The Asset Pricing Implications of Investment in Human and Physical Capital

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Motivation

“Why is relative distress a state variable of special hedging concern to investors? One possible explanation is linked to human capital... [A] negative shock to a distressed firm more likely implies a negative shock to the value of specialized human capital... Thus, workers [may] avoid the stocks of distressed firms.”

—“Multifactor Explanations of Asset Pricing Anomalies”, 1996, Fama and French
Motivation

I use the approach of Davis et al (2006).

Volatility of Employment Growth

Volatility of Employment Growth

Small

Large
Pushing beyond Value-Growth, Small-Large...

Figure 2 from Mendez and Sepulveda (2012). The authors also highlight important differences between high-skilled and low-skilled individuals as well as between the employed and unemployed.
Some recent literature which investigates the asset pricing implications of capital allocation or (more generally) asset pricing in a production economy:

There is also a (less extensive) literature which studies the implication of labor supply and labor income on asset pricing:
Veronesi and Santos (2005),...

This research shows that production and labor income have important implications for asset returns and the predictability of returns. They are also able to match key moments of asset markets using models with production and labor
I hope to contribute much of this work by highlighting some of the mechanisms through which investment in both human and physical capital affect asset prices.

For example, during a recession negative shocks to human capital or increasing uncertainty about the value human capital could make holding equities less attractive and lead to (counter-cyclical) variations in the equity premium. This may be a plausible alternative to time-varying risk-aversion.

Some of the questions I would like to address are as follows:

1. What are the implications for returns and predictability of returns?
2. What is the effect of a negative productivity shock on future human capital accumulation, wage, return on capital, equity and bond returns?
3. What are the interactions between human capital income, physical capital income and equity returns?
Primitives

The model uses elements from

Blanchard (1985), Panageas and Garleanu (2010): OLG so that no group of agents accumulates wealth indefinitely. The endogenous processes will be bounded and converge to stationary distributions

Panageas and Garleanu (2010): I will use the Martingale approach. It yields a tractable framework

Eberly and Wang (2009): there are adjustment costs. Empirical evidence shows that stock market variations is due mostly to variations in prices rather than the level of capital. Introducing an adjustment cost implies a (time-varying) wedge, $q_t$, between the two ($V = qK$) which helps to better match data
The economy is populated with a continuum of agents.

Each agent faces a constant hazard rate of death $\pi > 0$ and a cohort of mass $\pi$ is born per unit of time.

The size of the cohort born at time $s < t$ is $\pi e^{-\pi(t-s)}$ and the total population size at date $t$ is $\int_{-\infty}^t \pi e^{-\pi(t-s)} = 1$.

A fraction $\nu_A$ of agents are “high-skilled” and will be labelled type A.

The remaining agents are of type B (“low-skilled”) and account for a fraction $\nu_B = 1 - \nu_A$ of the population.

The utility function at time $t$ for an agent of type $i \in \{A, B\}$ born at time $s$ is

$$u\left(c^i_{s,t}, h^i_{s,t}\right) = \log\left(c^i_{s,t}h^i_{s,t}\right)$$
The human capital at time $t$ for an agent of type $i \in \{A, B\}$ born at time $s$ follows

$$\frac{dh_{s,t}^i}{h_{s,t}^i} = \Gamma_i \log \left( 1 + \frac{\epsilon_{s,t}^i}{\theta_i h_{s,t}^i} \right) dt - \delta_i dt + \sigma_i dZ_t$$

$\epsilon_{s,t}^i$: investment in human capital on date $t$

$\delta_i$: depreciation rate

$\theta_i$: a scaling constant and $\Gamma_i \equiv \frac{\delta_i}{\log \left( 1 + \frac{\delta_i}{\theta_i} \right)}$

so that the expected growth rate of human capital equals zero when $\epsilon_{s,t}^i = \delta_i h_{s,t}^i$

Lastly, $\theta_A > \theta_B$ and $\delta_A < \delta_B$ so it is more costly for low-skilled agents to accumulate human capital
The two types of agents only differ in the parameters of the adjustment cost of human capital.

