PPHA 41501 – GAME THEORY
University of Chicago, Harris School of Public Policy
Autumn Quarter 2021, Monday/Wednesday 8:30-9:50am (Central Time)
Syllabus: <http://home.uchicago.edu/~rmyerson/teaching/ppha41501.pdf>
Instructor: Roger Myerson
Teaching Assistants: Kisoo Kim and Jose Pascual Moreno
These slides:  <http://home.uchicago.edu/~rmyerson/teaching/ppha41501slides.pdf>
Extensive notes:  <http://home.uchicago.edu/~rmyerson/teaching/ppha41501nts.pdf>

Course grades will be based on homework assignments, midterm exam, and final exam. Homework is graded on basis of effort only. You may discuss problems with each other, but you must turn in your own work. Do not copy and paste answers from others' work. Assignments at <https://home.uchicago.edu/~rmyerson/teaching/ppha41501hwk.pdf>

Assignment 1 should be prepared for discussion in class on Sept 29, not to be handed in.

Instructor's office hours: Thursday 8:30-10:00 online, 11:00-noon hybrid.
A model of decisions under uncertainty is characterized by:
a set of alternative choices C, a set of possible states of the world S,
a utility function \( u: C \times S \to \mathbb{R} \), and a probability distribution \( p \) in \( \Delta(S) \).
Suppose that C and S are nonempty finite sets. We use the notation:
\( \Delta(S) = \{ q \in \mathbb{R}^S \mid q(s) \geq 0 \ \forall s \in S, \ \sum_{\theta \in S} q(\theta) = 1 \}. \) (\( \forall s \in S, \ [s] \in \Delta(S) \) with \([s](s)=1\).)
The expected utility hypothesis says that an optimal decision should
maximize expected utility \( Eu(c) = Eu(c \mid p) = \sum_{\theta \in S} p(\theta)u(c,\theta) \) over all \( c \) in \( C \),
for some utility function \( u \) that is appropriate for the decision maker.

Fact: Given utility function \( u: C \times S \to \mathbb{R} \) and some choice \( d \in C \), the set of probability
distributions that make \( d \) optimal is a closed convex (possibly empty) subset of \( \Delta(S) \).
This set (of probabilities that make \( d \) optimal) is empty if and only if there exists some
randomized strategy \( \sigma \) in \( \Delta(C) \) such that \( u(d,s) < \sum_{c \in C} \sigma(c)u(c,s) \) \( \forall s \in S \).
When these inequalities hold, we say that \( d \) is strongly dominated by \( \sigma \).

Here we use the same \( \Delta(\bullet) \) notation as above, so
\( \Delta(C) = \{ \sigma \in \mathbb{R}^C \mid \sigma(c) \geq 0 \ \forall c \in C, \ \sum_{c \in C} \sigma(c) = 1 \}. \)
Example 1. Consider an example with choices $C = \{T,M,B\}$, state $S = \{L,R\}$, and $u(c,s)$:  

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
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<tbody>
<tr>
<td>T</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>M</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>6</td>
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Let the parameter $r$ denote the probability of state $R$.  

- $\text{Eu}(T|r) = (1-r)7 + r2 = 7-5r$,  
- $\text{Eu}(M|r) = (1-r)2 + r7 = 2+5r$,  
- $\text{Eu}(T|r) = (1-r)5 + r6 = 5+r$.

Then $B$ is optimal when $5(1-r) + 6r \geq 2(1-r) + 7r$ and $5(1-r) + 6r \geq 7(1-r) + 2r$, which are satisfied when $3/4 = (5-2)/(5-2)+(7-6) \geq r \geq (7-5)/(7-5)+(6-2)) = 1/3$.  

- $T$ is optimal when $r \leq 1/3$.  
- $M$ is optimal when $r \geq 3/4$.  

![Graph of u(c,L), u(c,R)](image1.png)  

![Graph of (r, Eu(c|r))] (image2.png)
Example 2: As above, $C = \{T,M,B\}$, $S = \{L,R\}$, and $u$ is same except $u(B,R) = 3$.

$u(c,s)$: 

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
<th>P(R) = r.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>7</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>2</td>
<td>7</td>
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<tr>
<td>B</td>
<td>5</td>
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$B$ would be optimal when $5(1-r)+3r \geq 7(1-r)+2r$ & $5(1-r)+3r \geq 2(1-r)+7r$,
which are satisfied when $r \geq 2/3$ & $3/7 \geq r$, which is impossible! So $B$ cannot be optimal.

