The Effects Of Uncertainty And Learning On The Housing Cycle

Serginio Sylvain
Department of Economics
University of Chicago

October, 2012
Outline

Introduction
  Motivation

Model
  Non-Stationary Learning
  Stationary Learning
  Equilibrium
  Solution: Dynamic Programming Approach
  Solution steps
  Key Equations

Price of Housing
  The Mechanism
  Price of housing
  Elasticities
  Calibration

Summary/Extensions
Motivation

What caused the sharp boom and bust in the recent housing cycle?

Source: Figure 2.1 in Robert J. Shiller, "Irrational Exuberance", 2009

The relevant literature provides the following explanations:

- ** Insider vs Outsider:** Marzo and Duffie (1999), LaCour-Little (2004)
- ** Excessive lending and securitization of lenders:** Liu and Wang (2009), Keys et al (2008), Coval et al. (2007), Glaeser et al. (2010)
- ** Behavioural biases (e.g overly optimistic beliefs):** Foote, Gerardi, Willen (2012), Liu and Wang (2009)
- ** Heterogeneity in agents beliefs:** Liu and Wang (2009)
- ** Financial innovation:** Mulligan (2010), Favilukis, Luvigson, and Van Nieuwerburgh (2010), Atif and Sufi (2009)
- ** Limited Information/Learning:** Mulligan (2010), Liu and Wang (2009)

I hope to show that parameter learning by itself can help generate a pronounced housing cycle. The model extends Mulligan (2010) by incorporating a learning mechanism.
Figure 1: Index of Economic Policy Uncertainty

Source: Baker, Bloom, and Davis (2011)
Main Goals

Show that parameter learning can lead to substantial deviations of housing price from “fundamental” values

Quantify these effects of uncertainty and (parameter) learning on housing price.

Relate parameter uncertainty to housing market uncertainty (e.g. uncertainty about gov’t involvement in housing mkt, etc...)

Provide (empirical) evidence

Today, I would like to gauge the interest in such exercises
Model

Preferences: \( V_t = E_t \int_t^\infty e^{-As} \frac{(C_s H_s)^{1-\gamma}}{1-\gamma} ds \)

Agent is endowed with earnings \( y_t = \alpha Y_t \) (with \( \alpha \in (0, 1) \)) and \( \frac{dY_t}{Y_t} = \mu_y dt + \sigma_y dz_{y,t} \)

Agent can invest in a risk free bond with return \( r_t \) and a (risky) share of the representative firm (a “Lucas tree”), which pays out dividends \( D_t = (1 - \alpha) Y_t \)

Share price follows: \( dS_t = (\mu_t S_t - D_t) dt + \varsigma_t S_t dz_t \)

Budget constraint:
\[
\begin{aligned}
dW_t &= [r_t W_t + \omega_t W_t(\mu_t - r_t) + y_t - C_t - l_t - B_t H_t] dt + \omega_t W_t \sigma_{w,t} dz_t \\
LOM of housing: \quad dH_t &= \theta H_t ln \left( 1 + \frac{l_t}{\theta H_t} \right) dt - \delta H_t dt \\
Housing cost: \quad dB_t &= b(\bar{B} - B_t) dt + \sigma_b dz_{b,t}
\end{aligned}
\]
The agents do not know $\bar{B}$ but observe $B_t$ and receive a signal $\beta_t$.

$$d\beta_t = \bar{B}dt + \sigma_\beta dz_\beta,t$$

$dz_t = (dz_{y,t}; dz_{b,t}; dz_{\beta,t})$ is a three-dimensional standard BM increment under the complete-knowledge information set, $\mathcal{G}_t$.

The agents know all other parameters of the model and observe the realizations of all variables.

The agents form estimates $x_t = E(\bar{B}|\mathcal{F}_t)$; where $\mathcal{F}_t$ is the agent’s information set at time $t$

The agents update these estimates using the realizations of $\beta_t$ and $B_t$
Given \((x_0, O_0)\), the agents update their estimate of \(\bar{B}\) via Bayesian updating.