As $\theta_i \to \infty$ the adjustment cost goes to zero and $E_t\left(\frac{dh_{s,t}^i}{h_{s,t}^l}\right) \to \frac{e_{s,t}^i}{h_{s,t}^l} dt$.
The process for physical capital is similar to that of human capital but the parameters are independent of the agent type. A life insurance company also provides $\pi k_{s,t}^i$ to each agent in return for her physical capital at her death

$$\frac{d k_{s,t}^i}{k_{s,t}^i} = \Gamma \log \left(1 + \frac{e_{s,t}^i}{\theta k_{s,t}^i}\right) dt - \delta dt + \pi + \sigma_k dZ_t$$

$$\frac{d h_{s,t}^i}{h_{s,t}^i} = \Gamma_i \log \left(1 + \frac{e_{s,t}^i}{\theta_i h_{s,t}^i}\right) dt - \delta_i dt + \sigma_i dZ_t$$

$dZ_t$ introduces aggregate uncertainty and is a standard Brownian Motion increment.

For now, let us consider a scalar Brownian Motion increment, $dZ_t$. 
There is 1 firm in this economy and it uses human and physical capital in its production process.

For now, assume that the firm can perfectly identify the types of agents.

Aggregate production

\[ Y_t = AK_t^{1-\alpha_A-\alpha_B} (H_t^A)^{\alpha_A} (H_t^B)^{\alpha_B} \]

Aggregate physical capital

\[ K_t = \int_{-\infty}^{t} \pi e^{-\pi(t-s)} (v_A k_{s,t}^A + v_B k_{s,t}^B) \, ds \]

with \( k_{t,t}^A = k_{t,t}^B = K_0 \) for all \( t \)

Total human capital of type \( i \in \{A, B\} \)

\[ H_t^i = \int_{-\infty}^{t} \pi e^{-\pi(t-s)} h_{s,t}^i \, ds \]

with \( H_0^i = h_{t,t}^i \) for all \( t \)
Agents have access to a security that allows to hedge against aggregate risk. The security price is $S_t$ and follows

$$\frac{dS_t}{S_t} = \mu_t dt + \varsigma_t dZ_t$$

The agents can invest in a risk free bond with return $r_t$ as well as the risky security.

$\varsigma_t$, $r_t$ and $\mu_t$ will be determined in equilibrium.

Markets are dynamically complete since $\text{rank}(\text{diag}(\varsigma_t))$ is equal to the number of elements in the $dZ_t$ vector.

At birth, each agent of type $i$ is endowed with physical capital $k_{s,s}^i = K_0$, human capital $h_{s,s}^i = H_0^i$ and financial wealth $W_{s,s}^i = W_0$.

Following Blanchard (1985) and Panageas and Garleanu (2010), $W_{s,s}^i = 0 \quad \forall s, i.$
The total wealth is $W_{s,t}^i + q_t k_{s,t}^i + p_t h_{s,t}^i$

$\{q_t, p_t^i\}$: value (per unit) of physical and human capital, determined in equilibrium.

Rental rate of physical capital is $r_{k,t}$ and the wage rate is $\omega_t^i$

The life insurance company also provides $\pi W_{s,t}^i$ to each agent in return for her financial wealth when at her death. Each agent’s financial wealth process is thus

$$dW_{s,t}^i = \left[ \omega_{s,t}^i W_{s,t}^i \mu_t + (1 - \omega_{s,t}^i) W_{s,t}^i r_t + \pi W_{s,t}^i - c_{s,t}^i - \epsilon_{s,t}^i - \epsilon_{s,t}^i \right] dt$$

$$+ [r_{k,t}^i k_{s,t}^i + \omega_t^i h_{s,t}^i] dt + \omega_{s,t}^i W_{s,t}^i \varsigma_t dZ_t$$

Here $\omega_{s,t}^i$: fraction of financial wealth invested in the risk security.

