Assignment and Unemployment

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1 Introduction

Unemployment is a pervasive feature of market economies. Search and matching frictions are commonly invoked in order to explain why there is unemployment (Lucas and Prescott 1974, Pissarides 1985). The question of why there are search and matching frictions, however, has received somewhat less attention.[1] The frictions are typically introduced via an aggregate matching function. Pissarides (2000) writes in his textbook,

> Trade in the labor market is a nontrivial economic activity because of the existence of heterogeneities, frictions, and information imperfections. If all workers were identical to each other and if all jobs were also identical to each other, and if there was perfect information about their location, trade would be trivial.... The matching function ... is a modeling device that captures the implications of the costly trading process without the need to make the heterogeneities and other features that give rise to it explicit. (pp. 3–4)

This paper makes information imperfections about heterogeneity explicit and explores the implications of this for the assignment of workers to jobs and for the existence of unemployment.

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[1] A notable recent exception is Lagos (2000).

I examine an extremely simple environment. There are two workers and two jobs. Workers and jobs are heterogeneous, and heterogeneity matters in the sense that aggregate output depends on the assignment of workers to jobs. I introduce a single friction into this economy. Each worker knows her own type, but knows only that the other worker's type is drawn independently from a given distribution. Each worker has an opportunity to apply for one job, but must do so without knowing what the other worker has done. If one worker applies for one job and the other worker for the other job, both are employed. If both workers apply for the same job, it goes to the worker who is more productive at that job. The other worker is unemployed and the other job is unfilled.[2]

I begin by examining a social planner's problem in which workers act so as to maximize the expected present value of output in the economy. Unemployment is avoidable if, for example, worker 1 always applies for job A and worker 2 always applies for job B, independent of the realization of types. But while such rigid policies are feasible, they come at a cost, since workers are unable to make use of the ex post realization of their comparative advantage. I show that under fairly general conditions, having some unemployment is socially optimal (Proposition 1). Under more stringent conditions, symmetric application strategies, where a worker's behavior depends on her type but not on her identity, are socially optimal (Proposition 2). These conditions essentially require that output is sufficiently responsive to a worker's type.

I next turn to a method of decentralizing the social optimum via an appropriate incentive structure, effectively a second price auction. I prove a version of the second welfare theorem in this environment (Proposition 3), but find that typically there may be equilibria that are not socially optimal. In particular, symmetric application strategies always form an equilibrium (Proposition 4) and, under weak conditions, are locally stable under a Tattonement best response dynamic (Proposition 5). On the other hand, the opposite strategies do not always form an equilibrium. The same condition that guarantees the social optimum has some unemployment ensures that there is unemployment in any equilibrium (Proposition 6).

The most interesting question is under what conditions symmetric strategies are the unique equilibrium of the decentralized economy. I show that under the same sufficient conditions that ensure the social optimum is symmetric, the symmetric equilibrium is unique (Proposition 7). Moreover, I use the theory of supermodular games (Milgrom and Roberts 1990, Vives 1990) to show that there are good reasons to believe that individuals will play the symmetric equilibrium strategies. They are the unique strategies that survive iterated

[2]Naturally this extreme assumption could be relaxed in a dynamic version of the model.

elimination of strictly dominated strategies, the unique rationalizable strategy profiles, and the unique solution to a variety of adaptive learning dynamics (Proposition 8).

The next section discusses the related literature. Section 3 describes the social planner's problem. Section 4 characterizes the solution to that problem. Section 5 discusses a method of decentralizing the social optimum. Section 6 concludes.

2 Related Literature

This paper is closely connected to the literature on 'directed search' or 'coordination frictions' (Montgomery 1991, Peters 1991). The simplest models have the same basic structure, except that workers and jobs are homogeneous. In these models, the agents play a three stage game: each job owner announces a wage; then each worker applies for one job; finally, a job that receives at least one application hires one worker and pays her the agreed wage. A job that receives no applications remains idle, while a job that receives two applications rejects one, leaving the worker unemployed. These papers focus on equilibria in which the workers use symmetric strategies in every subgame (i.e. following every wage offer). That is, if worker 1 applies to job A with probability 1, worker 2 also applies to job A with probability 1. A main result is that there is a unique symmetric equilibrium. In it, the job owners offer identical wages and the workers use (independent) mixed strategies, applying with equal probability to each job opening. Half the time, the two workers apply to the same job, giving rise to unemployment.

There are several difficulties that arise in this environment that might make one reluctant to view it as a theory of unemployment. First, there is a two-dimensional continuum of asymmetric equilibria even in the simplest environment (Burdett, Shi and Wright 2001). In these equilibria, workers use pure strategies on the equilibrium path, with worker 1 applying to job A and worker 2 to job B, but off the equilibrium path, workers may need to use mixed strategies in order to discipline wage offers. There is no intrinsic reason to focus on the symmetric mixed strategy equilibrium in this environment, although Burdett et al. (2001) argue that information might play an important role: "All these pure-strategy equilibria require a lot of coordination, in the sense that every buyer has to somehow know where every other buyer is going." (p. 1066) They therefore conclude that the symmetric mixed strategy equilibrium is more natural, even though it fails to maximize the available output. By introducing heterogeneity and an explicit restriction on available information, I work in an environment in which the symmetric mixed strategy equilibrium may maximize output

(Proposition 2) and may be unique (Proposition 7).

A second difficulty arises upon introducing heterogeneity into the basic coordination friction model. Coles and Eeckhout (2000) show that if workers and jobs are heterogeneous and the matching problem is nontrivial, in the sense that aggregate output depends on the assignment of workers to jobs, there is a unique equilibrium of the basic model. In it, each worker is assigned to the job that is her comparative advantage. Coles and Eeckhout (2000) view this as a non-cooperative, decentralized solution to an assignment game, but it is also a strong critique of existing models with unemployment due to coordination frictions.[3] I address the Coles and Eeckhout's (2000) critique here by extending the basic model to allow for heterogeneity but then introducing an explicit information restriction on the realization of heterogeneity in a particular economy.