**Non-Stationary learning:** \(\bar{B}\) is constant

\[
dx_t = \left( O_t \frac{1}{\sigma_b} \right) d\tilde{z}_{b,t} + \left( O_t \frac{1}{\sigma_\beta} \right) d\tilde{z}_{\beta,t} \\
\]
\[
dO_t = - \left[ \left( \rho \sigma_\mu + O_t \frac{1}{\sigma_b} \right)^2 + \left( O_t \frac{1}{\sigma_\beta} \right)^2 \right] dt \\
\]
\[
d\tilde{z}_{\beta,t} = \frac{1}{\sigma_\beta} (d\beta_t - x_t dt) \\
\]
\[
d\tilde{z}_{b,t} = \frac{1}{\sigma_b} (dB_t - b(x_t - B_t) dt) \\
\]

\(O_t\) is both the posterior variance and the mean squared error
\[
O_t = E_t(\bar{B} - x_t)^2 = E_t(\bar{B} - E_t(\bar{B}))^2 = \frac{1}{\left( \frac{1}{\sigma_0} \right)^2 + t \left[ \left( \frac{1}{\sigma_b} \right)^2 + \left( \frac{1}{\sigma_\beta} \right)^2 \right]} \\
\]
\[
d\tilde{z}_t = (dz_{y,t}; d\tilde{z}_{b,t}; d\tilde{z}_{\beta,t})' \text{ is a three-dimensional standard BM increment under the agent information set, } \mathcal{F}_t.\]
Given \((x_0, O_0)\), the agents update their estimate of \(\bar{B}\) via Bayesian updating.

**Stationary learning: \(\bar{B}\) is stochastic**

\[
\begin{align*}
    d\bar{B} &= \vartheta (\mu_b - \bar{B}) dt + \sigma_\mu d\bar{z}_{\mu,t} \\
    dx_t &= \vartheta (\mu_b - x_t) dt + \left( O_t \frac{1}{\sigma_b} \right) d\bar{z}_{b,t} + \left( O_t \frac{1}{\sigma_\beta} \right) d\bar{z}_{\beta,t} + \rho \sigma_\mu d\bar{z}_{b,t} \\
    dO_t &= \left\{ \sigma_\mu^2 - 2\vartheta O_t - \left[ \left( \rho \sigma_\mu + O_t \frac{1}{\sigma_b} \right)^2 + \left( O_t \frac{1}{\sigma_\beta} \right)^2 \right] \right\} dt
\end{align*}
\]

In steady state \(dO_t = 0\) and \(O_t = \bar{O}\)

\[
\bar{O} = \frac{\sigma_\mu^2 (1 - \rho^2)}{\left( \vartheta + \rho \sigma_\mu \frac{1}{\sigma_b} \right) + \sqrt{\left( \vartheta + \rho \sigma_\mu \frac{1}{\sigma_b} \right)^2 + \left( \sigma_b^{-2} + \sigma_\beta^{-2} \right) \sigma_\mu^2 (1 - \rho^2)}}
\]

where

\[
\frac{\partial O_t}{\partial \sigma_\beta} > 0 \quad \forall t
\]

We can perturb the signal around learning steady state. This cannot be done with non-stationary learning.
Equilibrium

\[
\max_{\{C_t, H_t, I_t, \omega_t\}_{t=0}^\infty} E_0 \int_0^\infty e^{-At} \frac{(C_t H_t^\eta)^{1-\gamma}}{1-\gamma} \, dt \quad \text{subject to:}
\]

\[
dW_t = [r_t W_t + \omega_t W_t(\mu_t - r_t) + y_t - C_t - I_t - B_t H_t] \, dt + \omega_t W_t \sigma_{w,t} \, d\tilde{z}_t
\]

\[
dH_t = H_t \theta \ln \left(1 + \frac{l_t}{\theta H_t}\right) \, dt - \delta H_t \, dt
\]

\[
 dB_t = b(x_t - B_t) \, dt + \sigma_b d\tilde{z}_{b,t}
\]

\[
 dx_t = \vartheta(\mu_b - x_t) \, dt + \left(O_t \frac{1}{\sigma_b}\right) d\tilde{z}_{b,t} + \left(O_t \frac{1}{\sigma_\beta}\right) d\tilde{z}_{\beta,t}
\]

An equilibrium consists of a set of adapted processes \(\{C_t, I_t, H_t, \omega_t\}\) s.t.

1. \(C_t, H_t, I_t, \omega_t\) maximize agent’s objective s.t. dynamic constraints
2. the market for goods and housing clears: \(C_t + I_t + B_t H_t = Y_t\)
3. the market for stocks clears: \(\omega_t W_t = S_t\)
4. the market for bonds clears: \(\omega_t = 1\)
Equivalently...

