\mu$ , policy functions (for each agent  $z = 1,0$ ) for consumption  $c_z(k, m)$ , next period capital  $k'_z(k, m)$ , next period money holdings  $m'_z(k, m)$  and investment  $i(k, m)$ , probability measures  $\lambda_z(k, m)$ , and aggregate level of private capital  $N$ , and total aggregate capital  $K$ , such that:*

1.  $c_z(k, m)$ ,  $k'_z(k, m)$ ,  $m'_z(k, m)$  and  $i(k, m)$  maximize  $j$ 's utility subject to flow and "liquidity constraints"
2. at given  $r, w$  firms maximize profits:

$$r = F_K(K, L), \quad w = F_L(K, L)$$

3. with private demand of capital:

$$N' = \int k'_1(k, m)d\lambda_1(k, m) + \int k'_0(k, m)d\lambda_0(k, m),$$

and government demand of capital:  $H'$ ,

4. the capital market clears:

$$K' = N' + H'$$

$$K' = (1 - \delta)K + I$$

5. labor and money market clear:

$$L = \int d\lambda_1(k, m) + \int d\lambda_0(k, m) = 1$$

$$M' = \int m'_1(k, m)d\lambda_1(k, m) + \int m'_0(k, m)d\lambda_0(k, m)$$

6. Government satisfy its budget constraint:

$$T + q[H' - (1 - \delta)H] = rH + \mu(M' - M).$$

## Analysis

### The case $q > 1$

An equilibrium cannot be reached when  $q < 1$  since in this case the investor will not create any new units of capital (the cost of investment which is one would be higher than the market price of accumulating capital). The case  $q = 1$  will be analyzed later as a limit result from the case analyzed here.

To simplify notation I define:

$$\begin{aligned}\psi &\doteq \frac{1 - q\theta_s}{1 - \theta_s} \\ \eta &\doteq 1 - \theta_r + q\theta_r \frac{1 - \theta_s}{1 - q\theta_s}.\end{aligned}$$

When  $q > 1$ , then the following relationships hold in equilibrium<sup>10</sup>:

$$0 < \psi < 1 < q \quad \eta > 1. \tag{11}$$

When (11) are satisfied, the feasibility set for the investors can be written as:

$$c + \psi [k' - \eta(1 - \delta)k] + \mu(m' - m) \leq w + rk + \tau$$

$$k' \geq (1 - \theta_r)(1 - \delta)k, \quad m' \geq 0.$$

An explanation about this feasibility set is in order: note that this feasibility set is isomorphic to the one saver's face. The difference being that in the flow constraint, the effective price that investors pay for capital is  $\psi$  and the effective post-production capital per unit of capital rented to firms is  $\eta(1 - \delta)$ . This effective flow constraint is obtained by replacing out investment from the flow constraint with the value of investment from the liquidity constraint at equality. This constraint binds when  $q > 1$ , since investors find optimal to invest as much as possible. By the same token, investors are always selling capital under this scenario because their liquidity constraint is binding.

The per-unit effective capital after depreciation available for the investor is not  $(1 - \delta)$  but  $\eta > 1$ . Hence the investor has an advantage as the effective amount of capital after production

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<sup>10</sup>In any equilibrium when  $q > 1$  it is required also that  $1 - q\theta_s > 0$ , since as it will be seen shortly, investors would face a negative price for capital accumulation, and hence their problem wouldn't have a solution.

is greater than the un-depreciated portion of the capital used in production. This advantage arises because the investor can use the investment technology to profit in the market as follows: from the initial unit of capital, after production he has  $(1 - \delta)$  units of which up to  $\theta_r$  units can be sold at price  $q$ , this gives  $q\theta_r(1 - \delta)$  units of goods, retaining  $(1 - \theta_r)(1 - \delta)$  units of capital. Goods obtained with this transaction can in turn be used again as input to create *new* units of capital, which can be sold up to  $\theta_s$  at price  $q$ , getting  $\theta_s q^2 \theta_r (1 - \delta)$  goods and retaining  $(1 - \theta_s)q\theta_r(1 - \delta)$  units of capital. Similarly, the new units can be sold up to  $\theta_s$ , getting  $(\theta_s)^2 q^3 \theta_r (1 - \delta)$  goods, and keeping  $(1 - \theta_s)\theta_s q^2 \theta_r (1 - \delta)$ . All units of goods can be sold in this fashion, getting in the end, the total amount of capital:

$$\begin{aligned}
& (1 - \theta_r)(1 - \delta) + (1 - \theta_s)q\theta_r(1 - \delta) + (1 - \theta_s)\theta_s q^2 \theta_r (1 - \delta) + \dots \\
&= (1 - \theta_r)(1 - \delta) + q\theta_r(1 - \delta)(1 - \theta_s) [1 + q\theta_s + (q\theta_s)^2 + (q\theta_s)^3 + \dots] \\
&= \left(1 - \theta_r + q\theta_r \frac{1 - \theta_s}{1 - q\theta_s}\right) (1 - \delta) = \eta(1 - \delta).
\end{aligned}$$

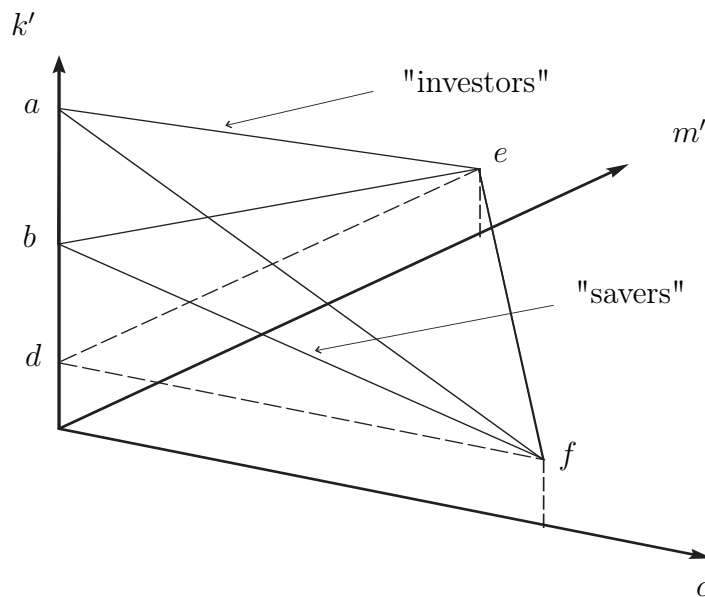
Increments above the level of capital  $\eta(1 - \delta)k$  are conducted at the "effective price"  $\psi$ , which under condition (11) above, is less than one. Hence again, the investor has an advantage relative to the saver who has to buy capital at price  $q > 1$ . This advantage arises due to the investment technology the investor possesses because within a period he can implement the following scheme: incur in a cost of one unit of good and produce 1 unit of capital, a fraction  $\theta_s$  of which can be sold at price  $q$ , buying  $q\theta_s$  units of goods and keeping  $1 - \theta_s$  of capital. The units of goods acquired can be used to produce  $q\theta_s$  units of capital, of which a fraction  $\theta_s$  can be sold getting  $q^2(\theta_s)^2$  goods and keeping  $q\theta_s - q(\theta_s)^2 = q\theta_s(1 - \theta_s)$  units of capital, continuing this process, the total amount of capital that the investor can get for his initial one unit of good is:

$$(1 - \theta_s) + q\theta(1 - \theta_s) + (q\theta_s)^2(1 - \theta) + \dots = \frac{1 - \theta_s}{1 - q\theta_s}.$$

hence, the "effective price" of one unit of capital for an investor can be defined as  $\psi$ .

Investors also face the non-negative constraint on investment which now takes the form:  $k' \geq (1 - \theta_r)(1 - \delta)k$ .

**Proposition 1.** *When  $q > 1$ , then investor's feasibility set is strictly larger than the saver's feasibility set. Proof: See Appendix A.*



When  $q > 1$ , the feasibility sets of both agents differ: investor's budget is the volume  $(a, d, e, f)$  and saver's budget is the volume  $(b, d, e, f)$ .

Figure 1: Opportunity sets in the  $(c, k', m')$  plane.

Some intuition might be obtained for the result in proposition 1 by examining Figure 1. In this figure, it is represented for given  $k$  and  $m$ , the feasible choices that agents have. The point  $d$  in the figure represents the minimum amount of capital agents are allowed to hold for the next period. This is the non-negativity constraint on investment for investors and the liquidity constraint for savers. In points where these constraints are binding such as  $e$  and  $f$ , agent's choices of  $m'$  coincide. The same is true for choices of  $c$  in such points. When agents acquire capital in excess of  $d$ , the opportunity cost in terms of consumption is always higher

for savers either because of reducing their money holdings or reducing their consumption levels. The maximum amount of capital savers can acquire is  $b$  with zero money holdings and zero consumption. Similarly with zero holdings of both goods, investors can hold for next period an amount  $a$ . As  $q \rightarrow 1$  the feasible set  $(a, d, e, f)$  for investors would coincide with the feasible set  $(b, d, e, f)$  for savers.

Given the facts stated above, in the  $q > 1$  case, the value functions can be written, for the investors:

$$\begin{aligned}
v_1(k, m) &= \max_{k', m'} \{w + rk + \mu m + \tau + \psi\eta(1 - \delta)k - \psi k' - \mu m' \\
&+ \beta\pi E[v_1(k', m')] + \beta(1 - \pi)E[v_0(k', m')]\} \\
&+ \lambda_1 [w + rn + \mu m + \tau + \psi\eta(1 - \delta)k - \psi k' - \mu m'] \\
&+ \mu_1 [k' - (1 - \theta_r)(1 - \delta)k] + \gamma_1 m' \}, \tag{12a}
\end{aligned}$$

and for the savers:

$$\begin{aligned}
v_0(k, m) &= \max_{k', m'} \{w + rk + \mu m + \tau + q(1 - \delta)k - qk' - \mu m' \\
&+ \beta\pi E[v_1(k', m')] + \beta(1 - \pi)E[v_0(k', m')]\} \\
&+ \lambda_0 [w + rk + \mu m + \tau + q(1 - \delta)k - qk' - p\mu m'] \\
&+ \mu_0 [k' - (1 - \theta_r)(1 - \delta)k] + \gamma_0 m' \}. \tag{12b}
\end{aligned}$$

Where  $\lambda_i$  is the multiplier on the non-negativity constraint on consumption and  $\gamma_i$  is the multiplier on the non-negativity constraint on money for  $i = 1, 0$ .  $\gamma_1$  is the multiplier on the non-negativity constraint on investment for investors and  $\gamma_0$  is the multiplier on the liquidity constraint for savers.

In order to solve the model, given the linear structure of preferences, it is assumed that:

**Assumption 1.**

$$\lambda_1 > 0, \lambda'_1 > 0, \gamma_1 > 0, \gamma'_1 > 0, \mu_1 = \mu'_1 = 0.$$

This is an strong assumption in that these values are assumed to hold for all possible realizations of the aggregate or idiosyncratic shocks. When solving the model numerically this assumption will be verified. The stochastic model will be solved numerically close to the steady state. Hence it is important to at least show formally that in the steady state the economy satisfy assumption 1, this will be proved rigorously.

In terms of the policy functions assumption 1 for each agent mean:

$$k'_1(k, m) = \frac{1}{\psi} [w + rk + \mu m + \tau + \psi\eta(1 - \delta)k] \quad (13a)$$

$$k'_0(k, m) \in \left( (1 - \theta_r)(1 - \delta)k, \frac{1}{q} [w + rk + \mu m + \tau + q(1 - \delta)k - \mu m'_0(k, m) - c_0(k, m)] \right) \quad (13b)$$

$$m'_1(k, m) = 0 \quad (13c)$$

$$m'_0(k, m) \in \left( 0, \frac{1}{\mu} [w + rk + \mu m + \tau + (1 - \delta)qn - qk'_0(k, m) - c_0(k, m) - \mu m'_0(k, m)] \right) \quad (13d)$$

$$c_1(k, m) = 0 \quad (13e)$$

$$c_0(k, m) \in (0, w + rk + \mu m + \tau + \theta_r(1 - \delta)k - \mu m'_0(k, m)). \quad (13f)$$

Referring back to Figure 1, assumption 1 imply that investors are at a corner (such as point  $a$  in Figure 1), with zero consumption and zero money holdings. All their resources are used to accumulate capital. Savers are at an interior solution, somewhere in the area delimited by  $(b, e, f)$ .

**Proposition 2.** *Under assumption 1, the aggregate equilibrium of the economy is characterized by the following stochastic dynamic system:*

$$\psi < \beta E \left\{ \left[ \frac{r'}{\psi'} + \eta'(1 - \delta) \right] q' \pi + [r' + q'(1 - \delta)](1 - \pi) \right\} \quad (14a)$$

$$q = \beta E \left\{ \left[ \frac{r'}{\psi'} + \eta'(1 - \delta) \right] q' \pi + [r' + q'(1 - \delta)](1 - \pi) \right\} \quad (14b)$$

$$\mu = \beta E \left[ \frac{q'}{\psi'} \mu' \pi + \mu'(1 - \pi) \right] \quad (14c)$$

$$N'_1 = \frac{1}{\psi} \left[ \frac{w}{N} + r + \psi \eta(1 - \delta) \right] \pi N + \frac{1}{\psi'} \mu \pi M + \frac{\pi T}{\psi} \quad (14d)$$

$$N'_0 = \theta^s I + [\theta_r(1 - \delta)\pi + (1 - \delta)(1 - \pi)]N \quad (14e)$$

$$K' = N'_1 + N'_0 + H' \quad (14f)$$

$$K' = (1 - \delta)K + I \quad (14g)$$

$$T + q[H' - (1 - \delta)H] = rH + \mu(M' - M), \quad M' = \gamma' M \quad (14h)$$

$$r = \alpha AK^{\alpha-1}, \quad w = (1 - \alpha)AK^\alpha. \quad (14i)$$

where  $N'_1$  is the aggregate demand of capital of investors, and  $N'_0$  is the aggregate demand of capital of savers.

*Proof:* See the Appendix.

The interpretations of these optimality conditions are not complicated, but it is perhaps worthwhile to spell them out, to explain better the mechanisms of the model. Note that so far it has been assumed that  $\mu > 0$ . As it will be shown in the next sections this is not necessarily so, money may not be valued in the economy. With this caveat we proceed to explain the optimality and equilibrium conditions.

First, note the RHS of (14a) and (14b) are equal. This is the expected gain that agents have, and they are equal independently of the current status of the agent because of the independent and identical distribution assumption of agent's status<sup>11</sup>. The expected gain of accumulating one unit of capital has two components, the first is the gain when the status of the agent is 1, that is an investor. When the agent's status next period is 1, the amount of goods obtained for the rental of his capital is  $r'$ , this can be transformed into capital at effective price  $\psi'$ . Recall that  $\eta(1 - \delta)$  is the amount of effective capital the investor gets post-production for one unit of capital rented to a firm. This amount of capital is sold at the effective price  $\psi'$ . Hence the term on square braces in (14a) is the total amount of capital that investors will have next period. This amount can then be used to buy capital at price  $q'$ . If the status of the agent is 0, then he can sell his un-depreciated amount of capital at price  $q'$  and hence gets  $[r' + q'(1 - \delta)]$  goods next period. These gains are weighted by the respective status probabilities and discounted at rate  $\beta$ .

Turn now to the LHS of (14a), this is the effective price investors pay for accumulating capital. Recall that by (11)  $\psi < 1$ , hence investors are accumulating capital at an effective price that is strictly lower than that faced by the savers, so indeed they want to accumulate as much as possible. Since consumption is restricted to be weakly greater than zero, the effective price will be smaller than the marginal benefit in equilibrium. The LHS of (14b) is the opportunity cost of accumulating capital for the saver. Since it has been assumed that savers are at an interior solution, then marginal cost equals marginal benefit of accumulating capital for them.

Next, (14c) is the condition expressing optimality in the demand of money for both investors and savers. The expected gain, the RHS of equation 14c has two components, when the status of the agent is 1, then the unit of money can be used to buy goods which gives  $\mu'$  amount of goods, this amount can be used as an input to create more capital, or in the effective price view,

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<sup>11</sup>Furthermore it is assumed that the idiosyncratic shock is not correlated with any of the aggregate shocks.

this amount can be used to buy capital at the price  $\psi$ , the total amount of capital so obtained is sold at price  $q'$ , to get a total amount of goods. When the agent has status 0, then he can only sell money at price  $\mu'$ . The LHS of (14c) is the marginal cost of taking one unit of money to the next period which is the same for savers and investors. Though the optimality condition (14c) is the same for investors and savers. Investors are constrained with zero money holdings. In fact in the Appendix it is shown that their first order condition is (14c) because  $\mu\lambda_1 = \gamma_1 > 0$  terms that appear in each of the sides of the first order condition for investor's money holdings. Hence it is precisely because the investor is constrained in their money constraint and consumption constraint that it is obtained (14c). For savers, this equation states that their individual money holdings are not determined because the marginal cost equals the marginal benefit and because their linear preferences. The same is true for investor's capital holdings.