$T$ is optimal when $r \leq 1/2$. $M$ is optimal when $r \geq 1/2$.

Now consider a randomized strategy that chooses $T$ with some probability $\sigma(T)$ and chooses $M$
with probability $\sigma(M) = 1-\sigma(T)$. This strategy is $\sigma(T)[T]+(1-\sigma(T))[M]$.

$B$ would be strongly dominated by this randomized strategy $\sigma$ if $5 < \sigma(T)7 + (1-\sigma(T))2$ (B worse than $\sigma$ in state L), and $3 < \sigma(T)2 + (1-\sigma(T))7$ (B worse than $\sigma$ in state R).

These are satisfied when $3/5 < \sigma(T) < 4/5$. For example, $\sigma(T) = 0.7$ works.

$B$ is strongly dominated by $0.7[T]+0.3[M]$ (5 < $0.7 \times 7 + 0.3 \times 2 = 5.5$, 3 < $0.7 \times 2 + 0.3 \times 7 = 3.5$).
Separating Hyperplane Theorem (MWG M.G.2):
Suppose X is a closed convex subset of \( \mathbb{R}^N \), and w is a vector in \( \mathbb{R}^N \). Then exactly one of the following two statements is true (but not both):
Either (1) \( w \in X \), or (2) there exists a vector \( y \in \mathbb{R}^N \) such that \( y^\top w > \max_{x \in X} y^\top x \).
(Here \( y^\top w = y_1x_1 + \ldots + y_Nx_N = \sum_{i \in \{1, \ldots, N\}} y_i x_i \), with \( y = (y_1, \ldots, y_N) \) and \( x = (x_1, \ldots, x_N) \).)

Supporting Hyperplane Theorem (MWG M.G.3):
Suppose X is a convex subset of \( \mathbb{R}^N \), and w is a vector in \( \mathbb{R}^N \). Then exactly one of these two statements is true (but not both): Either (1) w is in the interior of X (relative to \( \mathbb{R}^N \)), or (2) \( \exists y \in \mathbb{R}^N \) such that \( y \neq 0 \) and \( y^\top w \geq \max_{x \in X} y^\top x \). (Here \( 0 = (0, \ldots, 0) \).)

**Fact** If X is convex and compact (closed & bounded), then \( \max_{x \in X} y^\top x \) is finite, and this maximum must be achieved at some extreme point in X. (MWG p 946.)

**Fact** For any nonempty finite set \( C \) and any \( v \in \mathbb{R}^C \), \( \max_{\sigma \in \Delta(C)} \sum_{c \in C} \sigma(c) v_c = \max_{c \in C} v_c \), and \( \text{argmax}_{\sigma \in \Delta(C)} \sum_{c \in C} \sigma(c) v_c = \{ \sigma \in \Delta(C) | \{c| \sigma(c)>0\} \subseteq \text{argmax}_{c \in C} v_c \}. \)
**Strong domination Theorem.** Given nonempty finite sets $C=\{\text{choices}\}$ & $S=\{\text{states}\}$, utility function $u:C \times S \to \mathbb{R}$, and some $d \in C$, exactly of these two statements is true: either

(1) $\exists \sigma \in \Delta(C)$ such that $u(d,s) < \sum_{c \in C} \sigma(c)u(c,s) \ \forall s \in S$, \hspace{1cm} \text{[d dominated with randomization]}

or

(2) $\exists p \in \Delta(S)$ s.t. $\sum_{s \in S} p(s)u(d,s) = \max_{c \in C} \sum_{s \in S} p(s)u(c,s)$. \hspace{1cm} \text{[d optimal for some beliefs]}

**Proof.** Let $X = \{x \in \mathbb{R}^S | \exists \sigma \in \Delta(C) \text{ s.t. } x_s \leq \sum_{c \in C} \sigma(c)u(c,s) \ \forall s\}$. $X \subseteq \mathbb{R}^S$ is convex.