Let \( c = \frac{C}{W} \), \( h = \frac{H}{W} \), \( i = \frac{I}{W} \)

\[
\max_{\{c_t, h_t, i_t, \omega_t\}_{t=0}^\infty} E_0 \int_0^\infty e^{-At} \left( \frac{c_t h_t^{\eta}}{1-\gamma} \right) W_t^{(1-\gamma)(1+\eta)} dt \quad \text{subject to:}
\]

\[
dW_t/W_t = r_t + \omega_t (\mu_t - r_t) + y_t/W_t - c_t - i_t - B_t h_t dt + \omega_t \sigma_{w,t} d\tilde{z}_t
\]

\[
dh_t = h_t \left[ P\ln \left( 1 + \frac{i_t}{\theta h_t} \right) dt + (\omega_t^2 \sigma_t \sigma_t' - \delta) dt - \frac{dW_t}{W_t} \right]
\]

\[
 dB_t = b(x_t - B_t) dt + \sigma_b d\tilde{z}_{b,t}
\]

\[
dx_t = \vartheta (\mu_b - x_t) dt + \left( O_t \frac{1}{\sigma_b} \right) d\tilde{z}_{b,t} + \left( O_t \frac{1}{\sigma_\beta} \right) d\tilde{z}_{\beta,t}
\]

An equilibrium consists of a set of adapted processes \( \{c_t, i_t, h_t, \omega_t\} \) s.t.

1. \( c_t, h_t, i_t, \omega_t \) maximize the agent's objective s.t. dynamic constraints
2. the market for goods and housing clears: \( c_t + i_t + B_t h_t = Y_t/W_t \)
3. the market for stocks clears: \( \omega_t W_t = S_t \)
4. the market for bonds clears: \( \omega_t = 1 \)
Solution: Dynamic Programming Approach

\[
\max_{c, h, i, \omega} e^{-At} \frac{(c_t h_t^\eta)^{1-\gamma}}{1 - \gamma} W_t^{(1-\gamma)(1+\eta)} dt + V_t dt + V_w dW + \frac{1}{2} V_{ww} dW^2 + V_h dh \\
+ \frac{1}{2} V_{hh} dh^2 + V_x dx + \frac{1}{2} V_{xx} dx^2 + V_b dB + \frac{1}{2} V_{bb} dB^2 \\
+ V_{wh} dWdh + V_{wb} dWdB + V_{wx} dWdx \\
+ V_{xb} dBdx + V_{hb} dhdB + V_{xh} dhdx = 0
\]

Because of the homogeneity of the utility function

\[
VV(t, h_t, x_t, B_t, W_t) = F(t, h_t, x_t, B_t) e^{-At} \frac{W_t^{(1-\gamma)(1-\eta)}}{(1 - \gamma)(1 - \eta)}
\]
1. Take FOC's wrt $c, h, \omega$

2. Impose mkt clearing, $c + i + Bh = \frac{Y}{W}$, and $\omega = 1$

3. Differentiate SPD, $\Lambda_t = e^{-At}(V_w W - V_h h)$, to find $\sigma_{\Lambda,t}$ and $r_t$ as functions of $\sigma_{w,t}$ and the states.

4. Use resource constraint and again differentiate SPD, $\Lambda_t = e^{-At} u_c(h, \frac{Y}{W} - i - Bh)$, to find 2nd equation for $\sigma_{\Lambda,t}$ as a function of $\sigma_{w,t}$ and the states.

Last 2 equations yield $\sigma_{\Lambda,t}, r_t, \mu_t$ and $\sigma_{w,t}$ as functions of the states.