A third difficulty comes from extending this static model to a dynamic environment. If worker 1 attempts to get job A but instead goes to worker 2, what keeps her from getting job B ten minutes later? It remains a possibility coordination frictions can explain only very short duration unemployment. Unfortunately, my model does not address this shortcoming of the coordination friction literature. I return to this issue in the conclusion.

This paper is also related to a large literature on assignment models, pioneered by Tinbergen (1951), Roy (1951), and Koopmans and Beckmann (1957) and reviewed recently by Sattinger (1993). This literature typically examines a Walrasian equilibrium, in which workers are assigned to the job that is their comparative advantage. Although there has recently been some attempt to introduce search frictions into assignment models (Sattinger 1995, Burdett and Coles 1997, Shimer and Smith 2000), there does not appear to have been any previous attempt to formalize Pissarides's (2000) intuition that assignment problems may give rise to search frictions.

Finally, the results in this paper have the flavor of Harsanyi's (1973) famous purification theorem, which argues the mixed strategy equilibria of a generic game are pure strategy equilibria of a related game in which players payoffs are slightly perturbed and opposing players do not know each others' payoffs. Harsanyi's results do not apply here because I am not interested in small perturbations of payoffs. In my environment, the underlying symmetric information game is a coordination game with two asymmetric pure strategy equilibria and one symmetric mixed strategy equilibrium. Small perturbations would perturb

[3] Shi (2001), Shi (2002), and Shimer (2001) get around this problem by assuming that there are several types of workers but many workers of each type. Identical workers are restricted to using identical strategies. Such models retain the flavor of the basic coordination friction environment.

but not eliminate the pure strategy equilibria. This paper's main innovation is the proof that with a sufficient amount of payoff uncertainty, only the mixed strategy equilibrium survives.

3 Statement of Problem

There are two workers, 1 and 2, and two jobs, A and B. Each worker has a type $x \in [0,1]$, independently distributed with twice differentiable distribution function $G : [0,1] \to [0,1]$. Assume the distribution function is symmetric around $\frac{1}{2}$, so $G(x) + G(1-x) = 1$ and hence $G'(x) = G'(1-x)$. Also assume G' is strictly positive on its support, with $G(0) = 0$ and hence $G(1) = 1$. If a type x worker takes job A, she produces $f(x)$ units of output, while if she takes job B, she produces $f(1-x)$ units of output, where f is differentiable and strictly increasing.

Before observing the realization of the worker types, a social planner tells each worker which job to apply for as a function of that worker's realization. If one worker applies for each job, both are employed and produce. If both workers apply for the same job, the more productive worker is employed and the less productive worker is unemployed and produces nothing.[4] I do not assume the planner must give the two worker the same instructions. The planner's objective is to maximize output.

If the planner could condition the instructions on *both* realizations x_1 and x_2, she would tell worker 1 to apply to job A if $x_1 > x_2$ and to job B otherwise, she would give worker 2 similar instructions, and would use an arbitrary tie-breaking rule if $x_1 = x_2$. This would ensure that both workers are employed in the task in which they have a comparative advantage. But the informational friction prevents such conditioning. The only way to avoid any risk of unemployment is to tell one worker to always apply to job A and the other worker to always apply to job B. But this does not make use of the comparative advantage of the two workers.

4 Characterization of the Optimal Control

To begin, consider two extreme policies that the social planner might contemplate. First, the planner might tell worker 1 to go to job A and worker 2 to go to job B, regardless of the

[4]Suppose an unemployed worker produces output z, an employed worker produces $\tilde{f}(x)$ in job A and $\tilde{f}(1-x)$ in job B. Then we may view f as the net production function, $f(x) \equiv \tilde{f}(x) - z$. None of the results presented below would change.

realization of their types. Expected output from this policy is simply

$$\int_0^1 \big(f(x) + f(1-x)\big) dG(x).$$

Second, the planner might tell each worker to apply for the job that is her comparative advantage, i.e. to job A if $x > \frac{1}{2}$ and to job B otherwise. A worker with type $x < \frac{1}{2}$ is hired into job B as long as the other worker has a higher type, with probability $1 - G(x)$, in which event she produces $f(1-x)$. If her type is $x > \frac{1}{2}$, she is hired with probability $G(x)$ and produces $f(x)$. Putting this together, expected output per worker from this policy is

$$2 \int_0^{\frac{1}{2}} f(1-x)(1-G(x)) dG(x) + 2 \int_{\frac{1}{2}}^1 f(x)G(x) dG(x).$$

These two expressions clarify the tradeoff facing the social planner. The first policy results in lower output per employed worker, since workers do not always seek the job that is their comparative advantage. That is, output of $f(x) < f(\frac{1}{2})$ is possible. The second policy results in some unemployment risk, which is reflected in the additional terms $1 - G(x)$ and $G(x)$ in the two integrals. In general, we would expect the first policy to be preferred to the second if $f(x)$ is relatively insensitive to x, while the second policy is preferred to the first if the distribution function $G(x)$ puts most of its weight on values of x close to $\frac{1}{2}$.

