4.5 Computing Sequential Equilibria

Consider the game shown in Figure 4.9, adapted from Rosenthal (1981). This game can be interpreted as follows. After the upper chance event, which occurs with probability .95, the players alternate choosing between generous (g_k, for $k = 1,2,3,4$) and selfish (f_k) actions until someone is selfish or until both have been generous twice. Each player loses $1 each time he is generous, but gains $5 each time the other player is generous. Everything is the same after the lower chance event, which occurs with probability .05, except that 2 is then incapable of being selfish (perhaps because she is the kind of person whose natural integrity would compel her to reciprocate any act of generosity). Player 1 does not directly observe the chance outcome.

The move probabilities and belief probabilities that make up a sequential equilibrium are shown in Figure 4.9 in parentheses and angle brackets. To characterize the beliefs, let α denote the probability that player 1 would assign to the event that player 2 is capable of selfishness at the beginning of the game, and let δ denote the probability that 1 would assign to the event that 2 is capable of selfishness if he had observed her being generous once. To characterize the behavioral strategies, let β be the probability that 1 would be generous in his first move; let γ be the probability that 2 would be generous in her first move, if she is capable of selfishness; let ε be the probability that 1 would be generous in his second move; and let ζ be the probability that 2 would be generous at her second move, if she is capable of selfishness. We now show how to solve for these variables, to find a sequential equilibrium of this game.

Two of these variables are easy to determine. Obviously, $\alpha = .95$, because that is the probability of the upper alternative at the chance

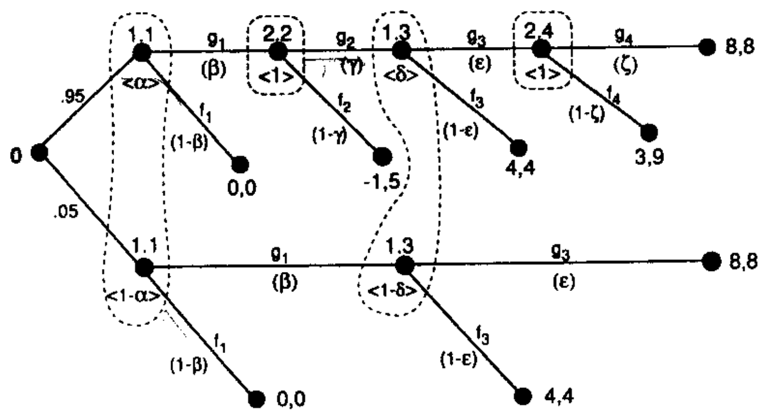

Figure 4.9

node, the outcome of which is unobservable to player 1. Also, it is easy to see that $\zeta = 0$, because player 2 would have no incentive to be generous at the last move of the game, if she is capable of selfishness.

To organize the task of solving for the other components of the sequential equilibrium, we use the concept of *support*, introduced in Section 3.3. At any information state, the support of a sequential equilibrium is the set of moves that are used with positive probability at this information state, according to the behavioral-strategy profile in the sequential equilibrium. We select any information state and try to guess what might be the support of the sequential equilibrium at this state. Working with this guess, we can either construct a sequential equilibrium or show that none exists with this support and so go on to try another guess.

It is often best to work through games like this backward. We already know what player 2 would do at information state 4, so let us consider now player 1's information state 3, where he makes his second move. There are three possible supports to consider: $\{g_3\}$, $\{f_3\}$, and $\{g_3,f_3\}$. Because $\zeta = 0$, player 1's expected payoff (or sequential value) from choosing g_3 at state 3 is $3\delta + 8(1-\delta)$; whereas player 1's expected payoff from choosing f_3 at state 3 is $4\delta + 4(1-\delta) = 4$. By Bayes's formula (4.5),

$$\delta = \frac{.95\beta\gamma}{.95\beta\gamma + .05\beta} = \frac{19\gamma}{19\gamma + 1}.$$

Even if $\beta = 0$, full consistency requires that $\delta = 19\gamma/(19\gamma + 1)$, because this equation would hold for any perturbed behavioral strategies in which β was strictly positive.