5. Plug FOC’s and formula for $\{\sigma_{w,t}, \sigma_{\Lambda,t}\}$ back into the BE

6. Lastly we obtain 2 PDEs for $F(t, h, x, B)$
\[ \chi_t \equiv \{ t, x_t, B_t, h_t, W_t \} \]

\[ q_t = \frac{V_h(\chi_t)}{V_w(\chi_t)W_t - V_h(\chi_t)h_t} \]

Given \( h_0, W_0, c_0, i_0, x_0, B_0 \) we sequentially update the variables using

1. \[ \frac{dh_t}{h_t} = \text{Pln} \left( 1 + \frac{i_t}{\theta h_t} \right) dt + (\omega_t^2 \sigma_t^2 - \delta) dt - \frac{dW_t}{W_t} \]
2. \[ dW_t = W_t[\mu_t + (\alpha - 1)(c_t + i_t + B_t h_t)] dt + W_t \sigma_{w,t} d\tilde{z}_t \]
3. \[ dx_t = \vartheta(\mu_b - x_t) dt + \left( O_t \frac{1}{\sigma_b} \right) d\tilde{z}_{b,t} + \left( O_t \frac{1}{\sigma_\beta} \right) d\tilde{z}_{\beta,t} + \rho \sigma_\mu d\tilde{z}_{b,t} \]
4. \[ dB_t = b(x_t - B_t) dt + \sigma_b d\tilde{z}_{b,t} \]
5. \[ c_t = \frac{h_t^{\frac{1}{\gamma} - 1}}{\eta} (F + F_h h_t / \left[(-1 + \gamma)(1 + \eta)\right])^{-1/\gamma} \]
6. \[ i_t = \theta h_t (q_t - 1) \]
7. \[ q_t = \frac{V_h(\chi_t)}{V_w(\chi_t)W_t - V_h(\chi_t)h_t} \]
A Side Note: Solving PDE’s

First, use Projection Method to approximate solution to PDEs and get intuition for the results (see Judd Chap 11).

PDE of interest is

\[ a_0 + A_1(\chi, t)D_\chi + a_2(t)D_{\chi\chi} = 0 \]

approximate \( D \) with 2nd degree polynomial in \( \chi \)

\[ D \approx c_0 + \chi \cdot c_1(t) + \chi C_2(t) \chi' = \hat{D} \]

define residual

\[ R(\chi, c_0, c_1, C_2) = a_0(t) + A_1(\chi, t)\hat{D}_\chi + a_2(t)\hat{D}_{\chi\chi} \]

find the coefficients using

\[ \int_{\mathcal{A}} R(\chi, c_0, c_1, C_2)\psi_j(\chi)d\chi = 0 \quad j = \{0, 1, \ldots, N\} \]

\( \psi_j(\chi) \) is a basis function (e.g. \( \psi_j(\chi) = \chi^j \)). \( c_0 \) is given by boundary condition, otherwise \( c_0 = 0 \)

Then, if results look promising, use other numerical solvers to solve PDE’s.
Why Learning Matters

Covariance between Realized and Expected housing costs

\[ \text{cov}(dB_t, d\tilde{B}) = \rho \sigma_\mu \sigma_b \]

In the case of parameter uncertainty, the corresponding covariance becomes

\[ \text{cov}(dB_t, dx_t) = \text{cov}(dB_t, d\tilde{B}) + O_t \]

When realized \( B_t \) is high, optimal learning implies upward update of estimate, \( x_t \); hence the additional positive covariance, \( O_t \), between the innovations of \( B_t \) and \( x_t \)

Covariance between Consumption Growth and Key Variables

\[
\begin{align*}
\text{cov}\left(\frac{dC_t}{C_t}, dB_t\right) &= \sigma_b^2 \zeta_{b,t} + (\rho \sigma_\mu \sigma_b + O_t) \zeta_{\mu,t} \\
\text{cov}\left(\frac{dC_t}{C_t}, dx_t\right) &= (\rho \sigma_\mu \sigma_b + O_t) \zeta_{b,t} + \zeta_{\mu,t} \left( \frac{\sigma_\mu^2}{\sigma_b^2} + \left(\frac{O_t}{\sigma_b}\right)^2 + 2 \frac{O_t}{\sigma_b} \rho \sigma_\mu + (\rho^2 - 1) \sigma_\mu^2 + \left(\frac{O_t}{\sigma_\beta}\right)^2 \right)
\end{align*}
\]
In a competitive equilibrium the price of housing equals the marginal rate of substitution between housing and wealth.

$$\frac{\partial V(t, x_t, W_t, H_t/W_t)}{\partial H_t} / \frac{\partial V(t, x_t, W_t, H_t/W_t)}{\partial W_t} = q_t = \frac{V_h(\chi_t)}{V_w(\chi_t)W_t - V_h(\chi_t)h_t}$$

To quantify the effects of parameter learning on housing price we first do a Martingale decomposition; which helps to isolate the long-term exposure of $q_t$ to perturbations in the learning mechanism from the transient or short-term effects.