Equation (14d) is the aggregate amount of capital demanded by investors and this comes from result equation (13a) which states that all resources that investors have are transformed into capital at effective price  $\psi$  and accumulated for next period. Note that because the i.i.d. assumption and the linearity of the policy, aggregation is easily done. So for example, beginning of period aggregate values of capital, money and transfers for the investors are fractions  $\pi$  of beginning of period aggregates,  $N$ ,  $M$  and  $T$ .

Equation (14e) is the aggregate amount of capital demanded by savers. In the Appendix it is shown formally how this is obtained. Recall that savers are indifferent in how much of capital to accumulate by equation (13b). But market clearing helps to determine the aggregate value since in the aggregate investors are selling  $\theta_s$  of the aggregate amount of new units of capital to savers and also they sell up to  $\theta_r$  of old units of capital they have. Besides, savers accumulate in excess of their initial aggregate holdings of capital.

Equation (14f) is the aggregate demand of capital which includes government capital besides

private demand of capital. Equation (14g) states the equilibrium in the capital market. Finally, equations (14h) and (14i) are the government budget constraint and the optimality conditions for firms respectively.

Before going on with the analysis. I turn to the stationary equilibrium. This is central to the analysis for two reasons. First, idiosyncratic heterogeneity renders the stationary state a non-trivial and interesting object to study because, the equilibrium will not be a system at rest, but there will be movements in this state. Second, and most important, the steady state analysis will be solved analytically, and conditions on the parameters will be found such that different sort of equilibria might be reached. It will also be analyzed what types of policies the government could implement in the steady state, and its consequences for the allocations and prices. The full dynamic analysis will then be undertaken numerically by approximating the system around a given steady state value for the variables.

### 3 Steady States

The steady state is a fixed point of the equations that describe the economy, equations (14). I use bars over the variables to denote the steady state values. Recall that so far it has been assumed that the price of capital is above unity and the price of money is strictly positive. We need to find conditions such that this is true. There is little sense in analyzing government, specially monetary interventions if money is not valued in the first place. As emphasized by KM, the liquidity frictions will be central to characterize when the economy achieve optimal allocations and when money is valued.

To begin with, it is assumed that government does not hold any capital, and does not create any new money. Hence  $\bar{H} = 0 = \bar{T} = 0$ , there exist a constant stock of money that was introduced

in the economy at the beginning of life and this is the only monetary authority intervention. It is also assumed that the price of money is zero  $\bar{\mu} = 0$ . Hence we first focus on showing under what conditions  $\bar{q} > 1$ .

Equation (14b) at steady state is<sup>12</sup>:

$$1 = \left[ \frac{\bar{r}}{\bar{\psi}} + \bar{\eta}(1 - \delta) \right] \beta\pi + \frac{\bar{r} + \bar{q}(1 - \delta)}{\bar{q}} \beta(1 - \pi). \quad (15a)$$

Equation (14c) is not an optimality condition for the saver when  $\bar{\mu} = 0$ . Equations (14f),(14g) and (14i) can be arranged into a single equation:

$$1 = \frac{1}{\bar{\psi}} \left[ \frac{\bar{r}}{\alpha} + \bar{\psi}\bar{\eta}(1 - \delta) \right] \pi + \theta_s\delta + (\theta_r\pi + 1 - \pi)(1 - \delta). \quad (15b)$$

Equations (15) form a system of 2 equations in the  $\bar{q}, \bar{K}$  unknowns<sup>13</sup>. This system can be manipulated and reduced to a single quadratic equation in  $\bar{q}$ :

$$\begin{aligned} & \{ \theta_s\pi[(1 - \beta) + \beta\delta(1 - \alpha)] + \beta\bar{\theta}_r\pi[\pi(1 - \alpha) + \alpha\theta_s](1 - \delta) + \beta\alpha\delta(\theta_s)^2 \} \bar{q}^2 \\ & - \{ \pi(1 - \beta) + \beta\delta\pi(1 - \alpha) + \beta\alpha\theta_s\delta(2 - \pi) + \beta\pi\bar{\theta}_r(1 - \delta)[\pi + \alpha(1 - \pi)] \} \bar{q} \\ & + \beta\alpha\delta(1 - \pi) = 0. \end{aligned} \quad (16)$$

Solving this equation will give two solutions. Conditions on at least one of them have to be imposed in order to find a solution such that  $\bar{q} > 1$  to be consistent with the assumptions that lead to this equation as a representation of the solution of the model. Before analyzing this

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<sup>12</sup>I work with the equations representing saver's first order condition equation (14b), leaving for later the verification that the inequality (14a) hold true for investors.

<sup>13</sup> $\bar{r}$  is a function of  $\bar{K}$  by (14i) and

$$\bar{\psi} = \frac{1 - \bar{q}\theta_s}{1 - \theta_s}, \quad \bar{\eta} = 1 - \bar{\theta}_r + \bar{q} \frac{1 - \theta_s}{1 - \bar{q}\theta_s}$$

equation in these terms, it is illustrative to analyze the limit case of  $\bar{q} = 1$ , which I consider next.

**The case  $\bar{q} = 1$**

Consider equations (15) as a representation of the solution of the model with  $\bar{\mu} = 0$ . The following proposition establishes conditions on the parameters such that the economy attains a first best equilibrium where optimal allocations are reached.

**Proposition 3.** *When parameters satisfy:*

$$\delta \leq \pi\bar{A} + \theta_s\delta + \bar{\theta}_r(1 - \delta)\pi \tag{17}$$

*then  $\bar{q}^* = 1$  and  $\bar{K}^* = 1$ ; the economy attains a first best equilibrium, i.e. the same as a representative agent model without frictions. Asterisks denote optimal equilibrium values.*

*Proof.* I start by taking the limit of the asset pricing equation (15a) as  $\bar{q} \rightarrow 1$ :  $1 = \beta(\bar{r} + 1 - \delta)$ . By equation (14i) this implies that  $\bar{K}^* = 1$  given the normalizing parameter:  $\bar{A} = \frac{1-(1-\delta)\beta}{\alpha\beta}$ . Equation (15b) then reduces to:

$$\delta = \pi\bar{A} + \theta_s\delta + \bar{\theta}_r(1 - \delta)\pi. \tag{18}$$

Equation (18) clarifies the sources of capital accumulation financing in the steady state. Given that  $\bar{K}^* = 1$ , the reposiotion of depreciated capital is financed by the amount of goods investors get by renting their capital and labor  $\pi\bar{A}$ , and the fraction of new units of capital transferred to savers  $\theta_s\delta$ , plus the fraction of old units of capital transferred to savers  $\bar{\theta}_r(1 - \delta)\pi$ .

The derivation above assumes that investor's consumption is zero. Consumption in the economy is given only by savers. The aggregate amount of consumption for this group is given by their

budget constraint:  $\bar{C} + \bar{q}[K'_0 - (1 - \delta)(1 - \pi)\bar{K}] = \bar{w}(1 - \pi) + \bar{r}(1 - \pi)\bar{K}$ . Replacing the first best levels of prices and capital, consumption is found to be:

$$C = \bar{A} - [\bar{A}\pi + \theta_s\delta + \bar{\theta}_r(1 - \delta)\pi]$$

substituting (18), consumption is  $\bar{C} = \bar{A} - \delta$ . Hence we require  $\bar{A} > \delta$ . Note that consumption could also be found by the resource constraint  $\bar{Y} = \bar{C} + \bar{I}$ , given that  $\bar{Y} = \bar{A}$  and  $\bar{I} = \delta$  in the Pareto equilibrium.

The efficient allocation can be supported even when the RHS of (18) is greater than the LHS. To see this let us compute the aggregate level of consumption for investors, from their budget constraint with the liquidity constraint at equality:  $C_1 = \pi\bar{A} - (1 - \theta_s)\delta + \bar{\theta}_r(1 - \delta)\pi$ . The condition of Proposition 3 guarantees that  $C_1 \geq 0$ <sup>14</sup>.  $\square$

To show formally that as long as  $\delta > \pi\bar{A} + \theta_s\delta + \bar{\theta}_r(1 - \delta)\pi$ ,  $\bar{q} > 1$ , we need to solve equation (16), the quadratic equation in  $\bar{q}$ .

### The case of $\bar{q} > 1$

The following proposition states conditions on the parameters such that  $\bar{q} > 1$ .

**Proposition 4.** *When parameters satisfy:*

$$\delta > \pi\bar{A} + \theta_s\delta + \bar{\theta}_r(1 - \delta)\pi \tag{19}$$

then  $\bar{q} > 1$ .

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<sup>14</sup>When investors are liquidity-constrained, they sell to savers as much as possible. The aggregate saver's consumption is:  $C_0 = (1 - \pi)\bar{A} - \theta_s - \bar{\theta}_r(1 - \delta)\pi$ . Condition that guarantees that aggregate savers's consumption is not negative is then:  $(1 - \pi)\bar{A} \geq \theta_s + \bar{\theta}_r(1 - \delta)\pi$ .  $\bar{q} = 1$  is supported even when this condition on the parameters is not satisfied, because we can relax the liquidity constraint of investors by having them accumulate more capital, and hence avoiding the situation when savers have negative consumption.

*Proof.* The quadratic equation in  $\bar{q}$  take the form:  $a\bar{q}^2 + b\bar{q} + c = 0$ , where

$$\begin{aligned} a &= \theta_s \pi [(1 - \beta) + \beta \delta (1 - \alpha)] + \beta \bar{\theta}_r \pi [\pi (1 - \alpha) + \alpha \theta_s] (1 - \delta) + \beta \alpha \delta (\theta_s)^2 \\ b &= - \{ \pi (1 - \beta) + \beta \delta \pi (1 - \alpha) + \beta \alpha \theta_s \delta (2 - \pi) + \beta \pi \bar{\theta}_r (1 - \delta) [\pi + \alpha (1 - \pi)] \} \\ c &= \beta \alpha \delta (1 - \pi) \end{aligned}$$

Since  $a > 0, b < 0, c > 0$ , both solutions to the equation are positive<sup>15</sup>. Only the lowest of these solutions is consistent with a positive interest rate and hence the solution we seek consistent with  $\bar{q} > 1$  is:

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} > 1$$

but this implies after simplification:  $0 > a + b + c$ . After substitution we get the condition:  $(1 - \theta_s)[\pi A + \theta_s \delta + \bar{\theta}_r (1 - \delta) \pi - \delta] < 0$ . □

Condition (19) says that the amount of goods investors get by renting their capital and labor ( $\pi \bar{A}$ ), plus the fraction of new units of capital transferred to savers ( $\theta_s \delta$ ), plus the fraction of old units of capital transferred to savers ( $\theta_r (1 - \delta) \pi$ ), is not enough to replenish the first best steady state accumulation of capital  $\delta \bar{K}^* = \delta$ . In this case the economy is constrained, the liquidity frictions prevent the flow of resources and the economy has a stationary level of capital that is below the unconstrained first best:  $\bar{K} < 1$ .

Let me address now the issue of existence of a monetary equilibrium with government purchasing capital. That is, I want to assess if a fixed point for the system (14) exist. To start with I first assume that condition (19) is satisfied and also that  $\bar{\mu} > 0$ . Once this equilibrium is found, then we will analyze conditions on the parameters such that money is indeed valued and the effects of government capital in the economy.

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<sup>15</sup>The minimum of the quadratic function  $f(\bar{q}) = a\bar{q}^2 + b\bar{q} + c$  is  $\bar{q} = -\frac{b}{2a}$ , hence the function intercept the "q axis" on the positive side.

The system (14) at steady state, is simple enough to be solved recursively and find closed form solutions. From (14c) the price of capital is found as:

$$\bar{q} = \frac{\bar{\gamma} - \beta(1 - \pi)}{\beta\pi + \theta_s(\bar{\gamma} - \beta)}, \quad (20a)$$

where it was used the fact that the steady state inflation rate equals the rate of money growth:  $\bar{\gamma} = \mu/\mu'$ .

The effective price for capital that investors face is found by using its definition and replacing the steady state value of the price of capital:

$$\bar{\psi} = \frac{1 - \bar{q}\theta_s}{1 - \theta_s} = \frac{\beta\pi}{\beta\pi + \theta_s(\bar{\gamma} - \beta)}. \quad (20b)$$

The value of the effective post production capital per unit of capital is:

$$\bar{\eta} = 1 - \bar{\theta}_r + \bar{q}\bar{\theta}_r \frac{1 - \theta_s}{1 - \bar{q}\theta_s} = \frac{\beta\pi + \bar{\theta}_r(\bar{\gamma} - \beta)}{\beta\pi} \quad (20c)$$

From (14b) the rental rate of capital is found as:

$$\bar{r} = \frac{[1 - (1 - \delta)(\beta + \bar{\theta}_r(\bar{\gamma} - \beta))][\bar{\gamma} - \beta(1 - \pi)]}{[\beta\pi + \theta_s(\bar{\gamma} - \beta)]\bar{\gamma}} \quad (20d)$$

Then (14i) give the equilibrium value of capital:

$$\bar{K} = \left( \frac{[\beta\pi + \theta_s(\bar{\gamma} - \beta)]\bar{\gamma}}{[1 - (1 - \delta)(\beta + \bar{\theta}_r(\bar{\gamma} - \beta))][\bar{\gamma} - \beta(1 - \pi)]} \bar{A}\alpha \right)^{\frac{1}{1-\alpha}} \quad (20e)$$

Aggregate private investor's capital (14d) as a fraction of steady state capital is:

$$\frac{N_1}{\bar{K}} = \frac{\pi}{\bar{\psi}} \left( \frac{\bar{w}}{\bar{N}} + \bar{r} \right) \frac{\bar{N}}{\bar{K}} + \frac{\pi}{\bar{\psi}} \frac{\bar{T} + \mu M}{\bar{K}} + \eta(1 - \delta)\pi \frac{\bar{N}}{\bar{K}}. \quad (20f)$$

Similarly, aggregate private saver's capital (14e) as a fraction of steady state capital is:

$$\frac{N_0}{\bar{K}} = \theta_s \frac{\bar{I}}{\bar{K}} + (\bar{\theta}_r \pi + 1 - \pi)(1 - \delta) \frac{\bar{N}}{\bar{K}}$$

Denoting by  $\lambda = \frac{\bar{H}}{\bar{K}}$ , the fraction of government capital to total capital, (14f) and (14g) at steady state imply the equation:

$$1 = \frac{\bar{r}}{\bar{\psi}} \frac{1 - \alpha \lambda}{\alpha} \pi + \bar{\eta}(1 - \delta) \pi (1 - \lambda) + \frac{\pi}{\bar{\psi}} \left[ \frac{\bar{T}}{\bar{K}} + \frac{\bar{m}}{\bar{\gamma} \bar{K}} \right] + \bar{\eta}(1 - \delta) \pi (1 - \lambda) + \theta_s \delta + (\bar{\theta}_r \pi + 1 - \pi)(1 - \delta)(1 - \lambda) + \lambda$$

where it has been defined  $\bar{m} = \mu M'$  as the steady state real balances<sup>16</sup>. Government budget at steady state, equation (14h) is:

$$\frac{\bar{T}}{\bar{K}} + \frac{\bar{m}}{\bar{\gamma} \bar{K}} = (\bar{r} - \delta \bar{q}) \lambda + \frac{\bar{m}}{\bar{K}}$$

Replacing this into the previous equation we have:

$$1 = \frac{\bar{r}}{\bar{\psi}} \frac{1 - \alpha \lambda}{\alpha} \pi + \bar{\eta}(1 - \delta) \pi (1 - \lambda) + \pi \left( \frac{\bar{r} - \delta \bar{q}}{\bar{\psi}} \lambda \right) + \pi \frac{\bar{m}}{\bar{\psi} \bar{K}} + \theta_s \delta + (\bar{\theta}_r \pi + 1 - \pi)(1 - \delta)(1 - \lambda) + \lambda \quad (20g)$$

which determines the equilibrium value of real balances.

## The price of money

I now turn to the question of positiveness of the price of money. To clarify the role of money, it is first assumed that  $\lambda = \bar{T} = 0$ , and that there is a fixed stock of money in the economy,  $\bar{\gamma} = 1$ .

It was shown that under condition (17) the equilibrium value of  $\bar{q}$  is one, for this it was assumed that the price of money was zero. I now show that this condition suffices for this to be the

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<sup>16</sup>Here is one instance where the elimination of the  $j$  superscript for individual  $j$  might create confusion. I omit the subscript  $j$  before to denote individual *nominal* balances as  $m$ . Here  $\bar{m}$  refers to aggregate real balances.

case. Furthermore I show that there exist a combination of parameters that do not satisfy this condition and for which the economy still displays an equilibrium value for the price of money of zero. In particular, I show that when the values of the parameters  $\theta_s$  and  $\bar{\theta}_r$  satisfy a more stringent version of equation (19), then the price of money is strictly positive.