Of course, there is no reason that the social planner has to opt for one of these extreme policies. In general, the optimal policy consists of two functions, $\alpha_i(x)$, the probability that worker i with type x applies to job A. Expected output is

$$\begin{aligned} Y(\alpha) = \int_0^1 \Big(f(x_1)\alpha_1(x_1)A_1(x_1) + f(1-x_1)\big(1-\alpha_1(x_1)\big)B_1(x_1)\Big) dG(x_1) \\ + \int_0^1 \Big(f(x_2)\alpha_2(x_2)A_2(x_2) + f(1-x_2)\big(1-\alpha_2(x_2)\big)B_2(x_2)\Big) dG(x_2), \quad (1) \end{aligned}$$

where $A_i(x_i)$ ($B_i(x_i)$) is the probability that worker i is hired by job A (B) conditional on applying for it and conditional on her type x_i. If worker 1 has type x_1, she applies to job A with probability $\alpha_1(x_1)$, is hired with probability $A_1(x_1)$, and in which event she produces $f(x_1)$. She applies to job B with probability $1 - \alpha_1(x_1)$, is hired with probability $B_1(x_1)$, in which event she produces $f(1 - x_1)$. This is multiplied by the density $G'(x_1)$ and then integrated over the support $[0, 1]$ to get expected output from worker 1. The second line gives a similar expression for the expected output produced by worker 2. The hiring probabilities A and B can in turn be expressed as functions of the other worker's policy α:

$$A_i(x) = G(x) + \int_x^1 (1 - \alpha_{i'}(y))dG(y) \tag{2}$$

$$B_i(x) = 1 - G(x) + \int_0^x \alpha_{i'}(y)dG(y), \tag{3}$$

$i' \neq i$. For example, worker 1 is hired if she applies to job A in the event that either worker 2 has a lower type or worker 2 applies to job B (or both).

Substitute equations (2) and (3) into the objective function (1) to get an unconstrained expression for output as a function of the policy functions α. Then regroup terms, noting that $\int_0^1 \int_{x_2}^1 \phi(x_1, x_2)dx_1dx_2 \equiv \int_0^1 \int_0^{x_1} \phi(x_1, x_2)dx_2dx_1$. This delivers

$$\begin{aligned} Y(\alpha) = &\int_0^1 \Big(f(x_1)\alpha_1(x_1)G(x_1) + f(1-x_1)\big(1-\alpha_1(x_1)\big)\big(1-G(x_1)\big)\Big)dG(x_1) \\ &+ \int_0^1 \Big(f(x_2)\alpha_2(x_2)G(x_2) + f(1-x_2)\big(1-\alpha_2(x_2)\big)\big(1-G(x_2)\big)\Big)dG(x_2) \\ &+ \int_0^1 \int_{x_1}^1 \alpha_1(x_1)\big(1-\alpha_2(x_2)\big)\big(f(x_1) + f(1-x_2)\big)dG(x_2)dG(x_1) \\ &+ \int_0^1 \int_0^{x_1} \alpha_2(x_2)\big(1-\alpha_1(x_1)\big)\big(f(1-x_1) + f(x_2)\big)dG(x_2)dG(x_1). \end{aligned} \tag{4}$$

The first line gives the output that worker 1 produces if she applies to job A and worker 2 has a lower type, plus the corresponding term if worker 1 applies to job B and worker 2 has a higher type. The second line gives similar expressions for worker 2. The third line gives the output produces when $x_1 < x_2$ but worker 1 applies to job A and worker 2 to job B, i.e. the assignment is incorrect. The fourth line is a similar expression for output with the incorrect assignment when $x_1 > x_2$.

Definition 1. *A social optimum is a pair (α_1, α_2) that maximizes $Y(\alpha)$ given in equation (4).*

Lemma 1. *Any social optimum is described by two thresholds x_i^*, where worker i applies to job A if $x_i > x_i^*$ and to job B if $x_i < x_i^*$, with an arbitrary tie-breaking rule.*

Proof. For any fixed type x_1, aggregate output $Y(\alpha)$ is a linear function of $\alpha_1(x_1)$. This implies a bang-bang control: when the integrand is increasing in $\alpha_1(x_1)$, worker 1 applies to job A and when it is decreasing worker 1 applies to job B:

$$\int_0^{x_1} \big(f(x_1) - \alpha_2(x_2) f(x_2)\big) dG(x_2) + \int_{x_1}^1 \big(1 - \alpha_2(x_2)\big) dG(x_2) f(x_1) \gtreqless$$
$$\int_0^{x_1} \alpha_2(x_2) dG(x_2) f(1-x_1) + \int_{x_1}^1 \Big(f(1-x_1) - \big(1-\alpha_2(x_2)\big) f(1-x_2)\Big) dG(x_2)$$
$$\Rightarrow \alpha_i(x) = \begin{cases} 1 \\ 0\,. \end{cases}$$

The left hand side is the net output produced by worker 1 if she applies to job A. The first term is the probability that $x_2 < x_1$, in which case worker 1 produces $f(x_1)$ but this comes at the expense of worker 2 not producing $f(x_2)$ if she applies to form A, an event with probability $\alpha_2(x_2)$. The second term is the probability that $x_2 > x_1$, in which case worker 1 produces $f(x_1)$ whenever worker 2 does not apply to job A. Straightforward differentiation shows that the left hand side is an increasing function of x_1. The right hand side gives similar expressions if worker 1 applies to job B and is easily shown to be a decreasing function of x_1. This implies a threshold rule, worker 1 applies to job A if $x_1 > x_1^*$ and to job B otherwise. The first order condition for the choice of $\alpha_2(x_2)$ is symmetric and also implies a threshold rule. □

This characterization of the optimal policy as a threshold reduces the dimensionality of the social planner's problem from two functions $\alpha_i(x)$ to two numbers x_i^*. Assuming without loss of generality that $x_1^* \leq x_2^*$ and substituting the threshold rules into equation (4) for aggregate output gives

$$y(x_1^*, x_2^*) = 2\int_0^{x_1^*} f(1-x)\big(1-G(x)\big) dG(x)$$
$$+ \int_{x_1^*}^{x_2^*} \big(f(1-x)(1-G(x_1^*)) + f(x)G(x_2^*)\big) dG(x) + 2\int_{x_2^*}^1 f(x)G(x) dG(x).$$

For either worker, if her type is below x_1^*, she applies to job B and is hired as long as the other worker's type is bigger than x, producing $f(1-x)$. Worker 2 still applies to job B if her type is between x_1^* and x_2^*, in which event she is hired as long as worker 1's type is above x_1^*. Worker 1 applies to job A in this range, and is hired if worker 2's type is below x_2^*, producing $f(x)$. Finally, if either worker's type is above x_2^*, she applies to job A and is hired if the other worker's type is lower.