Following Hansen and Sheikman (2009a) and Borovicka et al (2011) we do multiplicative martingale decomposition

$$q_t = \exp(\nu t)\tilde{q}_t \frac{\hat{e}_t}{\hat{e}_0}$$

where $\tilde{q}_t$ is the a multiplicative martingale, $\nu$ is the eigen-value and $\hat{e}(\cdot) = 1/e(\cdot)$ where $e(\cdot)$ is an eigen-function.
\[
\ln(q_t) = \int_0^t \mu_q(\chi_u; \sigma_\beta) du + \int_0^t \sigma_q(\chi_u; \sigma_\beta) d\tilde{z}_u
\]

Using notation from Borovicka et al (2011) we can introduce a perturbation \( \sigma_\beta \rightarrow (1 + \epsilon)\sigma_\beta \) in the noisiness of the signal as follows

\[
\ln \left( \tilde{H}_t(\epsilon) \right) = \int_0^t \beta_h(\chi_u, \epsilon) du + \int_0^t \alpha_h(\chi_u, \epsilon) d\tilde{z}_u
\]

\[
\beta_h(\chi_t, \epsilon) = \mu_q(\chi_t; (1 + \epsilon)\sigma_\beta) - \mu_q(\chi_t; \sigma_\beta)
\]

\[
\alpha_h(\chi_t, \epsilon) = \sigma_q(\chi_t; (1 + \epsilon)\sigma_\beta) - \sigma_q(\chi_t; \sigma_\beta)
\]

The perturbed price of housing is: \( \tilde{H}_t(\epsilon)q_t \)

The risk elasticity is

\[
\frac{1}{t} \frac{d}{d\epsilon} \ln \left\{ E_0 \left( \tilde{H}_t q_t \right) \right\} \bigg|_{\epsilon=0}
\]
Let $\beta_l \equiv \frac{\partial}{\partial \epsilon} \beta_h \bigg|_{\epsilon=0} \quad \alpha_l \equiv \frac{\partial}{\partial \epsilon} \alpha_h \bigg|_{\epsilon=0} \quad \Sigma_t \equiv \text{varcov}(\chi_t)$

Recall that $q_t = \exp(\nu t)\hat{q}_t \hat{e}_0$

We use a convenient change of measure $E_0(q_t \times f(\chi_t)) = \exp(\nu t) \times \hat{E}_0(\hat{e}_t \times f(\chi_t))/\hat{e}_0$

We can re-write the risk elasticity

$$\frac{1}{t} \frac{d}{d\epsilon} \ln \left\{ \hat{E}_0 \left( \hat{H}_t \hat{e}_t \right) \right\} \bigg|_{\epsilon=0} = \frac{1}{t} \frac{\hat{E}_0 \left( \hat{e}_t(\chi_t) \int_0^t \epsilon(\chi_u, t - u)du \right)}{\hat{E}_0 \left( \hat{e}_t \right)}$$

$$\epsilon(\chi_u, t - u) = \alpha_l(\chi_u, \epsilon) \left( \phi_{u,t} + \sigma_q + \Sigma \cdot \frac{\partial}{\partial \chi} \left( 1/\hat{e}_u \right) \right)' + \beta_l(\chi_u, \epsilon)$$

$$\phi_{u,t} = \Sigma \cdot \frac{\partial}{\partial \chi} \ln \hat{E}_u(e_t)$$
The shock elasticity is

\[ \varepsilon(\chi_u, t - u) = \alpha_l(\chi_u, \epsilon) \left( \phi_{u,t} + \sigma_q + \sum \frac{\partial}{\partial \hat{\epsilon}} (1/\hat{\epsilon}) \right)' + \beta_l(\chi_u, \epsilon) \]

where \( \phi_{u,t} \) captures the transitory effects on the price of housing of a deterioration of the signal \( \beta_t \)

\[ \lim_{t \to \infty} \phi_{u,t} = 0 \]