Aggregate financial wealth follows

$$dW_t = \pi W_0 dt - \pi W_t dt + \sum_{i \in \{A,B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left( dW_{s,t}^i \right) ds$$

new births

$$dW_t = \pi W_0 dt - \pi W_t dt + \sum_{i \in \{A,B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left( dW_{s,t}^i \right) ds$$

surviving agents
Static Constraint

∃ a unique state price density (SPD), $\Lambda_t$, such that $E_t (\Lambda_T S_T) = \Lambda_t S_t$

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \sigma_{\Lambda,t} dZ_t$$

Using Ito’s Lemma we can show that $\varsigma_t = \sigma_{\Lambda,t}$ and $\mu_t - r_t = \sigma_{\Lambda,t}^2$

Applying Ito’s Lemma to $\Lambda_t e^{-\pi(t-s)} (W_{s,t}^i + h_{s,t}^i p_t^i + k_{s,t}^i q_t)$ and integrating yields and using a transversality condition yields

$$W_{s,s}^i + h_{s,s}^i p_s^i + k_{s,s} q_s$$

$$= E_s \int_s^\infty \Lambda_t e^{-\pi(t-s)} \left( c_{s,t}^i + e_{s,t}^i + \epsilon_{s,t}^i - r_{k,t} k_{s,t}^i - \omega_t h_{s,t}^i - \pi W_{s,t}^i \right) dt$$

$$+ E_s \int_s^\infty \Lambda_t e^{-\pi(t-s)} p_t^i h_{s,t}^i \left( \frac{\psi_t^i}{p_t^i} - \Gamma_i \text{Log} \left( 1 + \frac{e_{s,t}^i}{\theta_i h_{s,t}^i} \right) + \delta_i - \pi \right) dt$$

$$+ E_s \int_s^\infty \Lambda_t e^{-\pi(t-s)} q_t k_{s,t}^i \left( \frac{\psi_t^i}{q_t} - \Gamma \text{Log} \left( 1 + \frac{e_{s,t}^i}{\theta_i k_{s,t}^i} \right) + \delta - \pi \right) dt$$
Agent Problem

Each agent solves

$$\max_{\{c^i_{s,t}, e^i_{s,t}, e^i_{s,t}, h^i_{s,t}, k^i_{s,t}\}_{t=s}^\infty} e^{\pi s} E_s \int_s^\infty e^{-(\rho + \pi) t} \log (c^i_{s,t} h^i_{s,t}) \, dt$$

s.t.:

$$W^i_{s,s} + h^i_{s,s} p_s + k^i_{s,s} q_s =$$

$$E_s \int_s^\infty \Lambda_t e^{-\pi (t-s)} \left( c^i_{s,t} + e^i_{s,t} + e^i_{s,t} - r^i_{k,t} k^i_{s,t} - \omega_t h^i_{s,t} \right) \, dt$$

$$+ E_s \int_s^\infty \Lambda_t e^{-\pi (t-s)} p^i_t h^i_{s,t} \left( \frac{\bar{\psi}^i_t}{p^i_t} - \Gamma_i \log \left( 1 + \frac{c^i_{s,t}}{\theta^i h^i_{s,t}} \right) + \delta_i - \pi \right) \, dt$$

$$+ E_s \int_s^\infty \Lambda_t e^{-\pi (t-s)} q^i_t k^i_{s,t} \left( \frac{\psi^i_t}{q^i_t} - \Gamma \log \left( 1 + \frac{e^i_{s,t}}{\theta k^i_{s,t}} \right) + \delta - \pi \right) \, dt$$
An equilibrium consists of a set of adapted processes
\(\{c^i_{s,t}, e^i_{s,t}, \epsilon^i_{s,t}, h^i_{s,t}, k^i_{s,t}, \omega^i_{s,t}, p^i_t, q_t, r_k, t, \omega^i_t\}\) \(\forall t\) such that

1. \(\{c^i_{s,t}, e^i_{s,t}, \epsilon^i_{s,t}, h^i_{s,t}, k^i_{s,t}, \omega^i_{s,t}\}\) solve the agent problem above

2. processes \(\{H^A_t, H^B_t, K_t\}\) maximize the firm's profit,
\[Y_t - r_k t K_t - \omega^A_t H^A_t - \omega^B_t H^B_t\]