The first order conditions from this optimization problem are

$$y_1(x_1^*, x_2^*) \propto f(1-x_1^*)(1-G(x_1^*)) - f(x_1^*)G(x_2^*) - \int_{x_1^*}^{x_2^*} f(1-x)dG(x) = 0 \tag{5}$$

$$y_2(x_1^*, x_2^*) \propto f(1-x_2^*)(1-G(x_1^*)) - f(x_2^*)G(x_2^*) + \int_{x_1^*}^{x_2^*} f(x)dG(x) = 0, \tag{6}$$

where the subscript i denotes the partial derivative with respect to x_i^*. If $y_1(x, x_2^*) < 0$ for all x, x_1^* is driven to the extreme value of zero, with similar generalizations to handle other corner solutions.

Lemma 2. *In any social optimum, the thresholds are symmetric about $\frac{1}{2}$: $x_1^* + x_2^* = 1$.*

Proof. Define $X_1(x_1^*)$ by $y_1(x_1^*, X_1(x_1^*)) = 0$ and $X_2(x_1^*)$ by $y_2(x_1^*, X_2(x_1^*)) = 0$. One can show by implicit differentiation that both functions are strictly decreasing and hence well-defined. In any social optimum, the first order conditions (5) and (6) are satisfied, and whenever both first order conditions are satisfied, $X_1(x_1^*) = X_2(x_1^*)$. I will prove that any solution to the first order conditions must be symmetric by showing that if $X_1(x_1^*) \gtreqless 1-x_1^*$, $X_2(x_1^*) \lesseqgtr 1-x_1^*$, so $X_1(x_1^*) = X_2(x_1^*)$ if and only if $X_1(x_1^*) = 1 - x_1^*$.

First assume that $1 - x_1^* > X_1(x_1^*)$; the three cases can be handled with identical proofs. Observe that $y_1(x_1^*, x_2^*)$ is a strictly decreasing function of x_2^*, so $y_1(x_1^*, 1-x_1^*) < y_1(x_1^*, X_1(x_1^*))$. Since this last expression is equal to zero by definition, we get

$$f(1-x_1^*)(1-G(x_1^*)) - f(x_1^*)G(1-x_1^*) - \int_{x_1^*}^{1-x_1^*} f(1-x)dG(x) < 0$$

In order to find a contradiction, I now assume that $1 - x_1^* \geq X_2(x_1^*)$. Since $y_2(x_1^*, x_2^*)$ is also a strictly decreasing function of x_2^*, this implies $y_2(x_1^*, 1-x_1^*) \leq y_2(x_1^*, X_2(x_1^*)) = 0$:

$$f(x_1^*)(1-G(x_1^*)) - f(1-x_1^*)G(1-x_1^*) + \int_{x_1^*}^{1-x_1^*} f(x)dG(x) \leq 0$$

Symmetry of G implies that $G(1-x_1^*) = 1 - G(x_1^*)$, and so the sum of the two inequalities is

$$\int_{x_1^*}^{1-x_1^*} \big(f(x) - f(1-x)\big)dG(x) < 0$$

But symmetry of G also implies that the left hand side of this inequality is equal to zero, a contradiction. I conclude that whenever $x_1^* + X_1(x_1^*) < 1$, $x_1^* + X_2(x_1^*) > 1$. Identical arguments for the other two cases complete the proof. □

This lemma further reduces the social planner's problem to a single control variable, the choice of $x_1^* \leq \frac{1}{2}$, with $x_2^* = 1 - x_1^*$. The remaining results follow from examining the first order conditions of that optimization problem.

Proposition 1. *If $f(0) = 0$, the social optimum involves some unemployment, $0 < x_1^* \leq \frac{1}{2}$, with $x_2^* = 1 - x_1^*$.*

Proof. Differentiate $\hat{y}(x_1^*) \equiv y(x_1^*, 1 - x_1^*)$ to get a necessary first order condition for an optimal threshold:

$$\hat{y}'(x_1^*) = 2G'(x_1^*)\left(\big(f(1 - x_1^*) - f(x_1^*)\big)\big(1 - G(x_1^*)\big) - \int_{x_1^*}^{1-x_1^*} f(x)dG(x)\right) \tag{7}$$

Evaluate this at $x_1^* = 0$ and simplify:

$$\hat{y}'(0) = 2G'(0)\left(f(1) - f(0) - \int_0^1 f(x)dG(x)\right)$$

Since $f(x) < f(1)$ for all $x \in [0, 1)$,

$$\int_0^1 f(x)dG(x) < f(1)\int_0^1 dG(x) = f(1)$$

or $\hat{y}'(0) > -2G'(0)f(0)$. If $f(0) = 0$, this implies $\hat{y}'(0)$ is positive, so raising x_1^* slightly above 0 must increase output. □

The exact amount of unemployment depends on the thresholds x_1^* and x_2^*. The unemployment rate is

$$u = \tfrac{1}{2}\big(G(x_1^*)G(x_2^*) + (1 - G(x_1^*))(1 - G(x_2^*))\big).$$

With probability $G(x_1^*)G(x_2^*)$, both workers realize a type below their thresholds and apply to job B, in which case one is unemployed. With probability $(1 - G(x_1^*))(1 - G(x_2^*))$, both workers apply to job A, with the same result. Since $x_2^* = 1 - x_1^*$ and $G(1 - x_1^*) = 1 - G(x_1^*)$, this expression simplifies considerably to $u = G(x_1^*)(1 - G(x_1^*))$. In the relevant parameter range, $G(x_1^*) \leq \frac{1}{2} \leq G(x_2^*)$, and so this is an increasing function of x_1^*. In other words, the more similar are the thresholds, the higher is the unemployment rate. Unemployment is minimized — equal to zero — if $x_1^* = 0$ and it is maximized at $\frac{1}{4}$ if $x_1^* = \frac{1}{2}$. The next proposition examines conditions under which unemployment is maximized.