(1) here is equivalent to: (1') the vector $u(d) = (u(d,s))_{s \in S}$ is in the interior of $X$ in $\mathbb{R}^S$. By the Supporting Hyperplane Thm, (1') is false iff

(2') $\exists p \in \mathbb{R}^S$ such that $p \neq 0$ and $\sum_{s \in S} p(s)u(d,s) \geq \max_{x \in X} \sum_{s \in S} p(s)x_s$.

We must have $p(s) \geq 0$ for all $s$, because $x$ in $X$ can have $x_s$ approaching $-\infty$.

So $\sum_{s \in S} p(s) > 0$, from $p \geq 0$ and $p \neq 0$. Dividing by this sum, we can make $\sum_{s \in S} p(s) = 1$. Furthermore, the maximum of the linear function $p'x$ over $x \in X$ must be achieved at one of the extreme points in $X$, which are vectors $(u(c,s))_{s \in S}$ for the various $c \in C$:

$\max_{x \in X} \sum_{s \in S} p(s)x_s = \max_{\sigma \in \Delta(C)} \sum_{s \in S} p(s) \sum_{c \in C} \sigma(c)u(c,s) = \max_{c \in C} \sum_{s \in S} p(s)u(c,s)$.

So (2') is equivalent to condition (2) in the theorem here.
**Expected Utility Theorem.** Let \( N \) be a finite set of *prizes*, and consider a finite sequence of pairs of lotteries \( p(i) \in \Delta(N) \) and \( q(i) \in \Delta(N) \), for \( i \in M = \{1,...,m\} \). Essentially, \( M \) is a finite set of *lottery-comparisons* of the form "\( p(i) \) preferred to \( q(i) \)." (Here \( p(i) = (p_j(i))_{j \in N} \), \( q(i) = (q_j(i))_{j \in N} \). So in the \( i \)'th comparison, the probability of getting prize \( j \) is \( p_j(i) \) in the more-preferred lottery, \( q_j(i) \) in the less-preferred lottery.) Then exactly one of these two statements is true (*not both*): Either

1. \( \exists \sigma \in \Delta(M) \) such that \( \sum_{i \in M} \sigma(i)p_j(i) = \sum_{i \in M} \sigma(i)q_j(i) \forall j \in N \), \hspace{1cm} [substitution axiom is violated]

or

2. \( \exists u \in \mathbb{R}^N \) such that \( \sum_{j \in N} p_j(i)u_j > \sum_{j \in N} q_j(i)u_j \forall i \in M \). \hspace{1cm} [preferences satisfy utility theory]

**Proof.** Let \( X = \{ \sum_{i \in M} \sigma(i)(q(i) - p(i)) \mid \sigma \in \Delta(M) \} \). \( X \) is a closed convex subset of \( \mathbb{R}^N \). Condition (1) here is equivalent to: (1') the \( N \)-vector \( 0 \) is in \( X \).

By the Separating Hyperplane Thm, (1') is false iff

(2') \( \exists u \in \mathbb{R}^N \) such that \( 0 = u'0 > \max_{x \in X} u'x \).

The extreme points of \( X \) are vectors \( (q(i) - p(i)) = (q_j(i) - p_j(i))_{j \in N} \).

The linear function \( u'x = \sum_{j \in N} x_j u_j \) must achieves its maximum over \( x \in X \) at some extreme point, so \( \max_{x \in X} u'x = \max_{i \in M} \sum_{j \in N} u_j (q_j(i) - p_j(i)) \).

So (2') is equivalent to (2) in the theorem here.

**Fact.** Suppose the utility-representation condition (2) is satisfied by \( u = (u_j)_{j \in N} \). Then (2) is also satisfied by \( \hat{u} \) if there exists \( A>0 \) and \( B \) such that \( \hat{u}_j = Au_j + B \forall j \in N \).
Linear Duality Theorem (Farkas' lemma, theorem of the alternatives)

Given any m×n matrix \( A = (a_{ij})_{i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}} \) and any vector \( b = (b_i)_{i \in \{1, \ldots, m\}} \) in \( \mathbb{R}^m \), exactly one of the following two conditions is true: Either

1. \( \exists x \in \mathbb{R}^n \) such that \( \sum_{j \in \{1, \ldots, n\}} a_{ij} x_j \geq b_i \ \forall i \in \{1, \ldots, m\} \);

or

2. \( \exists y \in \mathbb{R}^m \) s.t. \( y_i \geq 0 \ \forall i \), \( \sum_{i \in \{1, \ldots, m\}} y_i a_{ij} = 0 \ \forall j \in \{1, \ldots, n\} \), \( \sum_{i \in \{1, \ldots, m\}} y_i b_i > 0 \).