So that

\[ \lim_{t \to \infty} \varepsilon(\chi_u, t - u) = \alpha_l(\chi_u, \epsilon) \left( \sigma_q + \sum \frac{\partial}{\partial \hat{\epsilon}} (1/\hat{\epsilon}) \right)' + \beta_l(\chi_u, \epsilon) \]

\[ \varepsilon(\chi_u, 0) = \alpha_l(\chi_u, \epsilon)\sigma_q' + \beta_l(\chi_u, \epsilon) \]
<table>
<thead>
<tr>
<th>Variable</th>
<th>Name/Calculation</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_0$</td>
<td>real GDP</td>
<td>11.216</td>
<td>BEA: NIPA</td>
</tr>
<tr>
<td>$h_0$</td>
<td>net stock private res. fixed invest.</td>
<td>14.234</td>
<td>BEA: FAA</td>
</tr>
<tr>
<td>$i_0$</td>
<td>res. invest.</td>
<td>0.583</td>
<td>BEA: NIPA</td>
</tr>
<tr>
<td>$\mu_b$</td>
<td>$E(\hat{B})$ : MRT b/w $h_t$ and $\hat{c}_t$</td>
<td>0.100</td>
<td>Mulligan (2010)</td>
</tr>
<tr>
<td>$B_0 = B_0 = \mu_b$</td>
<td></td>
<td>0.100</td>
<td>X</td>
</tr>
<tr>
<td>$\sigma_b$</td>
<td>std dev of % chg in housing subs.</td>
<td>0.050</td>
<td>X</td>
</tr>
<tr>
<td>$\frac{\eta}{1+\eta}$</td>
<td>housing exp. share</td>
<td>0.150</td>
<td>BEA: NIPA</td>
</tr>
<tr>
<td>$\mu_Y$</td>
<td>avg of $%\Delta(GDP)$</td>
<td>0.034</td>
<td>BEA: NIPA</td>
</tr>
<tr>
<td>$\sigma_Y$</td>
<td>std dev of $%\Delta(GDP)$</td>
<td>0.049</td>
<td>BEA: NIPA</td>
</tr>
<tr>
<td>$\theta = P$</td>
<td>adj. cost convexity coeff.</td>
<td>0.400</td>
<td>X</td>
</tr>
<tr>
<td>$A$</td>
<td>opp. cost of housing</td>
<td>0.050</td>
<td>Mulligan (2010)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>housing dep.</td>
<td>0.025</td>
<td>Mulligan (2010)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$corr(dz_\mu, dz_b)$</td>
<td>0.500</td>
<td>X</td>
</tr>
<tr>
<td>$\theta$</td>
<td>mean-reversion coef. for $\hat{B}$</td>
<td>0.300</td>
<td>X</td>
</tr>
<tr>
<td>$b$</td>
<td>mean-reversion coef. for $B_t$</td>
<td>0.500</td>
<td>X</td>
</tr>
<tr>
<td>$\sigma_\beta$</td>
<td>local std.dev of signal, $\beta_t$</td>
<td>$\sigma_b$</td>
<td>X</td>
</tr>
<tr>
<td>$\sigma_\mu$</td>
<td>local std.dev of $\hat{B}$</td>
<td>$\sigma_\beta$</td>
<td>X</td>
</tr>
</tbody>
</table>
Summary

Goal: show that parameter uncertainty contributes significantly to sharp boom and bust cycles in the housing market.

Possible next steps:

- Investigate the effects of learning on endogenous variables (explain the mechanism and provide economic intuition)
- Identify testable implications of the model and “bring it to data”. Show evidence for link between economic policy uncertainty and housing market
- Modify the model to increase the effect of parameter learning on prices (see below)
- Introduce a government with housing subsidy, distortionary taxes and agent uncertainty over subsidies to housing
- Introduce more elaborate goods and housing production... e.g. Housing suppliers face uncertainty about housing demand leading to over-production of housing and contribute to severity of the cycle
Stochastic Differential Utility

Agents will be endowed with Epstein-Zin Kreps-Porteus (EZ-KP) utility.

We use the corresponding stochastic differential utility, $V$ where

$$dV = -f(C, H, V)\, dt,$$

with aggregator function

$$f(C, H, V) = \frac{1}{1-\rho} \left( \frac{(CH^\eta)^{1-\nu}}{1-\nu} \right)^{1-\rho} \left( (1-\gamma)V^{1-\rho} - A(1-\gamma)V \right)^{-1} - A(1-\gamma)V$$

Solution method is unchanged but → PDEs slightly different.

SDU with preference for earlier resolution of uncertainty ($\gamma > 1/\rho$) leads to large welfare effects from learning (Ai; 2007).