Proposition 2. *If* $\frac{f'(x)}{f(x)} \geq \frac{G'(x)}{G(x)}$ *for all* x*, the unique social optimum is symmetric,* $x_1^* = x_2^* = \frac{1}{2}$*. If* $2f(\frac{1}{2})G'(\frac{1}{2}) > f'(\frac{1}{2})$*, the social optimum is asymmetric,* $x_1^* < \frac{1}{2} < x_2^*$*.*

Proof. First assume $\frac{f'(x)}{f(x)} \geq \frac{G'(x)}{G(x)}$ for all x. Equation (7) implies $\hat{y}'(\frac{1}{2}) = 0$. Differentiating indicates that

$$\begin{aligned}
\hat{y}''(\tfrac{1}{2}) &= 2G'(\tfrac{1}{2})\big(2f(\tfrac{1}{2})G'(\tfrac{1}{2}) - f'(\tfrac{1}{2})\big) \leq 0, \\
\hat{y}'''(\tfrac{1}{2}) &= 8f'(\tfrac{1}{2})G'(\tfrac{1}{2})^2 > 0,
\end{aligned}$$

where the first inequality uses the assumption evaluated at $x = \frac{1}{2}$ (recall $G(\frac{1}{2}) = \frac{1}{2}$). Since the second derivative of $\hat{y}(x_1^*)$ is weakly negative and the third derivative is strictly positive at $x_1^* = \frac{1}{2}$, this represents a local maximum on the restricted $x_1^* \leq \frac{1}{2}$.

To show that $\frac{1}{2}$ is a global maximum when $\frac{f'(x)}{f(x)} \geq \frac{G'(x)}{G(x)}$, it suffices to preclude any local minima at $x_1^* < \frac{1}{2}$. I do this by proving that whenever $\hat{y}'(x_1^*) = 0$, $\hat{y}''(x_1^*) < 0$:

$$\begin{aligned}
\hat{y}''(x_1^*) = 2G''(x_1^*) &\left(\big(f(1-x_1^*) - f(x_1^*)\big)\big(1 - G(x_1^*)\big) - \int_{x_1^*}^{1-x_1^*} f(x)dG(x) \right) \\
&+ 2G'(x_1^*)\Big(2f(x_1^*)G'(x_1^*) - \big(f'(x_1^*) + f'(1-x_1^*)\big)\big(1 - G(x_1^*)\big)\Big).
\end{aligned}$$

If $\hat{y}'(x_1^*) = 0$, the first line evaluates to zero. In addition, if $x_1^* < \frac{1}{2}$, the second line is strictly negative, as can be seen from a simple series of inequalities:

$$\begin{aligned}
\big(f'(x_1^*) + f'(1-x_1^*)\big)\big(1 - G(x_1^*)\big) &> f'(x_1^*)G(x_1^*) + f'(1-x_1^*)G(1-x_1^*) \\
&\geq f(x_1^*)G'(x_1^*) + f(1-x_1^*)G'(1-x_1^*) \\
&> 2f(x_1^*)G'(x_1^*).
\end{aligned}$$

The first inequality follows from the symmetry assumptions on G, $G(1-x_1^*) = 1 - G(x_1^*)$ and $G(x_1^*) < G(1-x_1^*)$ when $x_1^* < \frac{1}{2}$. The second uses the Proposition's assumption on the relationship between f and G. The third again uses symmetry of G, $G'(1-x_1^*) = G'(x_1^*)$, and monotonicity of f, $f(1-x_1^*) > f(x_1^*)$ for $x_1^* < \frac{1}{2}$.

If $2f(\frac{1}{2})G'(\frac{1}{2}) > f'(\frac{1}{2})$, $\hat{y}'(\frac{1}{2}) = 0$ and $\hat{y}''(\frac{1}{2}) > 0$, so the symmetric solution represents a local minimum of the planner's problem. □

5 Decentralization

5.1 Second Price Auction

This section examines a simple method of decentralizing the social optimum. If one type x worker shows up at a job, she keeps the entire output, $f(x)$ if job A and $f(1-x)$ if job B. If two workers show up, the more productive worker is hired, but she keeps only the incremental output. That is, if $x > y$ show up at job A, x is hired but earns only $f(x) - f(y)$. If they show up at job B, y is hired and earns $f(1-y) - f(1-x)$. This is effectively the outcome of a second price auction with a zero reserve bid in which the workers bid for jobs (Julien, Kennes and King 2000). I look for a Nash equilibrium, in which each worker decides where to apply as a function of her own type, taking the other worker's strategy as given. Note that each worker only knows her own type and the distribution G from which the other worker's type is drawn.

Suppose worker 2 uses the strategy 'apply to job A with probability $\alpha_2(x)$ if my type is x'. Then if worker 1 applies to job A, her expected income is

$$u_1^A(x_1) = \int_0^{x_1} \big(f(x_1) - \alpha_2(x_2) f(x_2)\big) dG(x_2) + \int_{x_1}^1 \big(1 - \alpha_2(x_2)\big) dG(x_2) f(x_1).$$

If worker 2 has a lower type than worker 1, worker 1 gets $f(x_1)$ when worker 2 applies to job B and $f(x_1) - f(x_2)$ otherwise. If worker 2 has a higher type, worker 1 gets $f(x_1)$ when worker 2 applies to job B and nothing otherwise. Alternatively, if 1 applies for job B, her expected income is

$$u_1^B(x_2) = \int_0^{x_1} \alpha_2(x_2) dG(x_2) f(1-x_1) + \int_{x_1}^1 \Big(f(1-x_1) - \big(1 - \alpha_2(x_2)\big) f(1-x_2)\Big) dG(x_2).$$

$u_2^A(x)$ and $u_2^B(x)$ are defined in a similar manner.