**Proposition 5.** *Assume that  $\lambda = 0 = \bar{T}$  and  $\bar{\gamma} = 1$ . When parameters satisfy:*

$$\delta > \frac{1 - (1 - \delta)[\beta + \bar{\theta}_r(1 - \beta)]}{1 - (1 - \delta)\beta} [1 - \beta(1 - \pi)]\bar{A} + \theta_s\delta + \bar{\theta}_r(1 - \delta)\pi + \bar{\theta}_r \frac{1 - \beta}{\beta} (1 - \delta) \quad (21)$$

then  $\bar{\mu} > 0$ .

*Proof.* In order to avoid repetitions of some steps of this proof, I consider the case  $\bar{\gamma} \neq 1$ . Of course the proof will hold also for the special case analyzed in the proposition, which is an economy with a constant stock of money. Equation (20g) determines the equilibrium value of real balances, which will be strictly positive when the price of money is strictly positive. This equation can be written as:

$$\delta = \frac{\bar{r}}{\psi} \frac{\pi}{\alpha} + (\bar{\eta} - 1)(1 - \delta)\pi + \theta_s\delta + \pi\bar{\theta}_r(1 - \delta) + \pi \frac{\bar{m}}{\psi\bar{K}}$$

For the price of money (and hence real balances) to be positive, it is required then (after replacing (20d),(20b) and (20c)):

$$\delta > \frac{1 - (1 - \delta)[\beta + \bar{\theta}_r(\bar{\gamma} - \beta)]}{1 - (1 - \delta)\beta} \frac{[\bar{\gamma} - \beta(1 - \pi)]}{\bar{\gamma}} \bar{A} + \theta_s\delta + \bar{\theta}_r(1 - \delta)\pi + \bar{\theta}_r \frac{\bar{\gamma} - \beta}{\beta} (1 - \delta)$$

from which the proof follows after replacing  $\bar{\gamma} = 1$ . □

Next, it is showed that the liquidity constraint parameters must be smaller (a more constrained economy) than the ones that imply a steady state value of the price of capital above unity in

order for money to be valued.

**Proposition 6.** *Condition (21) is more stringent than condition (19) in the sense that the set of values of  $\theta_s$  and  $\bar{\theta}_r$  that satisfy  $\bar{\mu} > 0$  is strictly smaller than the one for which  $\bar{q} > 1$ .*

*Proof.* Let us take the limit of both inequalities and analyze them in the plane  $\bar{\theta}_r, \theta_s$ . First the line describing the limit of equation (19) is:

$$\delta = \pi\bar{A} + \theta_s\delta + \bar{\theta}_r(1 - \delta)\pi$$

The intercepts of this equation are:

$$\hat{\theta}_s = \frac{\delta - \pi\bar{A}}{\delta}, \quad \hat{\theta}_r = \frac{\delta - \pi\bar{A}}{\pi(1 - \delta)}$$

when  $\bar{\theta}_r$  is zero and when  $\theta_s$  is zero, respectively. The line describing the limit of equation (21) is:

$$\delta = \frac{1 - (1 - \delta)[\beta + \bar{\theta}_r(\bar{\gamma} - \beta)]}{1 - (1 - \delta)\beta} \frac{[\bar{\gamma} - \beta(1 - \pi)]}{\bar{\gamma}} \bar{A} + \theta_s\delta + \bar{\theta}_r(1 - \delta)\pi + \bar{\theta}_r \frac{\bar{\gamma} - \beta}{\beta} (1 - \delta)$$

The intercepts of this equation are:

$$\tilde{\theta}_s = \frac{\delta - [\bar{\gamma} - \beta(1 - \pi)]\bar{\gamma}^{-1}\bar{A}}{\delta}, \quad \tilde{\theta}_r = \frac{\delta - [\bar{\gamma} - \beta(1 - \pi)]\bar{\gamma}^{-1}\bar{A}}{\pi(1 - \delta) + \frac{\bar{\gamma} - \beta}{\beta}(1 - \delta) - \frac{(1 - \delta)(\bar{\gamma} - \beta)}{\alpha\beta}[\bar{\gamma} - \beta(1 - \pi)]\bar{\gamma}^{-1}}$$

when  $\bar{\theta}_r$  is zero and when  $\theta_s$  is zero respectively. Therefore we need to show that  $\hat{\theta}_s > \tilde{\theta}_s$  and  $\hat{\theta}_r > \tilde{\theta}_r$ . The first of these inequalities is easily shown as  $[\bar{\gamma} - \beta(1 - \pi)]\bar{\gamma}^{-1} > \pi$ , since this is implied by  $(\bar{\gamma} - \beta)(1 - \pi) > 0$ <sup>17</sup>. For the second, first we write the sum of the second and third

---

<sup>17</sup>As I will show shortly, the smallest value that  $\bar{\gamma}$  can take is  $\beta$ ; in this case,  $\bar{q} = 1$  and  $\bar{\mu} = 0$ . The inequality holds true for the present case with  $\bar{\gamma} = 1$ .

term in the denominator of  $\tilde{\theta}_r$  as:  $\frac{(1-\beta)(1-\delta)}{\beta} \left[ 1 - \frac{1-\beta(1-\pi)}{\alpha} \right]$ . Hence by showing that  $\frac{1-\beta(1-\pi)}{\alpha}$  is strictly less than one, then the result follow. To do this note that as long as the numerator of  $\tilde{\theta}_r$  is positive (and by definition of  $\bar{A}$ ) we have:

$$\frac{\delta\beta}{1 - (1 - \delta)\beta} > \frac{[\bar{\gamma} - \beta(1 - \pi)]\bar{\gamma}^{-1}}{\alpha}$$

the left hand side of this inequality is strictly less than one and, hence, the result follows.  $\square$

Figure 2 illustrates the combinations of parameters such that the economy attains a first best equilibrium and when money is valued.

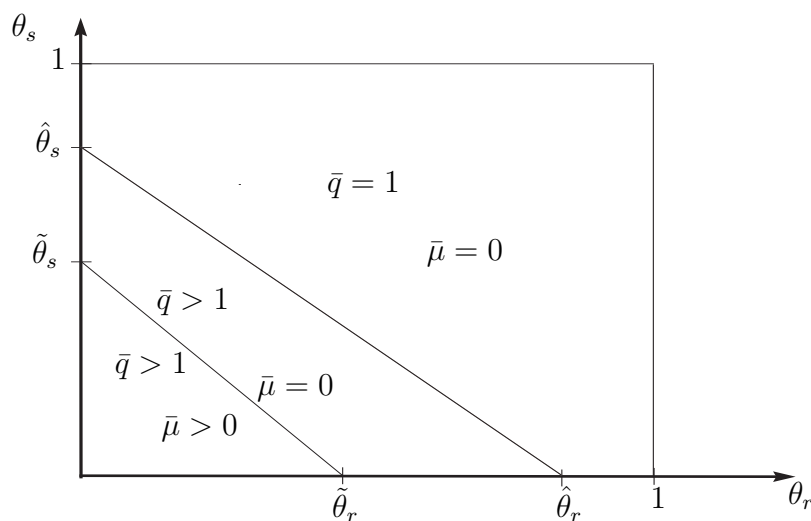


Figure 2: Regions of Efficiency

In the figure that considers a fixed stock of money, we can see that depending on a given combination of  $(\bar{\theta}_r, \theta_s)$ , there exist different regimes in the economy. For a relatively unconstrained economy, the first best is achieved with  $\bar{q} = 1$  and  $\bar{\mu} = 0$ . In this region money is not valued and heterogeneity is immaterial. Then, there is a region where there exist a non-monetary,

suboptimal equilibrium, where  $\bar{q} > 1$  and  $\bar{\mu} = 0$ . Finally when the parameters of liquidity are small enough, there exist a suboptimal monetary equilibrium.

The non-monetary suboptimal equilibrium region is interesting because money is not valued in spite of the economy being not Pareto optimal. The money stock has been introduced proportionately to all agents in the economy. As we know, investors supply inelastically all money they have. Hence the demand of money (created by savers) might fall short of the supply even at  $\bar{\mu} = 0$ .

The following numerical exercise has been performed to add to the insights on the steady state of the model. I set the following parameter values:  $\beta = 0.99$ ,  $\alpha = 0.36$  and  $\pi = 0.1$  (this parameter implying a 10% of investors in the economy), and plot how the price of capital, the effective price of capital for investors and capital vary with the parameters  $\theta_s$  and  $\bar{\theta}_r$ . Two exercises have been conducted the first is to compute the economy without money (i.e. there is no supply of it by the monetary authority). The second exercise is to introduce a stock of money, and to compute the steady state as before. In this second case we know that money may or may not be valued. Figures 3 through 7 show the results. Figure 3 shows the equilibrium price of

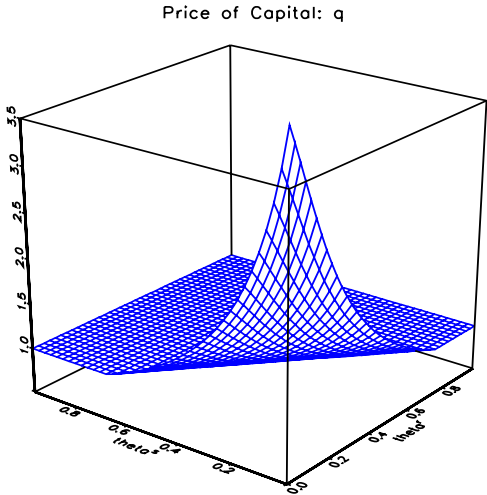


Figure 3: Steady state value of  $\bar{q}$ , no money.

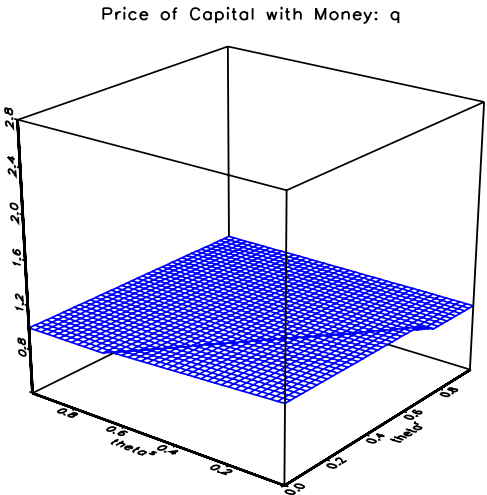


Figure 4: Steady state value of  $\bar{q}$ , with money.

capital  $\bar{q}$  as a function of the two liquidity parameters  $\theta_s$  and  $\theta_r$ . It is evident that for most of the combinations of both parameters the economy attains a first best outcome. In order to find  $\bar{q} > 1$  the economy must be relatively highly constrained. In this case it turns out that both parameters interact to rise the price of capital above unity. For relatively low levels of the parameter  $\theta_s$ ,  $\theta_r$  must be small as well to have  $\bar{q} > 1$ . We can see that once the economy is in the constrained region, further decreases of the re-selling constraint have a bigger impact on  $\bar{q}$ .

Figure 4 shows comparable computations for the case when money is added to the economy. In the region where money is valued, money function well to prevent the price of capital to rise too much above unity. Note that in the economy there exists only a fixed amount of money. When this kind of paper is valued, more transactions of capital in the economy are possible, and hence the price of capital does not rise as much. Figure 5 shows the equilibrium level of capital

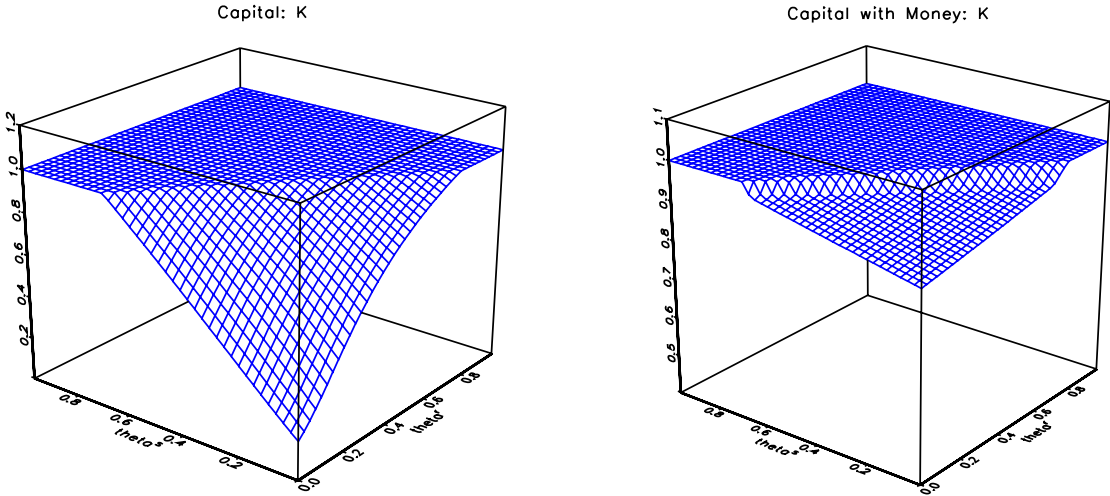


Figure 5: Steady state value of  $\bar{K}$ , no money.      Figure 6: Steady state value of  $\bar{K}$ , with money.

as a function of the liquidity parameters. This figure mirrors the price of capital. The first best is attained for most combinations of the liquidity parameters. Again it can be seen that once the economy is in the constrained region, further decreases in the liquidity parameters are detrimental for the economy. Figure 6 shows that when money is introduced the negative effects

of the liquidity constraints are hampered. Again we can see that money functions extremely well to prevent the economy to attain undesirable outcomes. Also, if the liquidity parameters are such that money is not valued in the economy, then the effects are of course as detrimental as when no money exists in the economy. Figure 7 shows the value of money (I have normalized

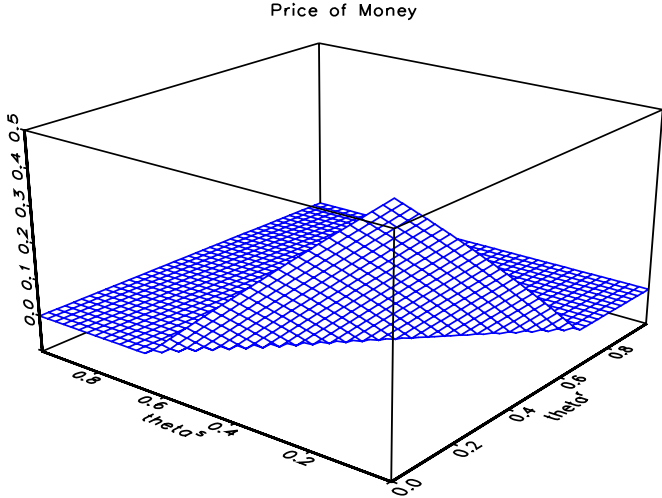


Figure 7: Steady state price of money  $\bar{\mu}$ .

the fixed stock of money to one:  $M = 1$ ). Here it can be seen that the economy has to be relatively constrained in order for money to be valued.

So far money has been introduced as a single stock given to agents at the start of life in equal amounts. I have shown that in the steady state, and only under specific conditions, this stock of money is valued. Note that a necessary (but not sufficient) condition for money to be valued is that the economy is liquidity-constrained in the sense that  $\bar{q} > 1$ . It is under these circumstances that the liquidity parameters prevent the flow of capital between investors and savers. Since money is valued, investors can use it to buy goods and use them in the investment technology to produce more capital, this offsetting some of the detrimental effects of the liquidity constraints. Savers are willing to receive the intrinsically useless pieces of paper because there is a chance that next period they will switch status so they can use money to transact with the savers of

that period.

### Some properties of the monetary model

When money is valued in the steady state of the economy, the model displays some properties that I now list. First I spell out some properties regarding the price of capital. The analysis of the previous section made clear that this price is key to understand whether the economy is liquidity constrained as well as how constrained it is. By simple differentiation of the equilibrium value for the price of capital (20a), we have the following properties:

$$\partial \bar{q} / \partial \pi < 0, \quad \partial \bar{q} / \partial \beta > 0$$

With a higher fraction of investors, the equilibrium price of capital falls as there is more supply of capital in the economy. Also the higher the discount factor, the higher the price of capital:  $\partial \bar{q} / \partial \beta > 0$ . More impatient agents will demand less capital and hence the equilibrium value of it would fall. Note also that as  $\theta_s$  decreases the equilibrium price of capital is higher. Interestingly once the economy is in the monetary suboptimal equilibrium, the equilibrium value of the price of capital does not depend on the reselling parameter  $\bar{\theta}_r$ .