**Sketch of Proof:** (1) & (2) cannot both be true, or else we could get \( 0 > y'(Ax-b) \geq 0 \).

If (1) is false then \( b \) is not in the closed convex set \( \{ Ax-z \mid x \in \mathbb{R}^n, z \in \mathbb{R}_+^m \} \), which by the Separating Hyperplane Theorem implies \( \exists y \) such that \( y'b > \max \{ y'(Ax-z) \mid x \in \mathbb{R}^n, z \in \mathbb{R}_+^m \} \). This vector \( y \) must satisfy condition (2).

So (1) & (2) cannot both be false.

**Weak domination Theorem.** Given nonempty finite sets \( C=\{\text{choices}\} \) & \( S=\{\text{states}\} \), utility function \( u:C \times S \rightarrow \mathbb{R} \), and some \( d \in C \), exactly of these two statements is true:

1. \( \exists p \in \Delta(S) \) such that \( p(s) > 0 \ \forall s \), and \( \sum_{s \in S} p(s)u(d,s) = \max_{c \in C} \sum_{s \in S} p(s)u(c,s) \);

2. \( \exists \sigma \in \Delta(C) \) s.t. \( u(d,s) \leq \sum_{c \in C} \sigma(c)u(c,s) \ \forall s \in S \), with strict \( < \) for some \( s \) in \( S \).

**Proof on page 4 of notes uses Linear Duality Thm.** ...With \( \varepsilon > 0 \), consider:

1'. \( \exists p \in \mathbb{R}^S \) such that \( \sum_{s \in S} (u(d,s) - u(c,s)) \ p(s) \geq 0 \ \forall c \in C \), \( \sigma(c) \)

and \( \ p(s) \geq \varepsilon \ \forall s \in S \), \( \delta(s) \)

2'. \( \exists (\sigma, \delta) \in \mathbb{R}_+^C \times \mathbb{R}_+^S \) s.t. \( \sum_{c \in C} \sigma(c)(u(d,s) - u(c,s)) + \delta(s) = 0 \ \forall s \in S \), \( p(s) \)
In **game theory** we assume that players are rational and intelligent. Here **rational** means that each player acts to maximize his own expected utility. Here **intelligent** means that the players know everything that we know about their situation when we analyze it game-theoretically. Intelligence implies that game model that we analyze must be **common knowledge** among the players, that is, all players know (that all players know)^k the model, ∀k={0,1,2,...}.

A **strategic-form game** is characterized by (N, (C_i)_{i∈N}, (u_i)_{i∈N}) where N = {1,2,...,n} is the set of players, and, for each player i:
- C_i is the set of alternative actions or (pure) strategies that are feasible for i in the game, and
- u_i:C_1×C_2×...×C_n→ℝ is player i's utility function in the game.

We generally assume that each player i independently chooses an action in C_i. If c = (c_1,c_2,...,c_n) is the combination (or profile) of actions chosen by the players then each player i will get the expected utility payoff u_i(c_1,c_2,...,c_n).

We let C = C_1×C_2×...×C_n = ×_{i∈N} C_i denote the set of all combinations or **profiles** of actions that the players could choose.

Let C_{-i} denote the set of all profiles of actions that can be chosen by players other than i. When c∈C is a profile of actions for the players, c_i denotes the action of each player i, c_{-i} denotes the profile of actions for players other than i where they act as in c, and (c_{-i};d_i) denotes the profile of actions in which i's action is changed to d_i but all others choose the same action as in c. So c = (c_{-i};c_i).
A randomized strategy (or mixed strategy) for player i is a probability distribution over \( C_i \), so \( \Delta(C_i) \) denotes the set of all randomized strategies for i. (pure=nonrandomized.)

An action \( d_i \) for player i is strongly dominated by a randomized strategy \( \sigma_i \in \Delta(C_i) \) if
\[
\forall c_i \in C_i, \quad u_i(c_i; d_i) < \Sigma_{c_i \in C_i} \sigma_i(c_i) u_i(c_i; c_i)
\]
An action \( d_i \) for player i is weakly dominated by a randomized strategy \( \sigma_i \in \Delta(C_i) \) if
\[
\forall c_i \in C_i, \quad u_i(c_i; d_i) \leq \Sigma_{c_i \in C_i} \sigma_i(c_i) u_i(c_i; c_i)
\]
with strict inequality (\(<\)) for at least one \( c_{-i} \).