Definition 2. *A decentralized equilibrium is a pair (α_1, α_2) that satisfies $\alpha_i(x) = 1$ if $u_i^A(x) > u_i^B(x)$ and $\alpha_i(x) = 0$ if $u_i^A(x) < u_i^B(x)$.*

Lemma 3. *Any decentralized is described by two thresholds x_i^*, where worker i applies to job A if $x_i > x_i^*$ and to job B if $x_i < x_i^*$, with an arbitrary tie-breaking rule.*

Proof. For fixed $\alpha_2(x_2)$, $u_1^A(x_1)$ is an increasing function of x_1 and $u_1^B(x_1)$ is a decreasing function of x_1, which immediately implies a threshold rule. □

Again, without loss of generality, assume $x_1^* \leq x_2^*$.[5] Worker 1 takes x_2^* as given and chooses a threshold x_1^* to satisfy $u_1^A(x_1^*) = u_1^B(x_1^*)$. With some algebra, this condition reduces to

$$f(x_1^*)G(x_2^*) = f(1-x_1^*)(1-G(x_1^*)) - \int_{x_1^*}^{x_2^*} f(1-x)dG(x).$$

This is identical to the social planner's first order condition (5). Similarly, worker 2 takes x_1^* as given and chooses a threshold x_2^* to satisfy $u_2^A(x_2^*) = u_2^B(x_2^*)$:

$$f(x_2^*)G(x_2^*) - \int_{x_1^*}^{x_2^*} f(x)dG(x) = f(1-x_2^*)(1-G(x_1^*)),$$

which is identical to the social planner's other first order condition (6).

A number of results follow immediately. I start with a version of the second welfare theorem.

Proposition 3. *The social optimum is a decentralized equilibrium.*

Proof. The first order conditions for the social planner's problem are necessary conditions for a social optimum and necessary and sufficient conditions for a decentralized equilibrium. □

On the other hand, the first welfare theorem does not necessarily hold, since the symmetric solution is always an equilibrium but is not necessarily socially optimal.

Proposition 4. *The symmetric solution $x_1^* = x_2^* = \frac{1}{2}$ is always a decentralized equilibrium.*

Proof. Follows immediately from $y_1(\frac{1}{2}, \frac{1}{2}) = y_2(\frac{1}{2}, \frac{1}{2}) = 0$. □

This result is intuitive, given the symmetry of the problem. If one player applies to job A whenever $x > \frac{1}{2}$ and otherwise to job B, the two jobs look identical to the other player if $x = \frac{1}{2}$, and so she follows the same rule.

Proposition 4 establishes that the symmetric solution $x_1^* = x_2^* = \frac{1}{2}$ is always a decentralized equilibrium, but this does not necessarily make it a reasonable solution to the decentralized economy. How would individual agents happen upon this symmetric solution? One possible answer is to look at a Tattonement adjustment process. Worker 1 takes worker 2's threshold x_2^* as given and chooses a best response $x_1^* = \chi_1(x_2^*)$, while worker 2 takes worker 1's threshold x_1^* as given and picks her best response, $x_2^* = \chi_2(x_1^*)$. This process

[5] If there is a decentralized equilibrium in which $x_1^* < x_2^*$, there must be another equilibrium in which the roles of the players is reversed. I ignore that trivial multiplicity in what follows.

repeats until convergence. The question is whether for an open set of initial conditions,[6] the process converges to the symmetric solution. The answer is affirmative if and only if the symmetric solution is a local maximum of the social planner's problem (see Proposition 2).

Proposition 5. *If* $2f(\frac{1}{2})G'(\frac{1}{2}) < (>) f'(\frac{1}{2})$*, the symmetric equilibrium is locally stable (unstable) under Tattonement adjustment dynamics.*

Proof. Recall that worker 1's best response function satisfies $y_1(\chi_1(x_2^*), x_2^*) = 0$ and worker 2's best response function satisfies $y_2(x_1^*, \chi_2(x_1^*)) = 0$. Under Tattonement dynamics, an initial pair of thresholds $(x_{1,n}^*, x_{2,n}^*)$ is updated to a new pair of thresholds $(x_{1,n+1}^*, x_{2,n+1}^*)$ via

$$x_{1,n+1}^* = \chi_1(x_{2,n}^*) \text{ and } x_{2,n+1}^* = \chi_2(x_{1,n}^*)$$

In particular, any decentralized equilibrium is a fixed point of this adjustment process. To examine convergence near a decentralized equilibrium, we linearize the pair of difference equations. Implicit differentiation of the first order conditions (5) and (6) yields

$$\chi_1'(x_2^*) = -\frac{\big(f(\chi_1(x_2^*)) + f(1-x_2^*)\big)G'(x_2^*)}{f'(\chi_1(x_2^*))G(x_2^*) + f'(1-\chi_1(x_2^*))\big(1-G(\chi_1(x_2^*))\big)}$$

$$\chi_2'(x_1^*) = -\frac{\big(f(x_1^*) + f(1-\chi_2(x_1^*))\big)G'(x_1^*)}{f'(\chi_2(x_1^*))G(\chi_2(x_1^*)) + f'(1-\chi_2(x_1^*))\big(1-G(x_1^*)\big)}$$

Since $\chi_1(\frac{1}{2}) = \chi_2(\frac{1}{2}) = \frac{1}{2}$,

$$\chi_1'(\tfrac{1}{2}) = \chi_2'(\tfrac{1}{2}) = -\frac{2f(\frac{1}{2})G'(\frac{1}{2})}{f'(\frac{1}{2})} \equiv k.$$

This implies that in a neighborhood of the symmetric steady state, the Tattonement dynamics may be approximated via[7]

$$\begin{bmatrix} x_{1,n+1}^* - \frac{1}{2} \\ x_{2,n+1}^* - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \cdot \begin{bmatrix} x_{1,n}^* - \frac{1}{2} \\ x_{2,n}^* - \frac{1}{2} \end{bmatrix} \equiv K \cdot \begin{bmatrix} x_{1,n}^* - \frac{1}{2} \\ x_{2,n}^* - \frac{1}{2} \end{bmatrix}$$

[6] The initial conditions need not be thresholds, but arbitrary behavior $\alpha_i(x)$. The first iteration of the Tattonement process will yield threshold rules, however, since a best response is always a threshold. I ignore that first iteration for notational simplicity.

