Appendix: Optical Pyrometry

The optical pyrometer works as follows: light passes from a blackbody source at temperature T, past a filament, and through a red filter before being observed through an eyepiece. As viewed through the filter, the filament appears to be superimposed on the source; the current flowing through the filament can then be adjusted until the filament and the source appear to be the same colour (i.e. have the same intensity at the wavelength λ_0 of the red filter.)

Suppose, though, that our source was not a perfect blackbody, but instead had an emissivity less than one, i.e. it emitted light with a spectrum

$$\mathcal{P}(\lambda) = a(\lambda) q \frac{\lambda^{-5}}{e^{hc/\lambda kT} - 1},$$

where $a(\lambda) \neq 1$ and q is an overall factor (depending only on fundamental constants.) If the pyrometer is calibrated so that its reading would be accurate for a perfect blackbody, then when we match the intensities at the wavelength λ_0, the pyrometer will observe a "brightness temperature" $\tilde{T}$ given by matching the intensity (at wavelength λ_0) of an ideal blackbody at $\tilde{T}$ to that of our source, i.e.

$$q \frac{\lambda_0^{-5}}{e^{hc/\lambda_0 k\tilde{T}} - 1} = a(\lambda_0) q \frac{\lambda_0^{-5}}{e^{hc/\lambda_0 kT} - 1}$$

Rearranging, we have

$$e^{hc/\lambda_0 kT} - 1 = a_0 \left(e^{hc/\lambda_0 k\tilde{T}} - 1 \right)$$

where we've defined $a_0 = a(\lambda_0)$ for convenience. Rearranging this, we find that the true temperature can be found from the brightness temperature via the relation

$$T = \frac{hc}{k\lambda_0 \ln \left[a_0 \left(e^{hc/\lambda_0 k\tilde{T}} - 1 \right) - 1 \right]}. \tag{19}$$

We can simplify this in the limit of "low temperature", i.e. in the limit where the dimensionless parameter $hc/k\lambda_0\tilde{T}$ is sufficiently large. We have

$$\begin{aligned}
\frac{hc}{\lambda_0 kT} &= \ln \left[a_0 \left(e^{hc/\lambda_0 k\tilde{T}} - 1 \right) - 1 \right] \\
&= \ln \left[\left(a_0 e^{hc/\lambda_0 k\tilde{T}} \right) \left(1 - (1 + a_0^{-1}) e^{-hc/\lambda_0 k\tilde{T}} \right) \right] \\
&= \ln \left(a_0 e^{hc/\lambda_0 k\tilde{T}} \right) + \ln \left(1 - (1 + a_0^{-1}) e^{-hc/\lambda_0 k\tilde{T}} \right) \\
&= \frac{hc}{\lambda_0 k\tilde{T}} + \ln a_0 + \mathcal{O}(e^{-hc/\lambda_0 k\tilde{T}})
\end{aligned}$$

For brightness temperatures on the order of 2000 K and red light at around 650 nm, we have $hc/\lambda_0 k\tilde{T} \approx 10$ and so we can discard the last term on the right-hand side to a very good approximation (even if 2000 K isn't usually considered "low temperature.") This means that we have

$$\frac{1}{T} \approx \frac{1}{\tilde{T}} + \frac{k\lambda_0}{hc} \ln a_0$$

or, rearranging again,

$$T \approx \frac{1}{1/\tilde{T} + k\lambda_0 \ln a_0 / hc} = \frac{\tilde{T}}{1 + \frac{k\lambda_0 \tilde{T}}{hc} \ln a_0}. \tag{20}$$