The set of player i's best responses to any profile of opponents' actions \( c_{-i} \) is
\[
\beta_i(c_{-i}) = \arg\max_{d_i \in C_i} u_i(c_i; d_i) = \left\{ d_i \in C_i \mid u_i(c_i; d_i) = \max_{c_i \in C_i} u_i(c_i; c_i) \right\}
\]
Similarly, if i's beliefs about the other players' actions can be described by a probability distribution \( \mu \) in \( \Delta(C_{-i}) \), then the set of player i's best responses to the beliefs \( \mu \) is
\[
\beta_i(\mu) = \arg\max_{d_i \in C_i} \Sigma_{c_i \in C_{-i}} \mu(c_{-i}) u_i(c_{-i}; d_i)
\]

Fact. If we iteratively eliminate strongly dominated actions for all players until no strongly dominated actions remain, then we get a reduced game in which each remaining action for each player is a best response to some beliefs about the other players' actions. These remaining actions are rationalizable.
If each player $j$ independently uses strategy $\sigma_j$ in $\Delta(C_j)$, then player $i$'s expected payoff is

$$u_i(\sigma) = u_i(\sigma_{-i};\sigma_i) = u_i(\sigma_1,\sigma_2,\ldots,\sigma_n) = \sum_{c \in C} \left( \prod_{j \in N} \sigma_j(c_j) \right) u_i(c) = \sum_{c_i \in C_i} \sigma_i(c_i) \sum_{c_i \in C_i} \left( \prod_{j \in N \setminus i} \sigma_j(c_j) \right) u_i(c_i; c_i) = \sum_{c_i \in C_i} \sigma_i(c_i) u_i(\sigma_{-i}; [c_i]).$$

Here $[c_i] \in \Delta(C_i)$ with $[c_i](c_i) = 1$, $[c_i](d_i) = 0$ if $d_i \neq c_i$. Notice $\sum_{c_i \in C_i} \sigma_i(c_i)(u_i(\sigma_{-i}; [c_i]) - u_i(\sigma)) = 0$.

**Fact** $\sigma_i \in \arg\max_{\tau_i \in \Delta(C_i)} u_i(\sigma_{-i}; \tau_i)$ if & only if $\{c_i \in C_i | \sigma_i(c_i) > 0\} \subseteq \arg\max_{d_i \in C_i} u_i(\sigma_{-i}; [d_i])$.

The set $\{c_i | \sigma_i(c_i) > 0\}$ of actions with positive probability under $\sigma_i$ is the support of $\sigma_i$.

A **Nash equilibrium** is a profile of actions or randomized strategies such that each player is using a best response to the others. That is $\sigma = (\sigma_1,\ldots,\sigma_n)$ is a Nash equilibrium in randomized strategies iff $\sigma_i \in \arg\max_{\tau_i \in \Delta(C_i)} u_i(\sigma_{-i}; \tau_i)$ for every player $i$ in $N$.

**Nash's Theorem.** Any finite strategic-form game has at least one Nash equilibrium in the set of all randomized strategies $(\times_{i \in N} \Delta(C_i))$.

**John Nash's 1950 proof:** Apply the Brouwer fixed point theorem to the function $f: \times_{i \in N} \Delta(C_i) \rightarrow \times_{i \in N} \Delta(C_i)$ such that $f(\sigma) = \tau$ iff $\forall i \in N, \forall c_i \in C_i$:

$$\tau_i(c_i) = (\sigma_i(c_i) + \max\{0, u_i(\sigma_{-i}; [c_i]) - u_i(\sigma)\}) / \sum_{d_i \in C_i} (\sigma_i(d_i) + \max\{0, u_i(\sigma_{-i}; [d_i]) - u_i(\sigma)\}).$$

**Brouwer Fixed Point Theorem:** If $X$ is a nonempty compact convex subset of some finite-dimensional $\mathbb{R}^m$ and $f:X \rightarrow X$ is continuous then $\exists x^*$ such that $f(x^*) = x^*$. 