3. market for goods clears:
\[
\sum_{i \in \{A,B\}} v_i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} (c^i_{s,t} + e^i_{s,t} + \epsilon^i_{s,t}) \, ds = Y_t \] where
\[Y_t = A K_t^{1-\alpha_A-\alpha_B} (H^A_t)^{\alpha_A} (H^B_t)^{\alpha_B}\]

4. market for human and physical capital clears: \(\int_{-\infty}^{t} \pi e^{-\pi(t-s)} h^i_{s,t} \, ds = H^i_t\) for \(i \in \{A, B\}\) and \(\sum_{i \in \{A,B\}} v_i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} k^i_{s,t} \, ds = K_t\)

5. market for risky security clears:
\[
\sum_{i \in \{A,B\}} v_i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \omega^i_{s,t} W^i_{s,t} \, ds = S_t
\]

6. market for bonds clears (zero net bond holdings):
\[
\sum_{i \in \{A,B\}} v_i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} W^i_{s,t} (1 - \omega^i_{s,t}) \, ds = 0\]


Solution

The optimality conditions from the firm problem are

\[ r_{k,t} = (1 - \alpha_A - \alpha_B)Y_t/K_t \]

\[ \omega_t^i = \alpha_i Y_t/H_t^i \]

Agent FOCs for investment in physical and human capital \((e_{s,t}^i \text{ and } \epsilon_{s,t}^i)\):

\[ \frac{e_{s,t}^i}{k_{s,t}^i} = \Gamma q_t - \theta \quad \Rightarrow \quad e_{s,t}^i \geq 0 \quad \text{iff} \quad q_t \geq \frac{\theta}{\Gamma} \]

\[ \frac{\epsilon_{s,t}^i}{h_{s,t}^i} = \Gamma_i p_t - \theta_i \quad \Rightarrow \quad \epsilon_{s,t}^i \geq 0 \quad \text{iff} \quad p_t \geq \frac{\theta_i}{\Gamma_i} \]
Thus, $p_t^i$ and $q_t$ are the Tobin Q from Q-Theory.

Taking the ratio of the FOCs for $c_{s,t}^i$ and $h_{s,t}^i$ yields

\[
\bar{\psi}_t^i = \omega_t^i + \frac{c_{s,t}^i}{h_{s,t}^i}
- \left( -p_t \left( \Gamma_i \log \left( p_t \frac{\Gamma_i}{\theta_i} \right) - \delta_i \right) + p_t \Gamma_i - \theta_i \right)
\]

where the price of human capital is

\[
p_t^i = E_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} \bar{\psi}_s^i \, ds
\]

The FOC for $k_{s,t}^i$ implies

\[
\psi_t = r_{k,t}^i
- \left( -q_t \left( \Gamma \log \left( q_t \frac{\Gamma_k}{\theta_k} \right) - \delta \right) + q_t \Gamma - \theta \right)
\]

where the price of physical capital is

\[
q_t = E_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} \psi_s \, ds
\]
Properties of the Equilibrium

\[
\frac{c^i_{s,u}}{c^i_{s,t}}, \frac{h^i_{s,u}}{h^i_{s,t}}, \frac{e^i_{s,u}}{e^i_{s,t}}, \frac{e^i_{s,u}}{e^i_{s,t}}, \frac{k^i_{s,u}}{k^i_{s,t}} \text{ are independent of } s. \text{ This allows for aggregation}
\]

\[q_t, \text{ depends only on time; } p^i_t, \text{ depends on time and the agent type}\]

Aggregate human and physical capital will

(endogenously) follow mean-reverting processes

\[
dH^i_t = \left(\pi - \Gamma_i \log \left(\frac{p^i_t \Gamma_i}{\theta_i}\right) + \delta_i\right) \left(\frac{\pi H^i_0}{\pi - \Gamma_i \log \left(\frac{p^i_t \Gamma_i}{\theta_i}\right) + \delta_i} - H^i_t\right) dt + H^i_t \sigma_i dZ_t
\]

\[
dK_t = \left(\pi - \Gamma \log \left(q_t \frac{\Gamma}{\theta}\right) + \delta\right) \left(\frac{\pi K_0}{\pi - \Gamma \log \left(q_t \frac{\Gamma}{\theta}\right) + \delta} - K_t\right) dt + K_t \sigma_k dZ_t
\]