Let me spell out the asset pricing equations for this economy in terms of the return on money and the return on capital. Defining as  $\mathcal{M}$  as the expected return on money,  $\mathcal{R}_0$  as the expected return on capital for the saver and  $\mathcal{R}_1$  as the expected return on capital for the investor:

$$\begin{aligned} \mathcal{M} &= \frac{\bar{q}}{\bar{\psi}} \pi + (1 - \pi) \\ \mathcal{R}_0 &= \left[ \frac{\bar{r}}{\bar{\psi}} + \bar{\eta}(1 - \delta) \right] \pi + \frac{\bar{r} + \bar{q}(1 - \delta)}{\bar{q}} (1 - \pi) \\ \mathcal{R}_1 &= \left[ \frac{\bar{r}}{\bar{\psi}} + \bar{\eta}(1 - \delta) \right] \frac{\bar{q}}{\bar{\psi}} \pi + \frac{\bar{r} + \bar{q}(1 - \delta)}{\bar{\psi}} (1 - \pi) \end{aligned}$$

We have by direct computation from (14a), (14b) and (14c) at steady state:

$$\frac{1}{\beta} = \mathcal{M} = \mathcal{R}_0 < \mathcal{R}_1$$

These relationships show that savers face a return on money that is equal to the return on capital. This shows the indifference they have on using any of those assets as a means of saving. Also the return on these assets equals  $1/\beta$ . Given their linear preferences this also mean that they are indifferent between consuming now or consuming later. Investors have an effective rate of return that is higher than the return on capital of money and higher than  $1/\beta$ . Given their linear utility , this implies that they are at a corner with no money holdings and no consumption.

### **Introducing government capital purchases**

In this subsection I analyze the steady state effect of introducing purchases of government capital. For this reason we allow transfers to be different than zero. Recall that in previous sections we have investigated under what conditions a given stock of money would be valued in the economy. It has been found that when money is valued, the performance of the economy is better than otherwise and money is valued only when liquidity constraints are tight enough. It has also been found that there exist a combination of liquidity parameters such that the economy display a sub-optimal equilibrium, yet the stock of money is not valued in the economy. I want to find out what features of such a result are modified once positive government capital purchases are introduced. It is found that once capital purchases are introduced, then the region of non-monetary, suboptimal equilibria increases. It is also shown that in the monetary region of the economy, the effects of capital purchases are null.

**Proposition 7.** *Assume that  $\lambda \neq 0$ ,  $\bar{T} \neq 0$  and  $\bar{\gamma} = 1$ . Recall that  $\lambda$  is the steady state*

government capital as a fraction of total capital. When parameters satisfy:

$$\begin{aligned} \delta > \frac{1 - (1 - \delta)[\beta + \bar{\theta}_r(1 - \beta)]}{1 - (1 - \delta)\beta} [1 - \beta(1 - \pi)] \bar{A}(1 - \alpha\lambda) + \theta_s \delta + \bar{\theta}_r(1 - \delta)(1 - \lambda)\pi \\ + \bar{\theta}_r \frac{1 - \beta}{\beta} (1 - \delta)(1 - \lambda) + [1 - (1 - \delta)(\beta + \bar{\theta}_r(1 - \beta)) - \delta] \frac{1 - \beta(1 - \pi)}{\beta} \lambda + \delta\lambda \end{aligned} \quad (22)$$

then  $\bar{\mu} > 0$ .

*Proof.* Equation (20g) determines the equilibrium value of real balances, which will be strictly positive when the price of money is strictly positive. This equation can be written as:

$$\delta = \frac{\bar{r}}{\bar{\psi}} \frac{\pi}{\alpha} + (\bar{\eta} - 1)(1 - \delta)(1 - \lambda)\pi + \pi \frac{\bar{r} - \delta\bar{q}}{\bar{\psi}} \lambda + \theta_s \delta + \pi \bar{\theta}_r(1 - \delta)(1 - \lambda) + \delta\lambda + \pi \frac{\bar{m}}{\bar{\psi}\bar{K}}$$

For the price of money (and hence real balances) to be positive, it is required then (after replacing (20d),(20b) and (20c)):

$$\begin{aligned} \delta > \frac{1 - (1 - \delta)[\beta + \bar{\theta}_r(\bar{\gamma} - \beta)]}{1 - (1 - \delta)\beta} \frac{[\bar{\gamma} - \beta(1 - \pi)]}{\bar{\gamma}} \bar{A}(1 - \alpha\lambda) + \theta_s \delta + \bar{\theta}_r(1 - \delta)(1 - \lambda)\pi \\ + \bar{\theta}_r \frac{\bar{\gamma} - \beta}{\beta} (1 - \delta)(1 - \lambda) + [1 - (1 - \delta)(\beta + \bar{\theta}_r(\bar{\gamma} - \beta)) - \delta\bar{\gamma}] \frac{\bar{\gamma} - \beta(1 - \pi)}{\bar{\gamma}\beta} \lambda + \delta\lambda \end{aligned} \quad (23)$$

from which the proof follows after replacing  $\bar{\gamma} = 1$ .  $\square$

Next, it is showed that the liquidity constraint parameters must be even lower (a higher constrained economy) than the ones that implied a price of capital above unity and positive price of money in the economy with  $\lambda = \bar{T} = 0$ , in order for money to be valued.

**Proposition 8.** *Assume that  $\lambda \neq 0$ ,  $\bar{T} \neq 0$  and  $\bar{\gamma} = 1$ . Then the set of values of  $\theta_s$  and  $\bar{\theta}_r$  that satisfy  $\bar{\mu} > 0$ ,  $\bar{q} > 1$  is strictly smaller than those that satisfy the same conditions when  $\lambda = \bar{T} = 0$  and  $\bar{\gamma} = 1$ .*

*Proof.* As before, let's start by taking the limit of the equation above (23). This equation would describe the limit values of both  $\theta_s$  and  $\bar{\theta}_r$  such that the economy has a positive price of money. The intercepts in the plane  $(\bar{\theta}_r, \theta_s)$  are:

$$\check{\theta}_s = \frac{\delta - [\bar{\gamma} - \beta(1 - \pi)]\bar{\gamma}^{-1}(1 - \alpha\lambda)\bar{A} - [1 - (1 - \delta)\beta - \delta\bar{\gamma}][\bar{\gamma} - \beta(1 - \pi)](\bar{\gamma}\beta)^{-1}\lambda - \delta\lambda}{\delta} \quad (24)$$

$$\check{\theta}_r = \frac{\delta - [\bar{\gamma} - \beta(1 - \pi)]\bar{\gamma}^{-1}\bar{A}\frac{1 - \alpha\lambda}{1 - \lambda} - [1 - (1 - \delta)\beta - \delta\bar{\gamma}][\bar{\gamma} - \beta(1 - \pi)](\bar{\gamma}\beta)^{-1}\frac{\lambda}{1 - \lambda}}{\pi(1 - \delta) + \frac{\bar{\gamma} - \beta}{\beta}(1 - \delta) - \frac{(1 - \delta)(\bar{\gamma} - \beta)}{\alpha\beta} \frac{\bar{\gamma} - \beta(1 - \pi)}{\bar{\gamma}} \frac{1 - \alpha\lambda}{1 - \lambda} - (1 - \delta)(\bar{\gamma} - \beta)\frac{[\bar{\gamma} - \beta(1 - \pi)]}{\bar{\gamma}\beta} \frac{\lambda}{1 - \lambda}} \quad (25)$$

when  $\bar{\theta}_r$  and  $\theta_s$  are zero respectively. Even though these values seem rather complicated the proof is simple. First compare the numerator of  $\check{\theta}_s$  with the numerator of  $\check{\theta}_r$ . It is straightforward that the first value is smaller than the second. It remains to show that  $\check{\theta}_r$  is strictly smaller than  $\check{\theta}_r$ . The numerator of  $\check{\theta}_r$  can be written as:

$$\delta - [\bar{\gamma} - \beta(1 - \pi)]\bar{\gamma}^{-1}\bar{A} - [\bar{\gamma} - \beta(1 - \pi)](\bar{\gamma}\beta)^{-1}\frac{\lambda}{1 - \lambda} \left( \frac{1 - (1 - \delta)\beta}{\alpha} - \delta\bar{\gamma} \right)$$

and its denominator as:

$$\pi(1 - \delta) + \frac{\bar{\gamma} - \beta}{\beta}(1 - \delta) - \frac{(1 - \delta)(\bar{\gamma} - \beta)}{\alpha\beta} [\bar{\gamma} - \beta(1 - \pi)]\bar{\gamma}^{-1} - \frac{(1 - \delta)(\bar{\gamma} - \beta)}{\alpha} [\bar{\gamma} - \beta(1 - \pi)](\bar{\gamma}\beta)^{-1}\frac{\lambda}{1 - \lambda}$$

By comparing with  $\check{\theta}_r$ , and given that this value is indeed a fraction, it suffices to show that the absolute value of the last term in the numerator of  $\check{\theta}_r$  is strictly higher than the absolute value of the last term in its denominator. This requires, after a bit of simplification:

$$(1 - \bar{\gamma}) + \delta\bar{\gamma} > \alpha\delta\bar{\gamma}$$

Replacing  $\bar{\gamma} = 1$  it is obtained:  $1 > \alpha$ . □

In the sub-optimal non-monetary region it has been shown that the effects of liquidity constraints

are highly detrimental. Hence this result is relevant in that it suggest that a minimal government intervention in acquiring capital in the market is desirable.

The intuition behind this result is not completely transparent to me. One possible explanation is the following: recall that is savers that demand capital in the economy. Then, when the government demand capital, the saver's holdings of capital must be reduced, but note that the proceeds from government holdings are distributed back to all agents in the economy. Hence this is equivalent to the government taking goods from the savers and giving them to the investors. In this case the demand of money from savers fall and the supply of money from investors increase. The result is a drop in the price of money which might result in an equilibrium with zero such price.

Note that the previous results have been shown for a given stock of money in the economy  $\bar{\gamma} = 1$ . When we consider possibly different money growth rates, the results will vary. In fact the next section shows that simply by setting  $\bar{\gamma} = \beta$  the government can attain the first best allocations, independently of how tight the liquidity constraints are.

### 3.1 Government Policy in the Steady State

In this section I investigate if the government could help the economy to achieve the first best steady state by some sort of monetary policy.

I restrict the analysis to a constant rate of growth of the money supply. We will see that the government could overcome, in the steady state, the negative effects of the liquidity constraints. Since prices are completely flexible, it is analyzed the effects of inflation in the economy. It is shown that the Friedman rule of deflating at rate  $\beta$  is optimal.

By simple examination of equations (20a), (20b), (20c), (20d) and (20e), it is seen that as  $\bar{\gamma} \rightarrow \beta$

then

$$[\bar{q}, \bar{\psi}, \bar{\eta}, \bar{r}, \bar{K}] \rightarrow \left[ 1, 1, 1, \frac{1}{\beta} - (1 - \delta), 1 \right]$$

Then the first best allocations are obtained. The remaining issue to address is to characterize the financing of the deflation by the government. To this end I show the following proposition which states that there exist a range  $\lambda \in [0, \hat{\lambda}]$  where  $\hat{\lambda}$  is strictly less than one, that the government could implement to finance the deflation satisfying its budget constraint.

**Proposition 9.**  *$\bar{\gamma} = \beta$  can be financed with any amount of government capital as a fraction of total capital:  $\lambda$ , belonging in a range  $[0, \hat{\lambda}]$  where  $\hat{\lambda} < 1$ .*

*Proof.* Consider equilibrium real balances in the economy, determined by equation (20g). When  $\bar{\gamma} = \beta$ , this equation states:

$$\delta = \bar{r} \frac{\pi}{\alpha} + \pi(\bar{r} - \delta)\lambda + \theta_s \delta + \pi \bar{\theta}_r (1 - \delta)(1 - \lambda) + \delta \lambda + \pi \bar{m}$$

in order for real balances to be positive it is then required that:

$$\delta > \bar{r} \frac{\pi}{\alpha} + \theta_s \delta + \pi \theta_r (1 - \delta) \left[ \pi \frac{1 - \beta}{\beta} + \delta - \pi \bar{\theta}_r (1 - \delta) \right] \lambda$$

The term in square braces can be shown to be strictly positive by replacing the maximum value that  $\bar{\theta}_r$  can take which is given by  $\hat{\theta}_r$ , and showing that the resulting expression is strictly positive. To show that the maximum possible value of  $\lambda$  is less than one, let's take the limiting value of  $\lambda$  such that  $\bar{m} \geq 0$  in the expression above. This value can be written as:

$$\hat{\lambda} = \frac{\delta - \pi \bar{\theta}_r (1 - \delta) - \left( \frac{\bar{r}}{\alpha} \pi + \theta_s \delta \right)}{\delta - \pi \bar{\theta}_r (1 - \delta) + \pi \frac{1 - \beta}{\beta}}$$

which is evidently less than one. □

The government can decide on what amount of capital to hold in steady state, up to  $\hat{\lambda}$ , its choice will determine the amount of transfers to the economy to satisfy its budget constraint, which under the deflationary policy is given by:

$$\bar{T} = \frac{1 - \beta}{\beta}(\lambda - \bar{m})$$

To take a concrete example, for the parametrization given in table 1, figure 8 shows the combinations of transfers and government capital holdings in steady state such that the Friedman rule is feasible to implement.

Table 1: **Parametrization**

$\beta$	$\delta$	$\theta_s$	$\bar{\theta}_r$	$\pi$
0.99	0.025	0.2	0.2	0.1

The government then have different ways to finance the deflation. It may not purchase any capital, in that case the agents need to be taxed to retire money from the market. The government could buy a positive amount of capital and set taxes to zero. In this case the government would be using the proceeds from the capital holdings to buy back money from the market. Finally, it may hold additional units of capital, and make positive transfers to agents.

The intuition behind the optimality of the Friedman rule is the following: Agents hold money because they expect it to be valued in the future as it will be used as an asset to obtaining units of goods for more capital creation, the lower the rate of money creation, the higher the expected price of it next period and hence the more goods can be exchanged for money to finance more capital creation.

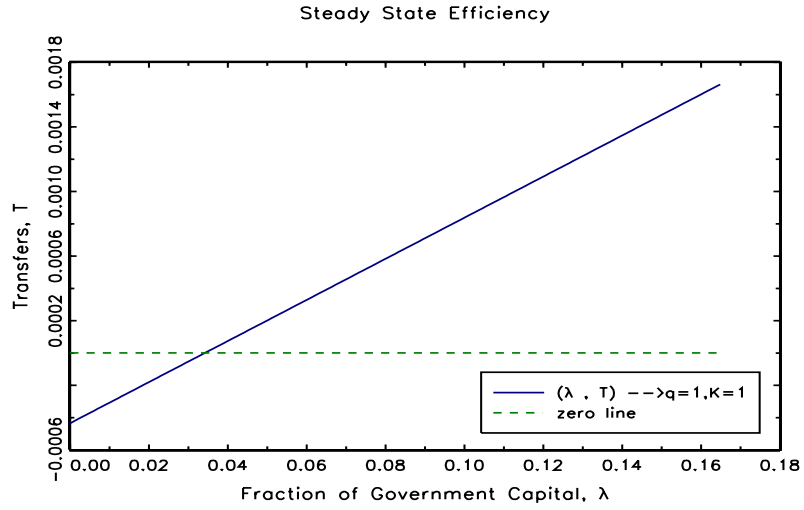


Figure 8: Ways to Implement the Friedman Rule

To illustrate the negative effects of different inflation rates, I compute the steady state value of different variables with different money creation rates (see table 2).

Table 2: **Steady State Values-Different Inflation Rates**

Annual inflation	-4%	0%	10%
$\gamma$	$\beta$	1.0000	1.0240
$q$	1.0000	1.0900	1.2980
$Y$	0.1669	0.1628	0.1566
$C$	0.1169	0.1161	0.1147
$I$	0.0500	0.0467	0.0419
$K$	1.0000	0.9340	0.8384

As it can be seen on table 2, deflating at rate  $\beta$  is optimal and higher inflation rates are detrimental for the economy

## 4 Transitional Dynamics

In the previous section I characterized the solution of the model in the steady state. I also showed that as long as the parameters of the model satisfy condition (19) the economy displays an equilibrium that is suboptimal. I also showed that when there is a fixed stock of money in the economy, the liquidity constraint parameters must be tighter for that stock to be valued. Finally, I analyzed the effects of government purchasing of capital and different inflationary policies. All that analysis were conducted in steady state. It is now time to start analyzing dynamics. In this section I do not yet conduct an (aggregate) stochastic dynamic analysis, hence all aggregate shocks are set to zero:  $\epsilon_A = \epsilon_\gamma = \epsilon_\theta = \epsilon_H = 0$ . Furthermore, I also assume that  $T = H = 0$  and  $\gamma = \bar{\gamma} = 1$ .