Let us try first the hypothesis that the support of the sequential equilibrium at 1's state 3 is $\{g_3\}$; so $\varepsilon = 1$. Then sequential rationality for player 1 at state 3 requires that $3\delta + 8(1-\delta) \ge 4$. This inequality implies that $0.8 \ge \delta = 19\gamma/(19\gamma + 1)$; so $\gamma \le 4/19$. But $\varepsilon = 1$ implies that player 2's expected payoff from choosing g_2 at state 2 (her first move) would be 9, whereas her expected payoff from choosing f_2 at state 2 would be 5. Thus, sequential rationality for player 2 at state 2 requires that $\gamma = 1$, when $\varepsilon = 1$. Since $\gamma = 1$ and $\gamma \le 4/19$ cannot both be satisfied, there can be no sequential equilibrium in which the support at 1's state 3 is $\{g_3\}$.

Let us try next the hypothesis that the support of the sequential equilibrium at 1's state 3 is $\{f_3\}$; so $\varepsilon = 0$. Then sequential rationality for player 1 at state 3 requires that $3\delta + 8(1-\delta) \le 4$. This inequality

R. Myerson, Game Theory, pages 177-181

implies that $0.8 \leq \delta = 19\gamma/(19\gamma + 1)$; so $\gamma \geq 4/19$. But $\varepsilon = 0$ implies that player 2's expected payoff from choosing g_2 at state 2 would be 4, whereas her expected payoff from choosing f_2 at state 2 would be 5. Thus, sequential rationality for player 2 at state 2 requires that $\gamma = 0$ when $\varepsilon = 0$. Because $\gamma = 0$ and $\gamma \geq 4/19$ cannot both be satisfied, there can be no sequential equilibrium in which the support at 1's state 3 is $\{f_3\}$.

The only remaining possibility is that the support of the sequential equilibrium at state 3 is $\{g_3,f_3\}$; so $0 < \varepsilon < 1$. Then sequential rationality for player 1 at state 3 requires that $3\delta + 8(1 - \delta) = 4$, or else player 1 would not be willing to randomize between g_3 and f_3. With consistent beliefs, this implies that $0.8 = \delta = 19\gamma/(19\gamma + 1)$; so $\gamma = 4/19$. Thus, player 2 must be expected to randomize between g_2 and f_2 at state 2 (her first move). Player 2's expected payoff from choosing g_2 at state 2 is $9\varepsilon + 4(1-\varepsilon)$, whereas her expected payoff from choosing f_2 at state 2 is 5. Thus, sequential rationality for player 2 at state 2 requires that $5 = 9\varepsilon + 4(1-\varepsilon)$; so $\varepsilon = 0.2$. It only remains now to determine 1's move at state 1. If he chooses s_1, then he gets 0; but if he chooses g_1, then his expected payoff is

$$.95 \times (4/19) \times .2 \times 3 + .95 \times (4/19) \times .8 \times 4 + .95 \times (15/19) \times (-1) + .05 \times .2 \times 8 + .05 \times .8 \times 4 = 0.25,$$

when $\alpha = .95$, $\gamma = 4/19$, $\varepsilon = .2$, and $\zeta = 0$. Because $0.25 > 0$, sequential rationality for player 1 at state 1 requires that $\beta = 1$. That is, in the unique sequential equilibrium of this game, player 1 should be generous at his first move.

Consider now the scenario $([f_1],[f_2],[f_3],[f_4])$, in which each player would always be selfish at any information state; so $\beta = \gamma = \varepsilon = \zeta = 0$. If the chance node and the nodes and branches following the lower .05-probability branch were eliminated from this game, then this would be the unique sequential-equilibrium scenario. That is, if it were common knowledge that player 2 would choose between selfishness and generosity only on the basis of her own expected payoffs, then no player would ever be generous in a sequential equilibrium. Furthermore, $([f_1],[f_2],[f_3],[f_4])$ is an equilibrium in behavioral strategies of the actual game given in Figure 4.9 and can even be extended to a weak sequential equilibrium of this game, but it cannot be extended to a full sequential equilibrium of this game. For example, we could make this scenario a weak sequential equilibrium, satisfying sequential rationality at all information states, by letting the belief probabilities α and δ both equal .95. However, the belief probability $\delta = .95$ would not be consistent (in the full sense) with this scenario because, if player 2 would be expected to be selfish at state 2 ($\gamma = 0$), then player 1 would infer at state 3, after he chose generosity and did not get a selfish response from player 2, that player 2 must be incapable of selfishness, so δ must equal 0. But with $\delta = 0$, selfishness (f_3) would be an irrational move for player 1 at state 3, because he would expect generosity (g_3) to get him a payoff of 8, with probability $1 - \delta = 1$.