\[k^A_{s,t} = k^B_{s,t} \ \forall\{s, t\} \text{ and } e^A_{s,t} = e^B_{s,t} \ \forall\{s, t\}: \text{ this feature of the model will allow us to focus on the effect of differences in human capital of high and low-skill agents while still permitting to study the effects of investment in physical capital at the aggregate level.}\]
Summarizing the Solution

given \( H_A^0, H_B^0, K_0, p_0, q_A^0, \) and \( q_B^0 \). we can sequentially update the variables \( C_t, H_t, K_t, p_t^i \) and \( q_t \) using

\[
\begin{align*}
\frac{dH_t^i}{dt} &= \pi \left( 1 - \frac{\Gamma_i}{\pi} \log \left( \frac{p_t^i}{\theta_i} \right) \right) \left( \frac{H_0^i}{1 - \frac{\Gamma_i}{\pi} \log \left( \frac{p_t^i}{\theta_i} \right) + \frac{\delta_i}{\pi}} - H_t^i \right) dt + H_t^i \sigma_i dZ_t \\
\frac{dK_t}{dt} &= \pi \left( 1 - \frac{\Gamma}{\pi} \log \left( \frac{q_t}{\theta} \right) \right) \left( \frac{K_0}{1 - \frac{\Gamma}{\pi} \log \left( \frac{q_t}{\theta} \right) + \frac{\delta}{\pi}} - K_t \right) dt + K_t \sigma_k dZ_t \\
\frac{dq_t}{dt} &= \left( -r_{k,t} + q_t \left( r_t + \Gamma + \delta - \Gamma \log \left( \frac{q_t}{\theta} \right) \right) - \theta + q_t \sigma_{\Gamma,t} \right) dt + q_t \sigma_{\Gamma,t} dZ_t \\
\frac{dp_t^i}{dt} &= \left( -\frac{\rho + \pi}{f_t^i} + p_t^i \left( r_t + \Gamma_i + \delta_i - \Gamma_i \log \left( \frac{p_t^i}{\theta_i} \right) \right) - \theta h - \omega_t + p_t^i \sigma_{\Gamma,t} \right) dt + p_t^i \sigma_{\Gamma,t} dZ_t \\
\frac{d\epsilon_t^i}{dt} &= H_t^i (\Gamma_i p_t^i - \theta_i), \quad e_t = K_t (\Gamma q_t - \theta), \quad C_t = Y_t - e_t - \sum_{i \in \{A,B\}} v_i \epsilon_t^i
\end{align*}
\]
\[ f_t^i = \frac{h^i_{s,t}}{\tilde{W}^i_{s,t}} p_t^i \] solves a linear PDE in \( \{ H^A_t, H^B_t, K_t, p^A_t, p^B_t, q_t \} \).

These PDEs are \textbf{linear}, and \textbf{non-stochastic}. They are not difficult to solve.

We can stack the PDEs into one equation and solve all at once (e.g. Miranda-Fackler).

Here, \( \tilde{W}^i_{s,t} \) is financial wealth plus the present value of the earnings stream:
\[
\tilde{W}^i_{s,t} \equiv W^i_{s,t} + \pi E_t \int_t^{\infty} e^{-\pi(\tau-t)} \frac{\Lambda_{\tau}}{\Lambda_t} \left( r_{k,\tau} k^i_{s,\tau} + \omega^i_{s,\tau} h^i_{s,\tau} - e^i_{s,\tau} - \epsilon^i_{s,\tau} \right) d\tau
\]
The Equity Premium

\[ E_t \left( \frac{dS_t}{S_t} \right) - r_t = \mu_t - r_t = \sigma^2_{\Lambda,t} \]