I focus on analyzing an economy with an initial stock of capital that is below the steady state and there is no money growth. I assume that equation (19) holds; I have shown that in the steady state this produces an equilibrium where  $\bar{q} > 1$ . Under the conjecture that the time path of the price of capital towards the steady state also presents this feature, I solve the model numerically and verify it ex-post.

I also compute the transition path for a frictionless economy, this serves as a benchmark to analyze the effects of the liquidity constraints on the time path of the variables.

When the economy is not liquidity constrained, e.g. in the case of  $\theta_s = \theta_r = 1$ , then agents allow swings in their consumption because of the linearity of their preferences. The solution in such a case will imply zero consumption for a finite lapse of time, until  $\beta(r + 1 - \delta) = 1$ , at which point consumption will jump to its steady state value. Until that point maximal accumulation of capital occurs. In the liquidity constraint economy, I show that maximal accumulation of capital is not possible. Investors consume zero, in fact not only in the transition but also in

the steady state (as shown before) but the liquidity constraint prevents capital from flowing to savers for maximal accumulation of capital. Given their resources, savers buy only a limited amount of capital, and there is positive consumption in the transition to the steady state.

I employ a guess-and-verify method to find the transition path to the steady state. Under assumption (1), a given stock of money normalized to unity ( $M = 1$ ) and with no government. The system can be reduced to the following equations:

$$\begin{aligned}
q &= \beta \left\{ \left[ \frac{r'}{\psi'} + \eta'(1 - \delta) \right] q' \pi + [r' + q'(1 - \delta)](1 - \pi) \right\} \\
\mu &= \beta \left[ \frac{q'}{\psi'} \mu' \pi + \mu'(1 - \pi) \right] \\
K' &= \frac{1}{\psi} \left[ \frac{r}{\alpha} + \psi \eta(1 - \delta) \right] \pi K + \frac{\pi}{\psi} \mu + \theta_s I + (\bar{\theta}_r \pi + 1 - \pi)(1 - \delta) K \\
K' &= (1 - \delta)K + I
\end{aligned} \tag{26}$$

The system that needs to be solved display interactions between prices and real variables. At this point it is used a numerical method to solve the system. I implement the parametric path method of Judd(2002) to solve the model. First I need to analyze the stability properties near the steady state.

### Analyzing stability conditions near the steady state

Here I start by assessing the type of stability near steady state. This is important not only to analyze the uniqueness of the equilibrium, but also because the asymptotic rate of convergence that is obtained by assessing the type of stability, is needed to apply the parametric path method later on.

Appendix B shows that the system (26) can be reduced to a system in  $[q, K, \mu]$  and the resulting linearized system around the steady state can be written as:

$$\begin{bmatrix} q' - \bar{q} \\ K' - \bar{K} \\ \mu' - \bar{\mu} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & 0 \\ 0 & 1 & 0 \\ v_{31} & 0 & v_{33} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ v_{21} & v_{22} & v_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q - \bar{q} \\ K - \bar{K} \\ \mu - \bar{\mu} \end{bmatrix}$$

where the  $v$ 's are complicated functions of the parameters of the model.

To assess the convergence properties of the system I use values of the parameters presented in table 1, these values imply that in steady state  $\bar{q} > 1, \bar{\mu} > 0$  and  $\bar{K} < 1$ . The eigenvalues associated with the matrix pre-multiplying the vector  $[q - \bar{q}, K - \bar{K}, \mu - \bar{\mu}]'$  are given by  $[1.05, 0.958, 1.005]'$ , therefore the linearized system displays saddle path stability. At this point I can derive the time path of the variables of the linearized system. Even though this is straightforward given the computations above, I opt to use the parametric path method presented in Judd (2002), because it respects the nonlinearities of the problem.

### Implementing the Parametric Path method to solve the model

Here I show a sketch of the idea of the method. For further details the reader is referred to Judd(2002). A solution to system (26), involves approximating the time path of  $q$  and  $K$  as follows:

$$\begin{aligned} q(t) &= \left( \sum_{j=0}^m a_j^q \phi_j(t) \right) e^{-\lambda t} + \bar{q}(1 - e^{-\lambda t}) \\ K(t) &= \left( \sum_{j=0}^m a_j^K \phi_j(t) \right) e^{-\lambda t} + \bar{K}(1 - e^{-\lambda t}) \\ \mu(t) &= \left( \sum_{j=0}^m a_j^\mu \phi_j(t) \right) e^{-\lambda t} + \bar{\mu}(1 - e^{-\lambda t}) \end{aligned}$$

where  $\phi_j(t)$  represent a family of orthogonal polynomials. In particular, I use Laguerre polynomials.  $\lambda$  measures the asymptotic rate of convergence of the linearized system, which is related

to the dominant stable eigenvalue of the system which is computed to be 0.95191916. The parameters  $a_j^q$ ,  $a_j^K$  and  $a_j^m$  are found by using projection conditions. The initial level of capital is:  $K_0 = a_0^K + \sum_{j=1}^m a_j^K \phi_j(0)$  which implies that  $K(t) = a_0^K e^{\lambda t} + \left( \sum_{j=1}^m a_j^K \phi_j(t) \right) e^{-\lambda t} + \bar{K}(1 - e^{-\lambda t})$ . The value of price of capital and price of money adjust from the beginning to equilibrate the system. To find the projection conditions we define the residual functions as the optimality conditions of the system (26), as follows:

$$\begin{aligned}
R_1(t, a) &= \beta \left\{ \left[ \frac{r(t+1)}{\psi(t+1)} + \eta(t+1)(1-\delta) \right] q(t+1)\pi + [r(t+1) + q(t+1)(1-\delta)](1-\pi) \right\} \\
&\quad - q(t) \\
R_2(t, a) &= \beta \left[ \frac{q(t+1)}{\psi(t+1)} \mu(t+1)\pi + \mu(t+1)(1-\pi) \right] - \mu(t) \\
R_3(t, a) &= \frac{1}{\psi(t)} \left[ \frac{r(t)}{\alpha} + \psi(t)\eta(t)(1-\delta) \right] \pi K(t) + \frac{\pi}{\psi(t)} \mu(t) + [1 - \theta_s - \pi(1 - \bar{\theta}_r)](1-\delta)K(t) \\
&\quad - (1 - \theta_s)K(t+1)
\end{aligned} \tag{27}$$

Following Judd(2002) I use a change of variable  $t(x) = L \frac{1+x}{1-x}$  and state the projection conditions:

$$\begin{aligned}
\langle R_s(t, a), P_j(t) \rangle_{j=0, \dots, m} &= \int_0^\infty R_s(t, a) \phi_j(t) dt = \int_{-1}^1 R_s(t(x), a) \phi_j(t(x)) t'(x) dx \\
&\doteq \sum_{\ell=1}^n R_s(t(x_\ell), a) \phi_j(t(x_\ell)) (1 - x_\ell^2)^{1/2} t'(x_\ell)
\end{aligned}$$

for  $s = 1, 2, 3$  and  $x_\ell = \cos\left(\frac{2\ell+1}{2k}\pi\right)$ ,  $\ell = 1, \dots, k$ .

In the graphs below I show the results of implementing the method. I also plot the values for an unconstrained economy (dotted lines, values denoted also with an asterisk). Note that the linear utility implies that there are "bang-bang" solutions in the unconstrained case; for example consumption is zero for a lapse of time after which it jumps to its steady state value.

Interestingly, liquidity constraints do not allow for such strong swings in the variables as can be seen in the graph. Figure 9 displays the time path of the price of capital. It can be seen

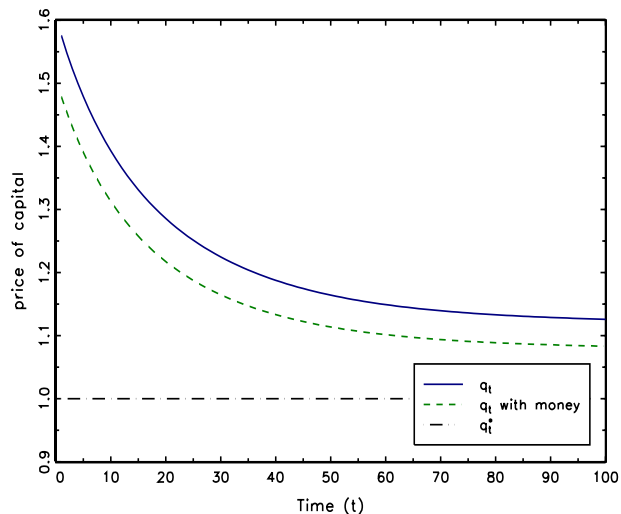


Figure 9: Time path for  $q$

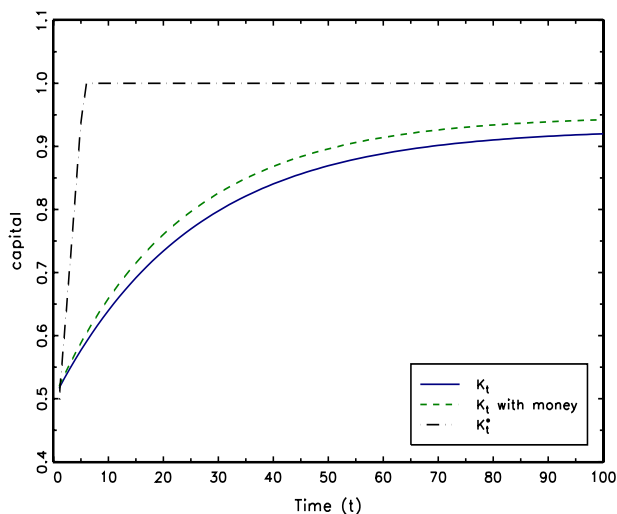


Figure 10: Time path for  $K$

that  $q > 1$  in the transition as well as in the steady state, confirming the initial guess that the liquidity constraint of the investor is always binding. The figure also shows the time path of the price of capital for the unconstrained economy; the value is always equal to one. When money is introduced in the economy, then the time path of the price of capital is lower at all points in time.

Figure 10 shows the time path of capital. Capital accumulation is monotonic. I have set the initial level of capital equal to 0.5. Note that the unconstrained economy also displays a monotonic accumulation of capital. In this case capital accumulates fast and then it remains equal to its steady state value of one. The steady state is reached in finite time. In the constrained economy the steady state is below unity and it is only reached asymptotically. When money is introduced, capital is all the time closer to the first best path. In Figure 11 it is displayed the time path of consumption. Note that in the unconstrained economy, consumption is initially zero and then it jumps to its steady state value. In the constrained economy such

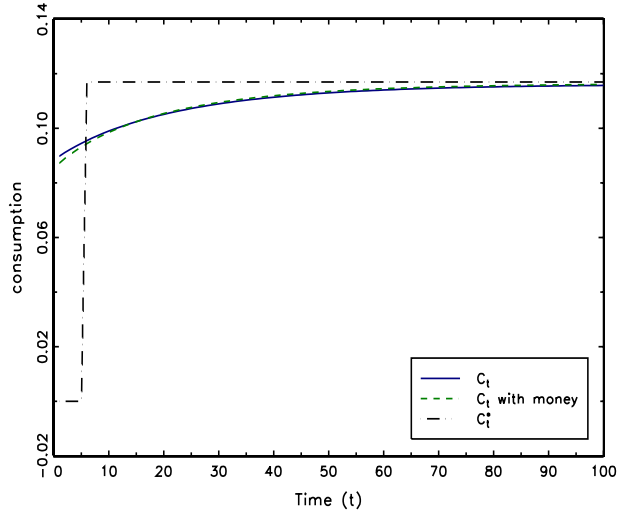


Figure 11: Time path for  $C$ , consumption

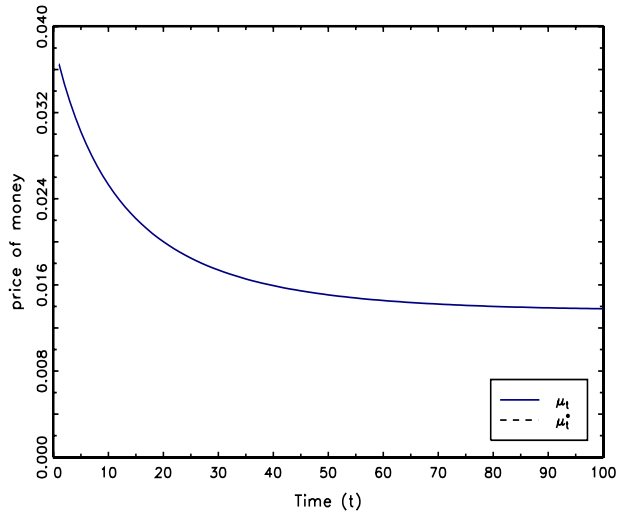


Figure 12: Time path for the price of money  $\mu$

oscillations are not present. Consumption is always positive and it is monotone increasing, reaching its steady state value asymptotically. Liquidity constraints prevent savers from buying more capital, and hence they consume all the time a positive amount. In figure 12, the time path of the price of money is presented. Note that when the economy starts far from the steady state, the value of money is higher. Finally, Figure 4 present the time path of investment. We

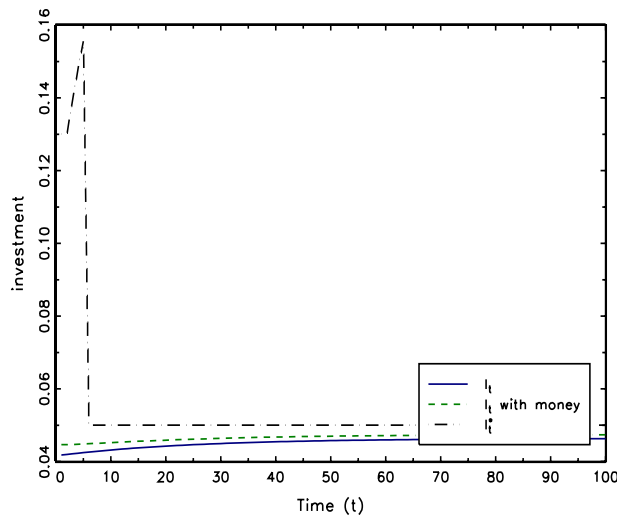


Figure 13: Time path of Investment  $I$

can see that the constrained economy presents a high level of investment initially, decreasing as time passes and reaching its steady state value that is below the one of the unconstrained economy, since the stock of capital to replenish is lower. When money is introduced, again the level of investment is all the time higher.

From this section we can learn that even when steady state effects of liquidity frictions are small, the transitional dynamic effects of such frictions can be sizeable. Notwithstanding, this affirmation needs to be qualified because it is required relatively high deviations from the steady state capital stock in order to have sizeable effects. We also learn some properties of the dynamic system. For example the fact that  $q$  is an indicator of how tight the liquidity constraints are. In the transition there exist more demand of capital than in the steady state, hence the liquidity friction is more relevant in that phase, the price of capital needs to be higher to equilibrate the market for capital. The next section analyzes the general model where all aggregate shocks are introduced.

## 5 Aggregate Stochastic Analysis

In this section, I return again to the general model outlined in the first section. The system (14) can be written as follows:

$$q = \beta E \left\{ \left[ \frac{r'}{\psi'} + \eta'(1 - \delta) \right] q' \pi + [r' + q'(1 - \delta)](1 - \pi) \right\}$$

$$1 = \beta E \left[ \frac{q' \pi}{\psi' \chi'} + \frac{1 - \pi}{\chi'} \right]$$

$$N'_1 = \frac{1}{\psi} \left\{ \frac{w}{N} + r + \psi \eta(1 - \delta) \right\} \pi N + \frac{\pi}{\psi \chi} m + \frac{\pi T}{\psi}$$

$$N'_0 = \theta^s I + [\theta_r(1 - \delta)\pi + (1 - \delta)(1 - \pi)]N$$

$$K' = N'_1 + N'_0 + H'$$

$$K' = (1 - \delta)K + I$$

$$T + q[H' - (1 - \delta)H] = rH + m' \left(1 - \frac{1}{\gamma'}\right), \quad m' = \frac{\gamma'}{\chi}m$$

$$r = \alpha AK^{1-\alpha}, \quad w = (1 - \alpha)AK^\alpha$$

$$\ln A' - \ln \bar{A} = \rho_A(\ln A - \ln \bar{A}) + \epsilon_A \quad E[\epsilon_A] = 0$$

$$\ln \gamma' - \ln \bar{\gamma} = \rho_\gamma(\ln \gamma - \ln \bar{\gamma}) + \epsilon_\gamma \quad E[\epsilon_\gamma] = 0$$

$$\ln \theta'_r - \ln \bar{\theta}_r = \rho_\theta(\ln \theta_r - \ln \bar{\theta}_r) + \epsilon_\theta \quad E[\epsilon_\theta] = 0$$

$$\ln H' - \ln \bar{H} = \rho_H(\ln H - \ln \bar{H}) + \epsilon_H \quad E[\epsilon_H] = 0$$

It has been defined the inflation rate as  $\chi = \mu/\mu'$  and real balances as  $m = \mu M'$ .