[7] Observe that these dynamics respect the ordering of $x_{i,n}^*$. That is, all my arguments are predicated on the normalization that $x_1^* \leq x_2^*$. If $x_{1,n}^* \leq x_{2,n}^*$, then $k(x_{1,n}^* - \frac{1}{2}) \geq k(x_{2,n}^* - \frac{1}{2})$ (since $k < 0$), and hence $x_{1,n+1}^* \leq x_{2,n+1}^*$.

The eigenvalues of the transition matrix K are $\pm k$. If $2f(\frac{1}{2})G'(\frac{1}{2}) < f'(\frac{1}{2})$, both lie within the unit circle, which indicates that the linearized system of difference equations is stable, i.e. for any initial condition $(x^*_{1,0}, x^*_{2,0})$, the Tattonement dynamics converge to the steady state $(\frac{1}{2}, \frac{1}{2})$. Of course, because this is a linearized system, the argument is only true for initial conditions in a neighborhood of the steady state. If the inequality is reversed, the linearized system is unstable. This implies that the Tattonement dynamics will not converge to the symmetric steady state unless the initial condition is exactly at the steady state. □

I now turn to results that correspond to the main propositions characterizing the social optimum. The first result parallels Lemma 2 and has a virtually identical proof, which is omitted.

Lemma 4. *Any decentralized equilibrium is symmetric about $\frac{1}{2}$: $x_1^* + x_2^* = 1$.*

Next, under conditions that ensure the extremal solution $x_1^* = 0 = 1 - x_2^*$ is not a local maximum of the social planner's problem (Proposition 1), it is also not a decentralized equilibrium:

Proposition 6. *If $f(0) = 0$, any decentralized equilibrium involves some unemployment, $0 < x_1^* \leq \frac{1}{2}$, with $x_2^* = 1 - x_1^*$.*

Proof. Suppose $x_2^* = 1$. Then $u_1^A(0) = f(0)$ and $u_1^B(0) = f(1) - \int_0^1 f(1-x)dG(x)$. Since f is increasing, $f(1-x) < f(1)$, which implies $u_1^B(0) > 0$. If $f(0) = 0$, worker 1 prefers to apply to job B when her type is zero — at least it gives her a chance of earning something. This proves $x_1^* = 0 = 1 - x_2^*$ is not an equilibrium. □

A third important result parallels the first part of Proposition 2. If the only local maximum of the social planner's problem is the symmetric solution $x_1^* = x_2^* = \frac{1}{2}$, then that is also the unique equilibrium of the decentralized economy.

Proposition 7. *If $\frac{f'(x)}{f(x)} \geq \frac{G'(x)}{G(x)}$ for all x, the symmetric equilibrium is unique.*

Proof. Since $x_2^* = 1 - x_1^*$ in any equilibrium, we may write the condition for an equilibrium simply as

$$\Phi(x_1^*) \equiv \big(f(x_1^*) - f(1-x_1^*)\big)(1 - G(x_1^*)) + \int_{x_1^*}^{1-x_1^*} f(1-x)dG(x) = 0,$$

Clearly $\Phi(\frac{1}{2}) = 0$. Differentiating gives

$$\Phi'(x_1^*) = \big(f'(x_1^*) + f'(1 - x_1^*)\big)(1 - G(x_1^*)) - 2f(x_1^*)G'(x_1^*).$$

Following the proof of Proposition 2, this is strictly positive when $x_1^* < \frac{1}{2}$, which implies $\Phi(x_1^*) < 0$ when $x_1^* < \frac{1}{2}$. □

Finally, note the choice of threshold rules form a supermodular game (Milgrom and Roberts 1990, Vives 1990), since after a trivial transformation of variables, the thresholds are strategic complementarities: an increase in x_1^* raises the best response $-x_2^*$, while an increase in $-x_2^*$ raises the best response x_1^*. A variety of corollaries follow immediately from those papers:

Proposition 8. *If $\frac{f'(x)}{f(x)} \geq \frac{G'(x)}{G(x)}$ for all x, the threshold rules $x_i^* = \frac{1}{2}$ are the unique serially undominated strategy profiles (i.e. the unique strategies that survive iterated elimination of strictly dominated strategies), the unique rationalizable strategy profiles (Bernheim 1984, Pearce 1984), and the unique solution to a variety of adaptive dynamics including best response dynamics.*

5.2 Alternative Decentralizations

The 'second price auction' is an important aspect of the decentralization. To show this, I consider two alternative plausible decentralizations.

First, suppose that a worker keeps all the output from a job if she is hired and nothing otherwise. Applying to job A gives $A_i(x)f(x)$ and applying to job B gives $B_i(x)f(1-x)$. It follows that workers use threshold rules, but now the thresholds must satisfy $x_1^* \leq x_2^*$,

$$f(x_1^*)G(x_2^*) = f(1 - x_1^*)(1 - G(x_1^*)), \text{ and } f(x_2^*)G(x_2^*) = f(1 - x_2^*)(1 - G(x_1^*)).$$

These thresholds are generally not socially optimal. To see what they *are*, combine the equations to get

$$\frac{f(x_1^*)}{f(1 - x_1^*)} = \frac{f(x_2^*)}{f(1 - x_2^*)} = \frac{1 - G(x_1^*)}{G(x_2^*)}.$$

If $x_1^* < x_2^*$, monotonicity of f ensures that the first equality is violated. That is, $x_1^* = x_2^*$. But since $f(x)/f(1-x)$ is an increasing function and $(1-G(x))/G(x)$ is a decreasing function, the second equality can only be satisfied at $x_1^* = x_2^* = \frac{1}{2}$. In other words, the unique decentralized equilibrium is symmetric in this case. If the social optimum is also symmetric, the first and second welfare theorems hold, while if it is asymmetric, the theorems are violated.