In general equilibrium with EZ-KP utility parameter learning has large effects on prices (Collin-Dufresne, Johannes, Lochstoer; 2012).

SDU should help increase short run effects of learning on housing price.
Heterogeneity in Housing Markets

There is a fair amount of evidence for heterogeneity of housing markets across US regions/states.

The framework presented above can be extended to allow for such heterogeneity. Given $N$ housing markets

**Heterogeneity in housing demand elasticity**

\[
\begin{align*}
    d\ln(H_t) &= \theta \ln(q_t) dt - \delta dt \\
    d\ln(q_t) &= \mu_{q,t} dt + \sigma_{q,t} d\tilde{z}_t
\end{align*}
\]

\[\Rightarrow \quad \frac{d\ln(H_t)}{E_t d\ln(q_t)} = \frac{\theta \ln(q_t) - \delta}{\mu_{q,t}}\]

We can use a different $\theta$ for each market ($\theta_i \forall i \in \{1, ..., N\}$) to parametrize (and calibrate) the price sensitivity of housing demand across different markets.
Heterogeneity in parameter uncertainty

Agents in the different markets receive market-specific (but correlated) signals.

Introduce Habit formation (“keeping up with the Jone’s”) where agents will modify their behavior based on other agent’s decisions and infer if other agent’s decision is due to good shock or additional information about housing market.
From Prof. Liptser Lecture 6, “Kalman-Bucy Filter”, the optimal $x_0$ and $O_0$ are

$$x_0 = E x_0 + (\text{cov}(x_0, B_0) \quad \text{cov}(x_0, \beta_0))$$
$$\times \text{varcov}(B_0 \quad \beta_0)^{-1} \times (B_0 - E B_0 \quad \beta_0 - E \beta_0)'$$

$$O_0 = \text{cov}(x_0, x_0) - \frac{[\text{cov}(x_0, B_0) + \text{cov}(x_0, \beta_0)]^2}{\text{cov}(B_0, B_0) + \text{cov}(\beta_0, \beta_0)}$$

In the graphical examples I use $x_0 = E(\bar{B}) - \sigma_b$, $B_0 = \bar{B}_0 = \mu_b$ and $O_0 = \left(1/\sigma_b^2 + 1/\sigma_\beta^2\right)^{-1}$ for the non-stationary case.

In the stationary learning case and I use $O_0 = \left(1/\sigma_b^2 + 1/\sigma_\beta^2\right)^{-1}$, $x_0 = \mu_b$, and $\beta_0 = E(\beta_0)$
It is important to note that with this change of variables the marginal value of wealth becomes

\[
\frac{\partial V(t, h, x, B, W)}{\partial W} = \frac{\partial V(t, H/W, x, B, W)}{\partial W} = V_w - V_h h/W
\]

Similarly, the marginal value of housing is

\[
\frac{\partial V(t, h, x, B, W)}{\partial H} = \frac{\partial V(t, H/W, x, B, W)}{\partial H} = V_h/W
\]

Thus we have

\[
q_t = \frac{V_h}{V_w W - V_h h} = \frac{V_h}{V_w W - V_h H/W}
\]

\[
= \frac{V_h/W}{V_w - V_h H/W^2} = \frac{\partial V(t, x, W, H/W)}{\partial H} / \frac{\partial V(t, x, W, H/W)}{\partial W}
\]
\[
\omega_t = -\frac{(F(-1 + \gamma)(1 + \eta) + F_h h_t)(r_t - \mu_t)}{(F_{hh} h_t^2 + (\gamma + (-1 + \gamma)\eta)(F(-1 + \gamma)(1 + \eta) + 2F_h h_t)) \sigma_{2ww}}
\]

\[
+ \frac{((-1 + \gamma)(1 + \eta)F_b + F_{bh} h_t) \sigma_{2wb}}{(F_{hh} h_t^2 + (\gamma + (-1 + \gamma)\eta)(F(-1 + \gamma)(1 + \eta) + 2F_h h_t)) \sigma_{2ww}}
\]

\[
+ \frac{((-1 + \gamma)(1 + \eta)F_x + F_{xh} h_t) \sigma_{2wx}}{(F_{hh} h_t^2 + (\gamma + (-1 + \gamma)\eta)(F(-1 + \gamma)(1 + \eta) + 2F_h h_t)) \sigma_{2ww}}
\]