It is not the purpose of this section to analyze thoroughly the time series properties of the model. The linearity of preferences assumption plus possibly some other features of the model make it hard to fully evaluate its properties by comparison with time series properties of the data. Rather, the objective here is to assess if this model have certain properties that show promise that the mechanisms incorporated could drive the results of more general and empirically relevant versions. To this end, I restrict the analysis to an impulse response computation and evaluation. I log-linearized the system around a deterministic sub-optimal monetary steady state. The following table shows relevant information on some of the parameters and steady states used for the approximation: Denoting by  $x$  the vector of state variables,  $y$  the vector of "jump" variables

Table 3: **Steady State Values for Approximation**

$\bar{\gamma}$	$\bar{\theta}_r = \theta_s$	$\lambda$	$\bar{K}$	$\bar{q}$
1	0.2	0.05	0.9487	1.08

and  $z$  the vector of stochastic shocks, we have:

$$x = \begin{bmatrix} \hat{K} \\ \hat{m} \end{bmatrix}, \quad y = \begin{bmatrix} \hat{I} \\ \hat{T} \\ \hat{q} \\ \hat{\chi} \\ \hat{\psi} \\ \hat{\eta} \\ \hat{r} \end{bmatrix}, \quad z = \begin{bmatrix} \hat{A} \\ \hat{\gamma} \\ \hat{\theta}_r \\ \hat{H} \end{bmatrix}$$

The variables with hat denote the variables as log deviations from the steady state. The system has been solved applying the method of Uhlig (1999), where four matrices  $P, Q, R$  and  $S$  are found such that the solution can be represented as:

$$x' = Px + Qz$$

$$y = Rx + Sz$$

At this point I turn to compute responses to the different shocks in the economy.

### Impulse Responses

The first set of figures show the responses of a one percent change relative to steady state in the productivity parameter  $A$ . It is set  $\rho_A = 0.95$ . To narrow the discussion it is presented the responses of six variables only, capital, price of capital, inflation, investment, consumption and

real balances. Figures 14 through 19 show the results.

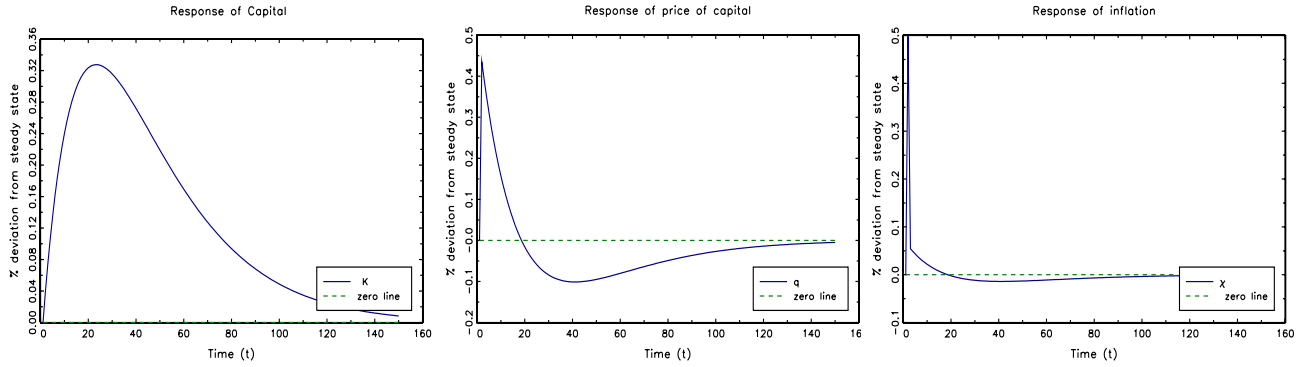


Figure 14:  $\uparrow \epsilon_A \rightarrow K, \rho_A = 0.95$

Figure 15:  $\uparrow \epsilon_A \rightarrow q, \rho_A = 0.95$

Figure 16:  $\uparrow \epsilon_A \rightarrow \chi, \rho_A = 0.95$

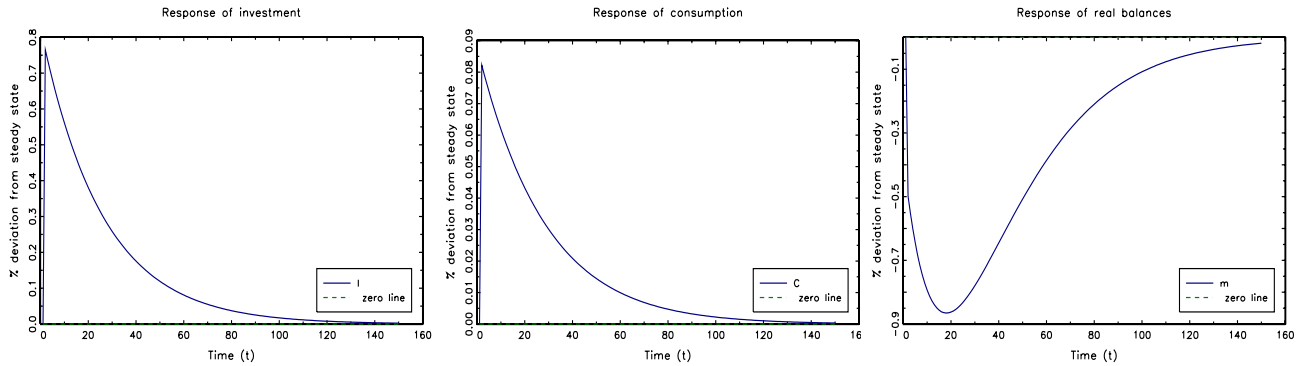


Figure 17:  $\uparrow \epsilon_A \rightarrow I, \rho_A = 0.95$

Figure 18:  $\uparrow \epsilon_A \rightarrow C, \rho_A = 0.95$

Figure 19:  $\uparrow \epsilon_A \rightarrow m, \rho_A = 0.95$

The first conclusion that we can draw regarding the performance of the model as compared to other models without liquidity frictions, is the small responses of the variables to the shock. It seems that liquidity frictions are responsible for such result. This feature was already anticipated in view of the transitional dynamic analysis of the previous section. There the comparison with the benchmark case of the model without frictions made the point more extreme. What is also revealing in the present section is that the responses of the variables are even less pronounced than in other related models where the maximum effect, as an exemplification, of a similar shock to productivity in the stock of capital of the economy reach usually 0.8 percent deviation or

more of the steady state value. Here in this economy the maximum response is in the order of a little more than 0.32 percent.

To gain insight on the mechanics of the model, let me explain the behavior of the variables that the graphs display. First, the increase in the productivity parameter rises the rental rate of capital, this increases the demand of it, whose price increases at impact to clear the market. Note also that agents reduce the demand of money because the return on capital increases, hence the price of money drops to equilibrate the money market. The more output at the time of the shock permits an increase in investment and consumption. Real balances decrease because given the fixed stock of money supplied, inflation erodes its value. Investment increases in spite of the fall in real balances. Real balances permit investors to get more goods for the investment technology, in this case the rise in the productivity is enough to create more goods to finance investment. Eventually, all the variables revert to their previous steady state values, as the shock is transitory.

I turn now to liquidity shocks. I consider a negative liquidity shock, a one percent negative deviation from the steady state value of the reselling constraint parameter,  $\bar{\theta}_r$ , which was set to 0.2. To narrow the discussion I present the responses to capital, the price of capital and inflation only. The value  $\rho_\theta = 0.95$  is used. Figures 20 through 22 show the results. When the negative

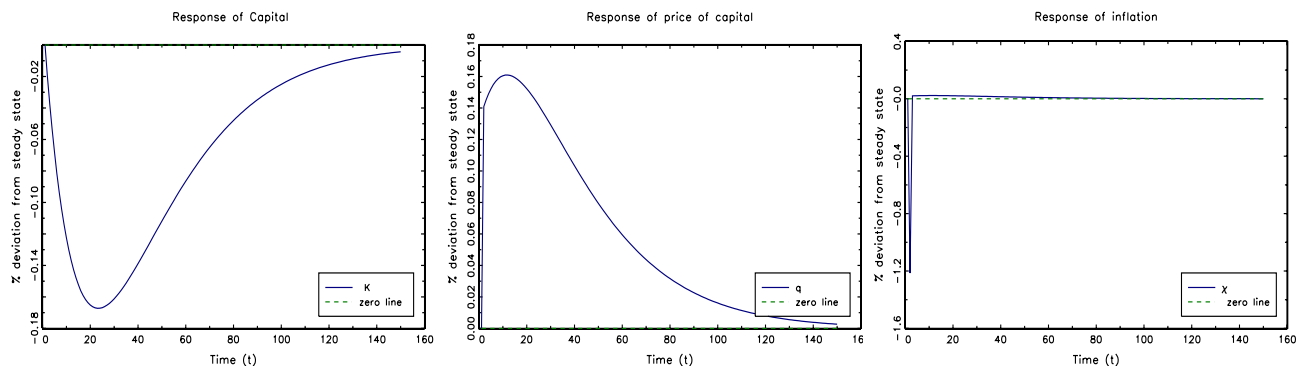


Figure 20:  $\downarrow \epsilon_\theta \rightarrow K, \rho_\theta = 0.95$

Figure 21:  $\downarrow \epsilon_\theta \rightarrow q, \rho_\theta = 0.95$

Figure 22:  $\downarrow \epsilon_\theta \rightarrow \chi, \rho_\theta = 0.95$

liquidity shock is transitory, the price of capital rises above steady state and returns slowly to it. The negative shock induce investors to supply less capital, and savers to demand less capital since they expect future impediments to the selling of the capital they acquire. The net effect in principle could be ambiguous. Here it turns out that since the shock is transitory, savers realize that next period the liquidity level of capital will tend to return to a less illiquid state, then demand does not fall too much, and the supply fall dominates making the price of capital higher. Capital itself of course display a transitory decline. Inflation also decreases at impact. When agents experience the negative liquidity shock they increase the demand for the asset that is not suffering liquidity problems, money, then its price must rise to clear the market.

It is now analyzed the responses to a positive money growth shock, the results are shown in Figures 23 through 25. It can be seen that the positive money growth shock is contractive

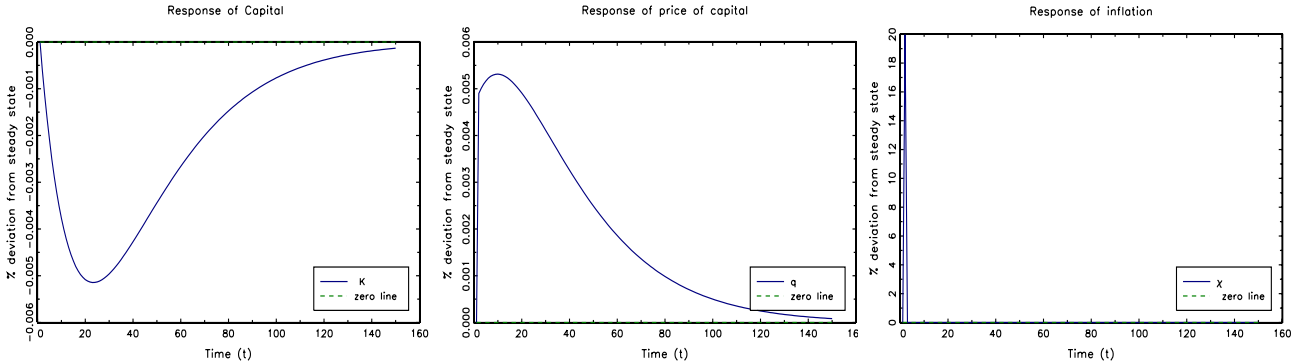


Figure 23:  $\uparrow \epsilon_\gamma \rightarrow K, \rho_\gamma = 0.95$     Figure 24:  $\uparrow \epsilon_\gamma \rightarrow q, \rho_\gamma = 0.95$     Figure 25:  $\uparrow \epsilon_\gamma \rightarrow \chi, \rho_\gamma = 0.95$

although in magnitude the effects are extremely small. The negative effect arises because the money injection is eroding the rentability of money which is the asset that is helping the economy to alleviate the liquidity constraints. Agents that expect further increases in the money supply, anticipate that money will be less valued and demand less of it further increasing inflation. Agents switch to capital as a means to saving, which rises its price to clear the market. Real balances (not shown) also decrease which imply that investors are getting less goods for the

money they sell and hence less input for producing investment, hence capital falls transitorily.

## 6 An exploration in the model to understand current crisis

The objective of this section is to use the model to understand the current crises and the effects that different government policies could have in the economy. Given the current stylized model that I have, it is not expected that the model would replicate quantitatively several aspects of the crises, nor that the simulated policies give quantitative directions for actual implementation. The objective is much more modest. The model is used to see if it can replicate some qualitative features in the fluctuations of the economic variables, and most importantly to improve our understanding of the possible mechanisms that underlie the current crises and the possible effects that several policy implementations could have.

I want first to focus on the liquidity shocks in the economy. These shocks are thought to be potential candidates as depictions of some aspects of the current crisis. In this paper, the liquidity parameter that fluctuates is  $\theta_r$ , the maximum fraction of old units of capital that can be sold per unit of time. Recall that by selling old units of capital, investors are getting more goods to finance their investment. The amount of old units of capital in a given period might be then the model analog of commercial paper in the economy, and when subjected to a negative shock, can be interpreted as a model analog of its sudden "dryness" in the market. Figures 20 through 22 already displayed the effects on some variables of a transitory shock to liquidity. As we saw, the negative liquidity shock is indeed contractive. The prediction of the model with regard to the price of capital might seem odd when one looks for a counterpart in reality where prices of assets in general had declined substantially. Therefore, I explore what would happen in the model if the liquidity parameter permanently falls to a lower level. The effect the variables of this permanent liquidity shock can be seen in Figures 26 through 28. When the shock

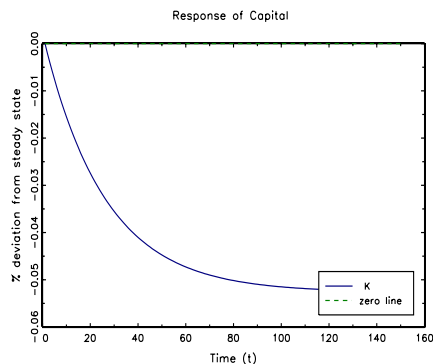


Figure 26:  $\downarrow \epsilon_\theta \rightarrow K, \rho_\theta = 1$

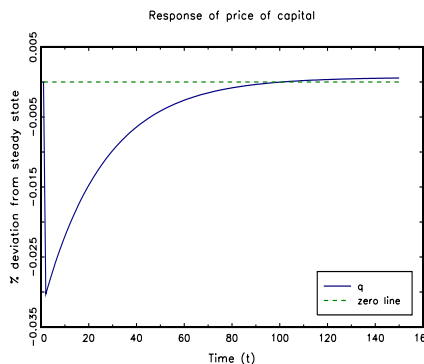


Figure 27:  $\downarrow \epsilon_\theta \rightarrow q, \rho_\theta = 1$

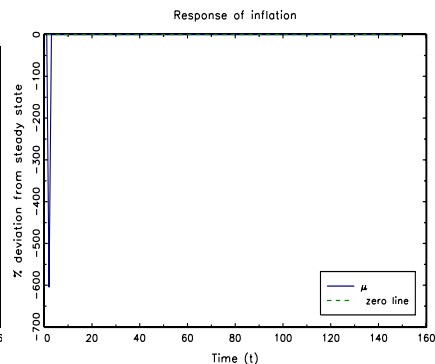


Figure 28:  $\downarrow \epsilon_\theta \rightarrow \chi, \rho_\theta = 1$

is persistent, the stock of capital and hence output decreases permanently. Supply of capital decreases and in this case, different from the transitory case, savers expect no recovery from the fall in the liquidity of the asset and hence the demand falls more than the supply, lowering the price of capital. The effects on deflation are even higher since the demand of money increases even more when agents expect no recovery.

### Some government policies

What are the effects of purchases of capital by the government? How do these effects depend on the way the purchase is financed? These are questions that can be addressed in the model. In this stylized model, government interventions in the capital market are operationalized as stochastic purchases of capital. Government acting like a private agent in that the services of capital acquired are rented to the firms in the economy.

I perform two exercises, first a given transitory purchase of capital financed with taxes and second, the same transitory purchase financed with money creation.

The effects on several variables are displayed in Figures 29 through 34, for a transitory shock financed with taxes, and where the persistence parameter is  $\rho_H = 0.95$ .