This example illustrates the fact that small initial doubts may have a major impact on rational players' behavior in multistage games. If player 1 had no doubt about player 2's capacity for selfishness, then perpetual selfishness would be the unique sequential equilibrium. But when it is common knowledge at the beginning of the game that player 1 assigns even a small positive probability of .05 to the event that player 2 may be the kind of person whose natural integrity would compel her to reciprocate any act of generosity, then player 1 must be generous at least once with probability 1 in the unique sequential equilibrium. The essential idea is that, even if player 2 does not have such natural integrity, she still might reciprocate generosity so as to encourage player 1 to assign a higher probability to the event that she may continue to be generous in the future. Her incentive to do so, however, depends on the assumption that player 1 may have at least some initial uncertainty about player 2's type in this regard. The crucial role of such small initial uncertainties in long-term relationships has been studied in other examples of economic importance (see Kreps, Milgrom, Roberts, and Wilson, 1982).

For a second example, consider the game in Figure 4.10. To characterize the sequential equilibria of this game, let α denote the belief probability that player 2 would assign in information state 3 to the upper 2.3 node, which follows player 1's x_1 move. With these beliefs at information state 3, player 2's conditionally expected payoff would be $8\alpha + 0(1 - \alpha)$ if she chose e_3, $7\alpha + 7(1 - \alpha) = 7$ if she chose f_3, or $6\alpha + 9(1 - \alpha) = 9 - 3\alpha$ if she chose g_3. So move e_3 would be optimal for player 2 at state 3 when both $8\alpha \geq 7$ and $8\alpha \geq 9 - 3\alpha$, that is, when $\alpha \geq 7/8$. Move f_3 would be optimal for 2 when both $7 \geq 8\alpha$ and $7 \geq 9 - 3\alpha$, that is, when $2/3 \leq \alpha \leq 7/8$. Move g_3 would be optimal for 2 when both $9 - 3\alpha \geq 8\alpha$ and $9 - 3\alpha \geq 7$, that is, when $\alpha \leq 2/3$. Notice that

For the game in Figure 4.9 of Myerson's Game Theory, the portion of the game after the upper branch with chance-probability 0.95 is an example of the centipede game. If this upper branch were the whole game, so that it would be common knowledge that player 2 is a rational actor who prefers f_4 at the last move, then the game would have the following normal representation in strategic form:

	f_2 •	g_2f_4	g_2g_4
f_1 •	0, 0	0, 0	0, 0
g_1f_3	−1, 5	4, 4	4, 4
g_1g_3	−1, 5	3, 9	8, 8

(Here f_1• denotes the two equivalent strategies f_1f_3 and f_1g_3, and f_2• denotes the two equivalent strategies f_2f_4 and f_2g_4.)

For the actual extensive-form game in Figure 4.9, where player 2's nice reciprocating behavioral type has probability 0.05, the normal representation in strategic form can be written:

	f_2 •	g_2f_4	g_2g_4
f_1 •	0, 0	0, 0	0, 0
g_1f_3	−0.75, 4.95	4, 4	4, 4
g_1g_3	−0.55, 5.15	3.25, 8.95	8, 8

This game has a Nash equilibrium at (f_1•, f_2•), but this equilibrium does not correspond to a sequential equilibrium of the extensive-form game.

The sequential equilibrium corresponds to the Nash equilibrium

$(0.8[g_1f_3] + 0.2[g_1g_3], (15/19)[f_2\bullet] + (4/19)[g_2f_4])$.