where

\[
\sigma_{\Lambda,t} = \frac{Y_t}{Y_t + H_t^Av_A\theta_A + H_t^Bv_B\theta_B + K_t\theta} \sigma_y + \frac{K_t(\Gamma q_t - \theta)}{Y_t + H_t^Av_A\theta_A + H_t^Bv_B\theta_B + K_t\theta} \sigma_k \\
+ \frac{H_t^A(\Gamma p_t^A - \theta_A) v^A}{Y_t + H_t^Av_A\theta_A + H_t^Bv_B\theta_B + K_t\theta} \sigma_A + \frac{H_t^B(\Gamma p_t^B - \theta_B) v_B}{Y_t + H_t^Av_A\theta_A + H_t^Bv_B\theta_B + K_t\theta} \sigma_B
\]

\[
\Rightarrow \sigma_{\Lambda,t} = \frac{C_t \sigma_y + \epsilon_t^A (1 + v^A \sigma_A) + \epsilon_t^B (1 + v_B \sigma_B) + e_t (1 + \sigma_k)}{\pi \tilde{W}_t + H_t^Ap_t^A\Gamma A v_A + H_t^Bp_t^B\Gamma B v_B + K_tq_t\Gamma}
\]

The denominator is a weighted average of financial wealth, human capital wealth and physical capital wealth

\[
\sigma_y = (1 - \alpha_A - \alpha_B) \sigma_k + \alpha_A \sigma_A + \alpha_B \sigma_B
\]
Extensions

We can introduce idiosyncratic shocks in the human capital processes

\[
\frac{dh_{s,t}^i}{h_{s,t}^i} = \Gamma_i \log \left( 1 + \frac{\epsilon_{s,t}^i}{\theta_i h_{s,t}^i} \right) dt - \delta_i dt + \sigma_i dZ_t + \sqrt{\eta_t^i} dB_{s,t}^i
\]

\[
\eta_t^i = m(\bar{\eta} - \eta_t^i) dt - \sigma_{\eta} \sqrt{\eta_t^i} dZ_t
\]

\(\sigma_{\eta} > 0 \Rightarrow\) volatility will be counter cyclical, following some of the research from Bloom.

\(\eta_t^i\) has loading on the aggregate uncertainty. This links idiosyncratic and aggregate risk; akin to Constantinides and Duffie (1996), Di Tella (2012)

LLN implies that idiosyncratic risk will not matter unless there is some friction: firm cannot perfectly distinguish between the types of agents...
We could then study the implications of idiosyncratic risk on low-skill vs high-skill agents as well as the effects of uncertainty and volatility shocks on the wage gap, etc...

The utility function can be changed with little cost. The more general CRRA utility function would yield the same solution but with the coefficient of risk aversion appearing is some of the equations.

Since a CRRA utility function ties the agent’s IES to her risk aversion, we can instead use recursive preferences. More specifically using a Epstein-Zin Kreps-Porteus (EZ-KP) utility function as in Panageas and Garleanu (2010) can be done and would introduce one more PDE.
Conclusion

There is an increasing interest in studying the implications of investment in physical capital on asset pricing.

I hope to complement this research by investigating the joint effects of investment in physical and human capital on asset pricing.

I also hope to study the complementarity between investment in human and physical capital in a stochastic setting.

The framework presented above is tractable and provides some structure which helps formulate interesting questions and determine what to look for in data.
\[ r_t = \mu_y + \pi - \sigma^2_{\Lambda,t} - \sum_{i \in \{A,B\}} v_i \pi \beta^i_t \]

\[ + \sum_{i \in \{A,B\}} \left( \frac{\Gamma_i p^i_t - \theta_i}{Y_t} \right) \left( H^i_t \left( v^2_i \pi - v_i \Gamma_i \right) - v_i H^0_i \pi \right) \]

\[ + \frac{\Gamma q_t - \theta}{Y_t} \left( K_t \left( \pi \left( v^2_B + v^2_A \right) - \Gamma \right) - K_0 \pi \right) \]

\[ - \sum_{i \in \{A,B\}} \frac{H^i_t}{Y_t} v_i \Gamma_i \left( -\omega^i_t - \frac{\pi}{f^i_t} + p^i_t \sigma_{\Lambda,t} \sigma_{\Lambda,t}' \right) \]