Alternatively, suppose that job owners (hereafter 'firms') sell jobs using a first price auction. That is, when a worker applies for a job, she must submit a wage demand without knowing whether the other worker has applied for the job or, if she has applied, what her type is. The firm observes the workers' types and hires the worker who promises it the most profit, output minus wage, assuming this maximal profit is nonnegative. One might conjecture a revenue equivalence theorem would apply in this environment, so that a first price and second price auction would have the same outcome.

If the optimal thresholds are equal, $x_1^* = x_2^* = \frac{1}{2}$, then this conjecture is correct. In a second price auction, a type $x \geq \frac{1}{2}$ worker is employed at job A when the other worker has a lower type, with probability $G(x)$, in which case she earns $f(x)$ if the other worker's type is less than $\frac{1}{2}$ and $f(x) - f(y)$ if the other worker's type $y \in (\frac{1}{2}, x)$. The expected income of a type $x \geq \frac{1}{2}$ conditional on getting a job in a second price auction is therefore

$$w(x) \equiv f(x) - \frac{1}{G(x)} \int_{\frac{1}{2}}^{x} f(y) dG(y).$$

Symmetrically, the expected income of a type $x < \frac{1}{2}$ worker is $w(1 - x)$.

In a first price auction, there is an equilibrium in which both workers demand this wage. To see this, note first that these wage demands leave firm A with profit

$$\pi(x) \equiv \frac{1}{G(x)} \int_{\frac{1}{2}}^{x} f(y) dG(y)$$

from hiring a type $x \geq \frac{1}{2}$ worker, increasing in x. In equilibrium, therefore, the most productive applicant is the most profitable applicant. Now suppose worker 1 with type $x_1 \geq \frac{1}{2}$ deviates, applying to job A but demanding a wage w' such that $f(x_1) - w' = f(x') - w(x')$ for some x'.[8] Assuming worker 2 plays the proposed equilibrium strategy, the deviating worker gets job A with probability $G(x')$, and so her expected utility is

$$w' G(x') = f(x_1) G(x') - \int_{\frac{1}{2}}^{x'} f(y) dG(y) \equiv u(x_1 | x').$$

[8] $w' > f(x_1)$ is always rejected, and so is dominated by $w' = f(x_1)$. $w' < f(x_1) - f(1) + w(1)$ is always accepted, but is dominated by demanding exactly $f(x_1) - f(1) + w(1)$. Also, applying to job B when $x_1 > \frac{1}{2}$ is worse than demanding $w' = f(x_1)$ from job A.

Differentiate this expression with respect to x':

$$\frac{du(x_1|x')}{dx'} = \big(f(x_1) - f(x')\big)G'(x')$$

Since f is increasing, $u(x_1|x')$ is increasing in x' for $x' < x_1$, decreasing for $x' > x_1$, and hence maximized at $x' = x_1$. This implies that it is optimal for a type x_1 worker to bid $w(x_1)$, confirming that the proposed wage demands form an equilibrium strategy.

But when the social optimum is asymmetric, the analogy between a first and second price auction breaks down. Take an extreme example, $x_1^* = 0$ and $x_2^* = 1$, so worker 1 always applies to job A and worker 2 always applies to job B. This cannot be decentralized through a standard first price auction. To see why, consider the optimal wage demands in a first price auction. Knowing that she will be the only applicant, each worker demands her full productivity. But this opens the doors to deviations. Take worker 1 and suppose $x_1 < \frac{1}{2}$. If she applies to job A, she gets $f(x_1) < f(\frac{1}{2})$. But if instead she applies to job B and demands a wage of $f(\frac{1}{2})$, leaving profit $f(1 - x_1) - f(\frac{1}{2})$ to the firm, firm B will hire worker 1 no matter what value x_2 takes. This breaks the proposed equilibrium.

In principle, one could prevent worker 1 from applying to job B even when it is her comparative advantage by handicapping the auction against her, but such handicapping is unlikely to occur in a decentralized economy. If a firm could handicap a first price auction, it would raise more revenue by handicapping the auction *in favor* of the low valuation bidder, worker 1 at job B, thereby encouraging the worker to apply for the job. Moreover, if a firm has the commitment power to handicap an auction, so that it does not necessarily hire the worker promising it more profit, the firm could presumably also impose a positive reservation bid, turning down workers who do not promise it sufficient profit. Such an action would clearly not decentralize the social optimum.

6 Conclusion

I conclude by discussing the possibility of extending these results to more interesting environments: more than two workers and jobs; and a dynamic environment.

Consider first a replication of the economy described in this paper. There are $2n$ workers and $2n$ jobs. Each worker is described by a type, which is a point in a $2n - 1$ dimensional simplex that describes her comparative advantage in each of the $2n$ jobs. All of the policies described in this paper are feasible; simply break the economy up ex ante into n separate economies, each populated by two workers and two jobs. But by allowing for greater interaction between workers and jobs, it should be possible to perform still better. This suggests that replicating the economy will relax the conditions under which the symmetric strategies are optimal and form the unique equilibrium of the model economy. If correct, this would confirm the intuition that allowing workers to make use of their comparative advantage is increasingly important in a larger economy.

Next, what happens if the assignment problem described here is repeated? Can this model generate a quantitatively significant coexistence of unemployment and vacancies if workers can quickly turn around and apply for another job? As written, the answer is probably no. But suppose that a worker observes her comparative advantage imperfectly before she applies for a job. Then she may find it optimal to apply for a job that ex ante looks promising, but later to refuse the job because ex post she realizes that there are better possibilities. This suggests that a careful model of the assignment problem with incomplete information may yield an interesting macroeconomic theory of unemployment.

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