When the transitory purchase of capital is financed with taxes agents consume and save less at

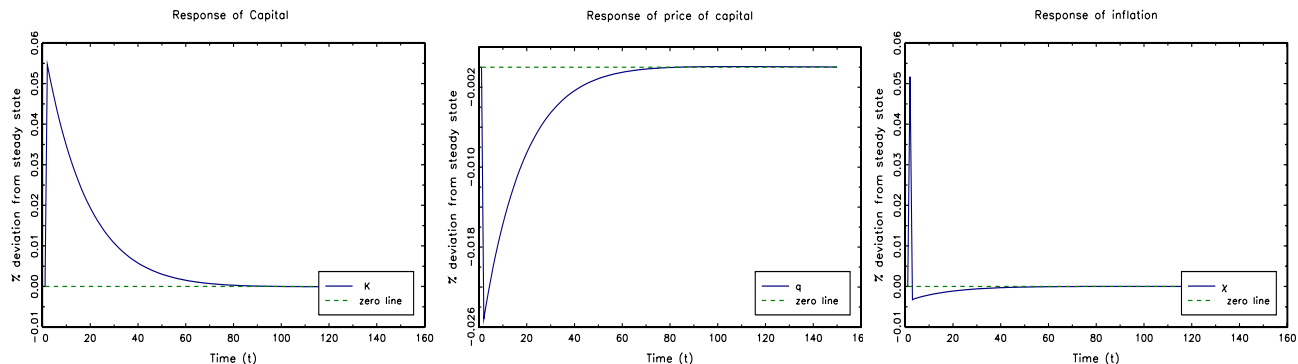


Figure 29:  $\uparrow \epsilon_H \rightarrow K, \rho_H = 0.95$     Figure 30:  $\uparrow \epsilon_H \rightarrow q, \rho_H = 0.95$     Figure 31:  $\uparrow \epsilon_H \rightarrow \chi, \rho_H = 0.95$

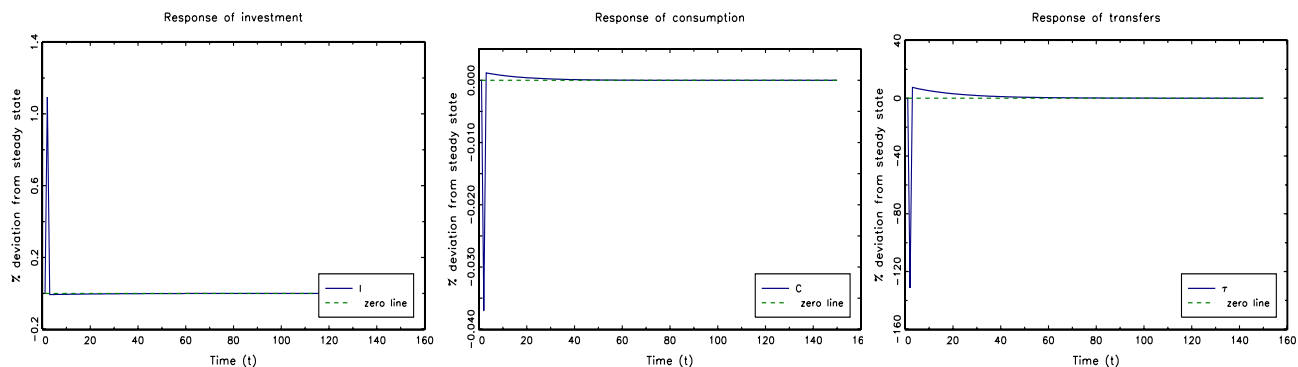


Figure 32:  $\uparrow \epsilon_H \rightarrow I, \rho_H = 0.95$     Figure 33:  $\uparrow \epsilon_H \rightarrow C, \rho_H = 0.95$     Figure 34:  $\uparrow \epsilon_H \rightarrow T, \rho_H = 0.95$

impact. The resources obtained by the government are used to buy capital from the investors which in this way finance the extra capital creation. The policy then not only substitute the demand of capital previously undertaken by the agents without the investment technology but also, by reducing consumption from them, enable investors to expand their activity. Note that in this case, the next period, the government has to sell some of the capital acquired at the time of the policy. Then, the effects on investment, consumption and transfers are reversed next period. Capital depreciates and then it slowly returns to its steady state value. Inflation increases at impact since demand of money falls as agents (particularly savers) are taxed and their net income decreases.

From the description of the effects in the paragraph above, it follows that what the government

does in the following periods after the initial purchase might be relevant for the dynamic response of the variables. For this particular policy it turns out that how persistent the purchase is matters only for the persistence of the effects in the economy. If the government purchases are permanent, then similar effects are obtained, to not clutter the exposition the impulse responses are not shown. In this case the government does not sell any capital and it only rebates the proceeds from its rental. The stock of capital and output, similarly to the previous graphs, return to their previous steady state level. The policy remains to be inflationary. If the purchase of capital is more transitory than what is implied by the parameter  $\rho_H = 0.95$ , then similar results are obtained, only that the persistence of the effects are reduced. The policy of purchasing capital succeeds in increasing investment and output, although consumption decreases as well. The higher the persistence in the purchase of capital, the more lasting are the effects.

Turning now to the policy of purchasing capital financed with money creation, I allow the rate of growth of money to adjust in order to finance any purchases of capital by the government. Transfers are set constant to a value close to zero. The results are presented in Figures 35 through 40 :

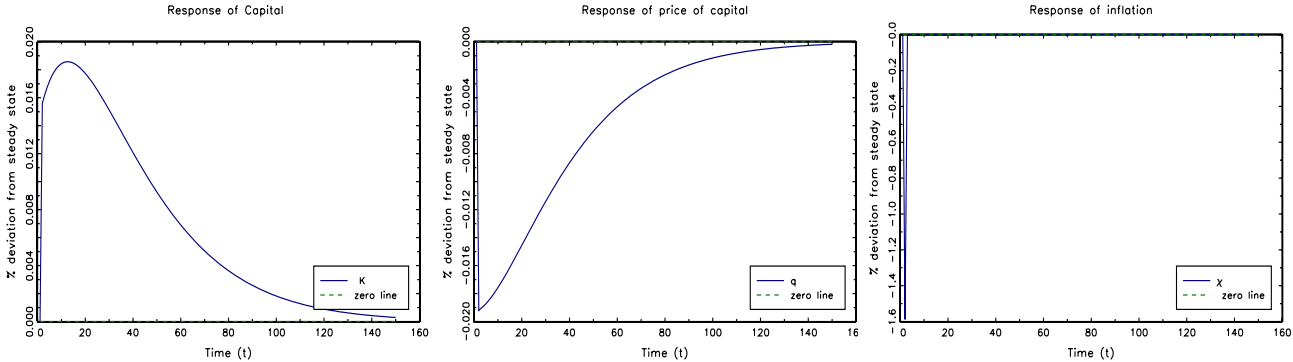


Figure 35:  $\uparrow \epsilon_H \rightarrow K, \rho_H = 0.95$     Figure 36:  $\uparrow \epsilon_H \rightarrow q, \rho_H = 0.95$     Figure 37:  $\uparrow \epsilon_H \rightarrow \chi, \rho_H = 0.95$

It is possible to see that this policy also succeeds in boosting the economy. It is interesting that consumption fell by orders of magnitude less than with the previous policy, while the difference in

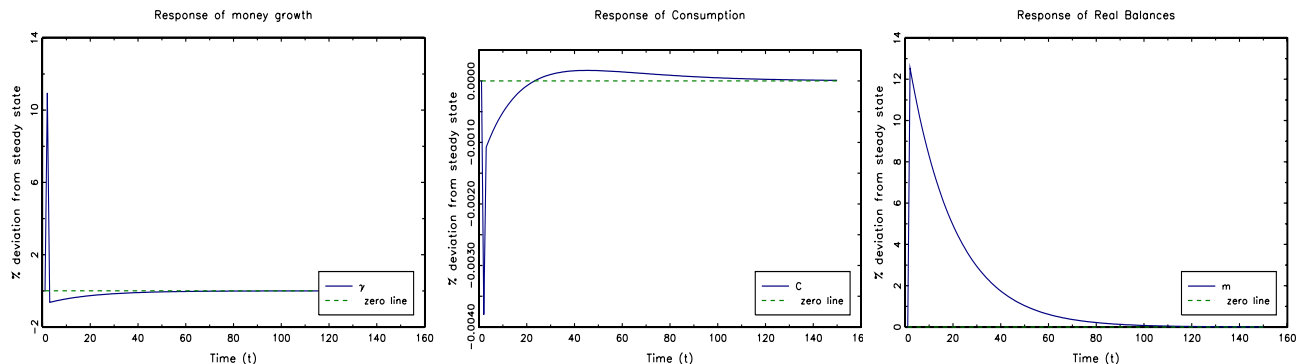


Figure 38:  $\uparrow \epsilon_H \rightarrow \gamma, \rho_H = 0.95$     Figure 39:  $\uparrow \epsilon_H \rightarrow C, \rho_H = 0.95$     Figure 40:  $\uparrow \epsilon_H \rightarrow m, \rho_H = 0.95$

the increase in capital with respect to the previous policy is not as large. The key to understand this result is that once the government print more money to finance its capital purchases, then the next period it needs to retire money since the proceeds from the buying of capital are large. Hence at impact, even though there is an inflationary pressure because of the cash injection, the expectation that the government will retire money leads to a higher demand of it, which makes actually the price of money to rise and hence a deflation is produced. This deflation is key for real balances to increase and hence allow for more input for the investment technology.

Different from the previous policy of financing with taxes, there is a stark difference whether the purchase under the current policy is permanent or not. The graphs 41 through 46 show the effects of a permanent purchase financed with money creation.

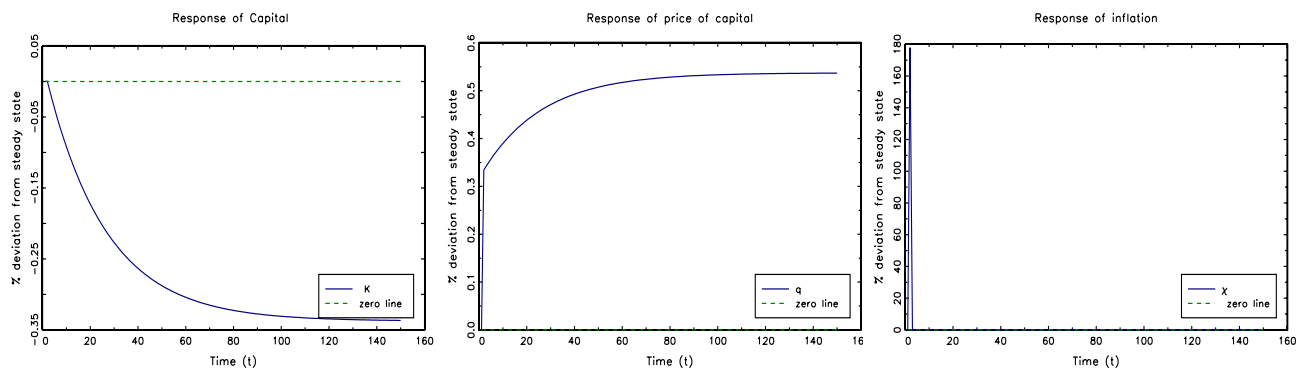


Figure 41:  $\uparrow \epsilon_H \rightarrow K, \rho_H = 1$     Figure 42:  $\uparrow \epsilon_H \rightarrow q, \rho_H = 1$     Figure 43:  $\uparrow \epsilon_H \rightarrow \chi, \rho_H = 1$

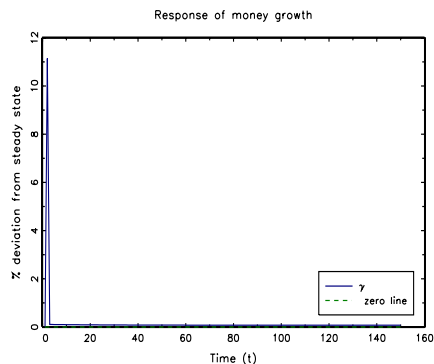


Figure 44:  $\uparrow \epsilon_H \rightarrow \gamma, \rho_H = 1$

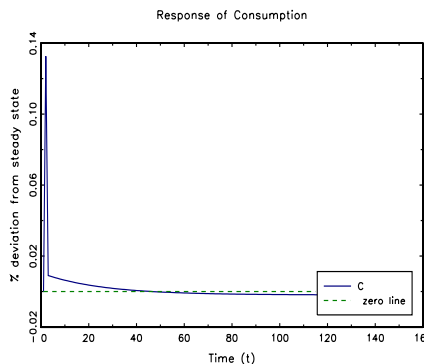


Figure 45:  $\uparrow \epsilon_H \rightarrow C, \rho_H = 1$

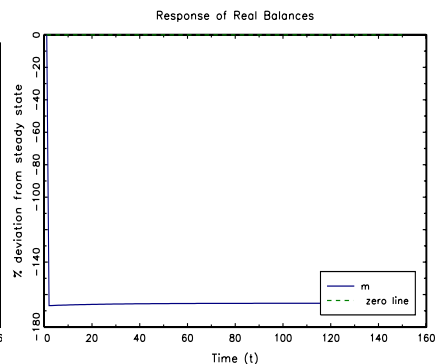


Figure 46:  $\uparrow \epsilon_H \rightarrow m, \rho_H = 1$

The graphs show that the policy in this case is detrimental for output and investment. If we compare figures 38 and 44, we can see that in the second case there is no withdrawal of cash from the economy from the period following the shock on. Instead, the rate of growth of money for this second exercise, is above its initial steady state value. This also affects real balances that can be seen to decrease permanently. Hence the policy is negatively affecting the value of the asset that helps the economy to alleviate the liquidity frictions which causes detrimental effects for capital accumulation and output.

I conclude this section by noting that the overall effects of the aggregate shocks are small in magnitude. The degree of liquidity constraints at which the approximation has been conducted is large, I used  $\theta_s = \bar{\theta}_r = 0.2$ , and this was necessary to approximate the economy around a steady state that display a positive price for money and where liquidity frictions matter. It seems that the tightness of these frictions prevent the aggregate shocks to have large effects in the economy. Still, the analysis presented illustrates the properties of the model and may lay structure to think about issues regarding the current crises.

## 7 Conclusions

The model is helpful to think about frictions that may be present in the transfer of resources among agents, how these frictions affect their behavior and its decisions in terms of investment, consumption and production as well as the interaction between the liquidity frictions and the prices of the assets in the economy. I found that the modifications to Kiyotaki and Moore (2008), that help to clarify the mechanisms through which these effects take place, also make transparent an interesting result they found related to the endogenous nature of money in this setup: when the economy displays a situation with relative "dryness" in terms of the assets that are used by agents to transfer resources, other assets become valued in the economy. In particular, the model treats money as the other asset. Therefore, conferring money with different liquidity properties than those assumed in current models of monetary policy.

As for the long-run analysis, I expanded the analysis on the determination of allocations and asset prices by considering government policies. I showed that the Friedman rule achieves optimality and its implementation can be reached with a variety of instruments that include taxes and purchases of capital. When the Friedman rule is not applied, I showed that a minimalist government intervention in purchasing capital in the market is desirable.

On the empirical exploration of the long run properties of the model I have shown that the degree of tightness of liquidity constraints have to be substantially large in order for negative effects to be present in the economy. Also, given an already illiquid economy, further tightening is incrementally detrimental. In this situation, money is important to alleviate the negative effects of liquidity constraints.

When the transitional dynamics were analyzed, I found that the effects of liquidity constraints are much more important in the transition than in the steady state and I also showed the role

of money as an asset that helps the time path of the economy's variables to be closer to the optimal transition. Notwithstanding, it is required relatively large deviations from the steady state to have significant detrimental effects due to liquidity frictions.

Aggregate stochastic shocks are analyzed in the model. Productivity shocks having qualitative effects on the variables that resemble results in typical DSGE models of the literature. I also showed that positive shocks to money growth have contractive effects in the economy, a result that is due to the role of money and its value as an asset to funnel resources in the economy.

When aggregate liquidity shocks were incorporated, it was possible to generate a temporary recession and also, under some specifications on the persistence of the liquidity shocks, to obtain a fall in the price of capital in the economy.

In general, even though the effects of the aggregate shocks had reasonable qualitative effects, quantitatively, it was shown that their effects are minor in the economy. I interpret these results as an indication that further elaborations on the model can be useful as a framework for quantitative policy analysis in situations where liquidity issues are thought to be important. While this is a task for future research, I considered that we could learn something by thinking about the effects of government policy in the present model. With this in mind, I performed some exercises regarding potential policies that could be implemented to enhance economic activity in the economy. I showed how transitory direct purchases of capital by the government can have temporary expansionary effects in the economy either when financed with taxes or money creation. Although the policy of financing the temporary purchase with money creation appears to be a better option along some dimensions. Finally, I also showed that a permanent purchase of capital by the government financed with money creation, have detrimental effects for investment and output.