\[ - \frac{K_t}{Y_t} \Gamma \left( -r_{k,t} + q_t \sigma_{\Lambda,t}' \sigma_{\Lambda,t} \right) \]

\[ \beta^i_t = \pi \phi^i_t + \frac{H^i_t \left( \Gamma_i p^i_t - \theta_i \right) + K_t \left( \Gamma q_t - \theta \right)}{Y_t} \]
\[ 0 = \sum_{i \in \{A, B\}} \frac{\partial \phi^i}{\partial H^i_t} H^i_t (\Gamma_ip^i_t - \theta_i + \sigma_i(\sigma_y + \sigma_{\Lambda,t})) + \sum_{i \in \{A, B\}} \frac{1}{2} \frac{\partial^2 \phi^i}{\partial (H^i_t)^2} H^i_t^2 \sigma^2_i \\
+ \frac{\partial \phi^i}{\partial K_t} K_t (\Gamma_q t - \theta + \sigma_k(\sigma_y + \sigma_{\Lambda,t})) + \frac{1}{2} \frac{\partial^2 \phi^i}{\partial (K_t)^2} K^2_t \sigma^2_k \\
+ \sum_{i \in \{A, B\}} \frac{\partial^2 \phi^i}{\partial H^i_t \partial K_t} H^i_t \sigma_i K_t \sigma_k + \frac{\partial^2 \phi^i}{\partial H^A_t \partial H^B_t} H^A_t \sigma_A \sigma_B H^B_t \sigma_B \\
+ \phi^i (-r_t + \mu_{y,t} - \sigma_{\Lambda,t}\sigma_y - \pi) \]
\[ 0 = \sum_{i \in \{A, B\}} \frac{\partial f^i}{\partial H^i_t} H^i_t (\Gamma p^i_t - \theta_i) + \sum_{i \in \{A, B\}} \frac{1}{2} \frac{\partial^2 f^i}{\partial (H^i_t)^2} (H^i_t)^2 \sigma^2_i \\
+ \frac{\partial f^i}{\partial K_t} K_t (\Gamma q_t - \theta) + \frac{1}{2} \frac{\partial^2 f^i}{\partial (K_t)^2} (K_t)^2 \sigma^2_k \\
+ \sum_{i \in \{A, B\}} \frac{\partial^2 f^i}{\partial H^i_t \partial K_t} H^i_t \sigma_i K_t \sigma_k \\
+ \frac{\partial^2 f^i}{\partial H^A_t \partial H^B_t} H^A_t \sigma_A \sigma_B H^B_t \sigma_B - (\rho + \pi)f^i + f^i_t (\omega^i_t - p^i_t \Gamma_i + \theta_i) + \pi \]
\[
\sigma_y = (1 - \alpha_A - \alpha_B)\sigma_k + \alpha_A\sigma_A + \alpha_B\sigma_B
\]

\[
\mu_{y,t} = \frac{(-1 + \alpha_A + \alpha_B) \left( -\pi K_0 + K_t \left( \pi + \delta - \Gamma \log \left[ \frac{q_t q}{\theta} \right] \right) \right)}{K_t} \\
+ \frac{\alpha_A \left( \pi H_0^A - H_t^A \left( \pi + \delta_A + \Gamma_A \log \left[ \frac{p_t^A q_A}{\theta_A} \right] \right) \right)}{H_t^A} \\
+ \frac{\beta \left( \pi H_0^B - H_t^B \left( \pi + \delta_B + \Gamma_B \log \left[ \frac{p_t^B q_B}{\theta_B} \right] \right) \right)}{H_t^B} \\
+ \frac{1}{2} (-1 + \alpha_A) \alpha_A \sigma_A^2 + \alpha_A \alpha_B \sigma_A \sigma_B + \frac{1}{2} (-1 + \alpha_B) \alpha_B \sigma_B^2 \\
+ (-\alpha_A (-1 + \alpha_A + \alpha_B) \sigma_A - \alpha_B (-1 + \alpha_A + \alpha_B) \sigma_B) \sigma_k \\
+ \frac{1}{2} (-1 + \alpha_A + \alpha_B) (\alpha_A + \alpha_B) \sigma_k^2
\]