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# Appendix A

## Proof of Proposition 1

*Proof.* Let me first derive what would be the amount of capital accumulated by any agent when he decides not to consume anything and not to take any money for the next period. In this case the amount of capital investors hold is:

$$k'_1 = \frac{1}{\psi} [w + rn + \mu m + \tau + \psi\eta(1 - \delta)k,]$$

and the amount of capital savers hold is:

$$k'_0 = \frac{1}{q} [w + rn + \mu m + \tau + q(1 - \delta)k].$$

Given the relationships expressed in (11), it can be seen that  $k'_1 > k'_0$ . Next, let me compute the amount of money each agent would take for the next period when consumption is zero and capital takes its minimum constrained value,  $k' = (1 - \theta_r)(1 - \delta)k$ . In this case money holdings satisfy:

$$m'_1 = m'_0 = \frac{1}{\mu} [w + rn + \mu m + \tau + q\theta_r(1 - \delta)k]$$

Finally let me compute the level of consumption each agent would hold when no money is taken to the next period and the amount of capital carried over is the minimum as above, then:

$$c_1 = c_0 = w + rn + \mu m + \tau + q\theta_r(1 - \delta)k$$

□

## Proof of Proposition 2

*Proof.* The first order conditions associated with agent's problem (12) are:

$$\begin{aligned}
k' &: -\psi + \beta\pi Ev_{1k}(k', m') + \beta(1 - \pi)Ev_{0k}(k', m') - \psi\lambda_1 + \mu_1 = 0, & z = 1 \\
m' &: -\mu + \beta\pi Ev_{1m}(k', m') + \beta(1 - \pi)Ev_{0m}(k', m') - \mu\lambda_1 + \gamma_1 = 0, & z = 1 \\
k' &: -q + \beta\pi Ev_{1k}(k', m') + \beta(1 - \pi)Ev_{0k}(k', m') - q\lambda_0 + \mu_0 = 0, & z = 0 \\
m' &: -\mu + \beta\pi Ev_{1m}(k', m') + \beta(1 - \pi)Ev_{0m}(k', m') - \mu\lambda_0 + \gamma_0 = 0, & z = 0.
\end{aligned}$$

The envelope conditions are:

$$\begin{aligned}
v_{1k}(k, m) &= r + \psi\eta(1 - \delta) + \lambda_1 [r + \psi\eta(1 - \delta)] - \mu_1(1 - \theta_r)(1 - \delta) \\
v_{1m}(k, m) &= \mu + \lambda_1\mu \\
v_{0k}(k, m) &= r + q(1 - \delta) + \lambda_0[r + q(1 - \delta)] - \mu_0(1 - \theta_r)(1 - \delta) \\
v_{0m}(k, m) &= \mu + \lambda_0\mu.
\end{aligned}$$

next period envelope conditions are:

$$\begin{aligned}
v_{1k}(k', m') &= r' + \psi'\eta'(1 - \delta) + \lambda'_1 [r' + \psi'\eta'(1 - \delta)] - \mu'_1(1 - \theta'_r)(1 - \delta) \\
v_{1m}(k', m') &= \mu' + \lambda'_1\mu' \\
v_{0k}(k', m') &= r' + q'(1 - \delta) + \lambda'_0[r' + q'(1 - \delta)] - \mu'_0(1 - \theta'_r)(1 - \delta) \\
v_{0m}(k', m') &= \mu' + \lambda'_0\mu'.
\end{aligned}$$

Under assumption 1 the first order conditions can be written as:

$$k' : \frac{1 - q\theta_s}{1 - \theta_s}(1 + \lambda_1) = \beta\pi Ev_{1k}(k', m') + \beta(1 - \pi)Ev_{0k}(k', m') \quad (29a)$$

$$m' : \mu(1 + \lambda_1) = \beta\pi Ev_{1m}(k', m') + \beta(1 - \pi)Ev_{0m}(k', m') + \gamma_1 \quad (29b)$$

$$k' : q = \beta\pi Ev_{1k}(k', m') + \beta(1 - \pi)Ev_{0k}(k', m') \quad (29c)$$

$$m' : \mu = \beta\pi Ev_{1m}(k' m') + \beta(1 - \pi)Ev_{0m}(k', m') \quad (29d)$$

By combining equations (29a) and (29c) we have:  $(1 + \lambda_1) = q/\psi$ . We need to show that this value is greater than 1 for  $\lambda_1$  to be strictly positive, but under (11), we know this is true. To verify that  $\gamma_1$  is strictly positive. Compare equations (29b) and (29d). We learn that:  $\mu(1 + \lambda_1) = \mu + \gamma_1$ , and then as long as  $\mu > 0$ ,  $\gamma_1 > 0$ . Then the first order conditions can be written, using the envelope conditions as equations (14a),(14b) and (14c).

Denoting by  $N'_1$  and  $N'_0$  aggregate demand for capital for both the investors and the savers respectively, investor's aggregate demand of capital,

$$N'_1 = \int k'_1(k, m)d\lambda_1(k, m) = \pi \int k'_1(k, m)d\lambda(k, m),$$

given (13a) and (13c), we have:

$$\begin{aligned} N'_1 &= \pi \int \frac{1 - \theta_s}{1 - q\theta_s} \left\{ w + rn + \mu m + \tau + \frac{1 - q\theta_s}{1 - \theta_s} \left[ 1 - \theta_r + q\theta_r \frac{1 - \theta_s}{1 - q\theta_s} \right] (1 - \delta)k \right\} d\lambda(k, m) \\ &= \pi \frac{1 - \theta_s}{1 - q\theta_s} \left\{ w + r \int nd\lambda(k, m) + \int (\mu m + \tau)d\lambda(k, m) \right. \\ &\quad \left. + \frac{1 - q\theta_s}{1 - \theta_s} \left[ 1 - \theta_r + q\theta_r \frac{1 - \theta_s}{1 - q\theta_s} \right] (1 - \delta) \int nd\lambda(k, m) \right\} \\ &= \pi \frac{1 - \theta_s}{1 - q\theta_s} \left\{ w + r + \mu M + T + \frac{1 - q\theta_s}{1 - \theta_s} \left[ 1 - \theta_r + q\theta_r \frac{1 - \theta_s}{1 - q\theta_s} \right] (1 - \delta)N \right\} \end{aligned}$$

Hence:

$$N'_1 = \frac{1}{\psi} \left\{ \frac{w}{N} + r + \psi\eta(1 - \delta) \right\} \pi N + \frac{1}{\psi} \pi M + \frac{\pi T}{\psi}$$

To get the aggregate amount of capital that savers hold for next period, first notice that the liquidity

constraint of investors imply  $k'_1(k, m) = (1 - \theta_s)i + (1 - \theta_r)(1 - \delta)k$ , hence:

$$\begin{aligned} N'_1 &= \int k'_1(k, m)d\lambda_1(k, m) = (1 - \theta_s) \int i(k)d\lambda_1(k, m) + (1 - \theta_r)(1 - \delta) \int nd\lambda(k, m) \\ &= (1 - \theta_s)I + (1 - \theta_r)(1 - \delta)\pi N \end{aligned}$$

Aggregate capital holdings of savers are given then by a fraction  $\theta_s$  of new units of capital, a fraction  $\theta_r$  of old units of capital and the aggregate amount of capital savers had at the beginning of the period<sup>18</sup>:

$$N'_0 = \theta^s I + \theta_r(1 - \delta)\pi N + (1 - \delta) \int nd\lambda_0(k, m) = \theta^s I + [\theta_r(1 - \delta)\pi + (1 - \delta)(1 - \pi)]N$$

By aggregate private capital (3) and market clearing (14g), we have that:

$$\begin{aligned} K' &= N'_1 + N'_0 + H' \\ &= N' + H' \end{aligned}$$

$$K' = (1 - \delta)K + I$$

Government is described by the money growth rate:

$$M' = \gamma M$$

and the government budget constraint:

$$pT + q[H' - (1 - \delta)H] = rH + \mu(M' - M)$$

---

<sup>18</sup>The liquidity constraint in the aggregate for savers is given by:  $N'_0 \geq (1 - \theta_r)(1 - \delta)(1 - \pi)N$ , then the liquidity constraint will hold for them if and only if:

$$\theta_s\delta + \theta_r(1 - \delta)\pi + (1 - \delta)(1 - \pi) \geq (1 - \theta_r)(1 - \delta)(1 - \pi)$$

and this is satisfied because  $\theta_s\delta + 2\theta_r(1 - \delta)(1 - \pi) \geq 0$  is always satisfied.

And finally the optimality conditions for firms are:

$$r = \alpha AK^{1-\alpha}, \quad w = (1 - \alpha)AK^\alpha$$

□

In deriving the aggregate version of the economy, I have made the assumption that the mean values of the variables for given  $\lambda_1$  and  $\lambda_0$  are well defined. This remains to be proved and I leave this for future work. For completeness, I show that when idiosyncratic shocks are iid, and mean values exist, it is possible to work with aggregates. Assume first that the idiosyncratic shocks follow a Markov process:  $Q(z, z')$ . Define  $\lambda_1(k, m)$  and  $\lambda_0(k, m)$  the measures of investors and savers respectively, equations of motion for these measures are:

$$\lambda'_1(\tilde{k}, \tilde{m}) = \int_{\{k:\gamma_0(k,m)\leq\tilde{k}, m:h_0(k,m)\leq\tilde{m}\}} Q(0, 1)d\lambda_0(k, m) + \int_{\{k:\gamma_1(k)\leq\tilde{k}, m:h_1(k,m)\leq\tilde{m}\}} Q(1, 1)d\lambda_1(k, m) \quad (31)$$

$$\lambda'_0(\tilde{k}, \tilde{m}) = \int_{\{k:\gamma_0(k)\leq\tilde{k}, m:h_0(k,m)\leq\tilde{m}\}} Q(0, 0)d\lambda_0(k, m) + \int_{\{k:\gamma_1(k)\leq\tilde{k}, m:h_1(k,m)\leq\tilde{m}\}} Q(1, 0)d\lambda_1(k, m) \quad (32)$$

Under the i.i.d. case, we can define:  $Q(0, 1) = Q(1, 1) = \pi$  and  $Q(0, 0) = Q(1, 0) = 1 - \pi$ . In this case, the above equations can be written as:

$$\lambda'_1(\tilde{k}, \tilde{m}) = \pi \left[ \int_{\{k:\gamma_0(k,m)\leq\tilde{k}, m:h_0(k,m)\leq\tilde{m}\}} d\lambda_0(k, m) + \int_{\{k:\gamma_1(k)\leq\tilde{k}, m:h_1(k,m)\leq\tilde{m}\}} d\lambda_1(k, m) \right] = \pi \lambda'(\tilde{k})$$

$$\lambda'_0(\tilde{k}) = (1 - \pi) \left[ \int_{\{k:\gamma_0(k,m)\leq\tilde{k}, m:h_0(k,m)\leq\tilde{m}\}} d\lambda^0(k, m) + \int_{\{k:\gamma_1(k)\leq\tilde{k}, m:h_1(k,m)\leq\tilde{m}\}} d\lambda_1(k) \right] = (1 - \pi) \lambda'(\tilde{k}, \tilde{m})$$

where it has been defined the aggregate measure:

$$\lambda(\tilde{k}, \tilde{m}) = \int_{\{k:\gamma_0(k)\leq\tilde{k}, m:h_0(k,m)\leq\tilde{m}\}} d\lambda_0(k, m) + \int_{\{k:\gamma_1(k)\leq\tilde{k}, m:h_1(k,m)\leq\tilde{m}\}} d\lambda_1(k, m).$$

## Appendix B

### Implementing the Parametric Path Method

The system can be written more succinctly, as a system in 3 equations (it is normalized  $M = 1$ ):

$$q = \frac{1 - \theta_s}{1 - q'\theta_s} \left[ r' + \frac{1 - q'\theta_s}{1 - \theta_s} \left( 1 - \theta_r + q'\theta_r \frac{1 - \theta_s}{1 - q'\theta_s} \right) (1 - \delta) \right] q' \beta \pi + [r' + q'(1 - \delta)] \beta (1 - \pi)$$

$$\mu = \frac{1 - \theta_s}{1 - q'\theta_s} q' \mu' \beta \pi + \mu' \beta (1 - \pi)$$

$$K' = \frac{1}{1 - q\theta_s} \left[ \frac{r}{\alpha} + \frac{1 - q\theta_s}{1 - \theta_s} \left( 1 - \theta_r + q\theta_r \frac{1 - \theta_s}{1 - q\theta_s} \right) (1 - \delta) \right] \pi K + \frac{1}{1 - q\theta_s} \pi \mu + \frac{\theta_r \pi + (1 - \pi) - \theta_s}{1 - \theta_s} (1 - \delta) K$$

Linearizing the system around the steady state I obtained:

$$\begin{aligned} q &= \bar{q} + \left[ \frac{(1 - \theta_s) \beta \pi A \alpha \bar{K}^{\alpha-1}}{(1 - \bar{q}\theta_s)^2} + (1 - \theta_r)(1 - \delta) \beta \pi \right. \\ &\quad \left. + \frac{\theta_r (1 - \theta_s)(1 - \delta) \beta \pi (2\bar{q} - \bar{q}^2 \theta_s)}{(1 - \bar{q}\theta_s)^2} + (1 - \delta) \beta (1 - \pi) \right] (q' - \bar{q}) \\ &\quad + \left[ \frac{(1 - \theta_s) \beta \pi \bar{q}}{1 - \bar{q}\theta_s} + \beta (1 - \pi) \right] A \alpha (\alpha - 1) \bar{K}^{\alpha-2} (K' - \bar{K}) \end{aligned}$$

$$\mu = \bar{\mu} + \frac{1 - \theta_s}{(1 - \bar{q}\theta_s)^2} \bar{\mu} \beta \pi (q' - \bar{q}) + \left[ \frac{1 - \theta_s}{1 - \bar{q}\theta_s} \bar{q} \beta \pi + \beta (1 - \pi) \right] (\mu' - \bar{\mu})$$

$$\begin{aligned} K' &= \bar{K} + \left[ \frac{\pi A \bar{K}^{\alpha-1}}{1 - \bar{q}\theta_s} + \frac{\pi}{1 - \theta_s} \left( 1 - \theta_r + \bar{q}\theta_r \frac{1 - \theta_s}{1 - \bar{q}\theta_s} \right) (1 - \delta) \right. \\ &\quad \left. + \left( \frac{\theta_r \pi + (1 - \pi) - \theta_s}{1 - \theta_s} \right) (1 - \delta) + \frac{\pi A (\alpha - 1) \bar{K}^{\alpha-1}}{1 - \bar{q}\theta_s} \right] (K - \bar{K}) \\ &\quad + \left[ \frac{\pi A \bar{K}^{\alpha} \theta_s + \pi \theta_r (1 - \delta) \bar{K} + \theta_s \pi \bar{\mu}}{(1 - \bar{q}\theta_s)^2} \right] (q - \bar{q}) + \frac{\pi}{1 - \bar{q}\theta_s} (\mu - \bar{\mu}) \end{aligned}$$

Next, I use the definitions:

$$\begin{aligned}
v_{11} &= \frac{(1 - \theta_s)\beta\pi A\alpha\bar{K}^{\alpha-1}}{(1 - \bar{q}\theta_s)^2} + (1 - \theta_r)(1 - \delta)\beta\pi + \frac{\theta_r(1 - \theta_s)(1 - \delta)\beta\pi(2\bar{q} - \bar{q}^2\theta_s)}{(1 - \bar{q}\theta_s)^2} + (1 - \delta)\beta(1 - \pi) \\
v_{12} &= \left[ \frac{(1 - \theta_s)\beta\pi\bar{q}}{1 - \bar{q}\theta_s} + \beta(1 - \pi) \right] A\alpha(\alpha - 1)\bar{K}^{\alpha-2} \\
v_{21} &= \left[ \frac{\pi A\bar{K}^\alpha\theta_s + \pi\theta_r(1 - \delta)\bar{K} + \theta_s\pi\bar{\mu}}{(1 - \bar{q}\theta_s)^2} \right] \\
v_{22} &= \frac{\pi A\bar{K}^{\alpha-1}}{1 - \bar{q}\theta_s} + \frac{\pi}{1 - \theta_s} \left( 1 - \theta_r + \bar{q}\theta_r \frac{1 - \theta_s}{1 - \bar{q}\theta_s} \right) (1 - \delta) + \left( \frac{\theta_r\pi + (1 - \pi) - \theta_s}{1 - \theta_s} \right) (1 - \delta) + \frac{\pi A(\alpha - 1)\bar{K}^{\alpha-1}}{1 - \bar{q}\theta_s} \\
v_{23} &= \frac{\pi}{1 - \bar{q}\theta_s} \\
v_{31} &= \frac{1 - \theta_s}{(1 - \bar{q}\theta_s)^2} \bar{\mu}\beta\pi \\
v_{33} &= \frac{1 - \theta_s}{1 - \bar{q}\theta_s} \bar{q}\beta\pi + \beta(1 - \pi)
\end{aligned}$$