11
**Some basic 2×2 games.** \( N=\{1,2\} \), each player i has two possible actions, tables show how players' payoffs \((u_1,u_2)\) depend on their actions. (1 chooses row, 2 chooses column)

1. **Cain/Abel game**
   
   \[
   \begin{array}{ccc}
   & 1 & 2 \\
   L & 3,7 & 0,4 \\
   K & 4,0 & 2,2 \\
   \end{array}
   \]

2. **Prisoners' dilemma**
   
   (friendly) \[
   \begin{array}{ccc}
   & 1 & 2 \\
   f_1 & 5,5 & 0,6 \\
   g_1 & 6,0 & 1,1 \\
   \end{array}
   \]

   (aggressive) \[
   \begin{array}{ccc}
   & 1 & 2 \\
   f_1 & 5,5 & 0,6 \\
   g_1 & 6,0 & 1,1 \\
   \end{array}
   \]

3. **Stag hunt**
   
   (go for stag) \[
   \begin{array}{ccc}
   & 1 & 2 \\
   s_1 & 5,5 & 0,4 \\
   h_1 & 4,0 & 2,2 \\
   \end{array}
   \]

   (go for hares) \[
   \begin{array}{ccc}
   & 1 & 2 \\
   s_1 & 5,5 & 0,4 \\
   h_1 & 4,0 & 2,2 \\
   \end{array}
   \]

4. **Battle of sexes**
   
   (football) \[
   \begin{array}{ccc}
   & 1 & 2 \\
   f_1 & 2,1 & 0,0 \\
   s_1 & 0,0 & 1,2 \\
   \end{array}
   \]

   (shopping) \[
   \begin{array}{ccc}
   & 1 & 2 \\
   f_1 & 2,1 & 0,0 \\
   s_1 & 0,0 & 1,2 \\
   \end{array}
   \]

5. **SymmetricBoS**
   
   \[
   \begin{array}{ccc}
   & 1 & 2 \\
   b_1 & 0,0 & 2,1 \\
   a_1 & 1,2 & 0,0 \\
   \end{array}
   \]

   \[
   (1-p) \]

   \[
   \begin{array}{ccc}
   & 1 & 2 \\
   b_1 & 0,0 & 2,1 \\
   a_1 & 1,2 & 0,0 \\
   \end{array}
   \]

   \[
   \begin{align*}
   u_1([b_1],\sigma_2) &= 0q+2(1−q), \\
   u_1([a_1],\sigma_2) &= 1q+0(1−q). \\
   \end{align*}
   \]

6. **OnMinimax**
   
   \[
   \begin{array}{ccc}
   & 1 & 2 \\
   T & 0, 0 & 0,−1 \\
   B & 1, 0 & −1, 3 \\
   \end{array}
   \]

Nonrandom equilibria: 1 \((K,k)\). 2 \((g_1,g_2)\). 3 \((s_1,s_2),(h_1,h_2)\). 4 \((f_1,f_2),(s_1,s_2)\). 5 \((b_1,a_2),(a_1,b_2)\). 6 none.
On Minimax: \[ \begin{array}{c|c|c|c|c}
1: & 2: & \text{L} (q) & \text{R} (1-q) \\
(\text{p}) & \text{T} & 0, 0 & 0, -1 \\
(1-\text{p}) & \text{B} & 1, 0 & -1, 3 \\
\end{array} \]

With \( \sigma_2(\text{L}) = q \),

\[ u_1(\text{T}, \sigma_2) = 0q + 0(1-q) = 0, \]

\[ u_1(\text{B}, \sigma_2) = 1q + (-1)(1-q) = 2q - 1. \]

With \( \sigma_1(\text{T}) = p \), \( u_2(\sigma_1, \text{[L]}) = 0p + (1-p)0 = 0 \), \( u_2(\sigma_1, \text{[R]}) = -1p + 3(1-p) = 3 - 4p \).

1's best responses: \( q < 1/2 \rightarrow \text{T} (p=1) \), \( q > 1/2 \rightarrow \text{B} (p=0) \), \( q = 1/2 \rightarrow \{ \text{all } 0 \leq p \leq 1 \} \).

2's best responses: \( p < 3/4 \rightarrow \text{R} (q=0) \), \( p > 3/4 \rightarrow \text{L} (q=1) \), \( p = 3/4 \rightarrow \{ \text{all } 0 \leq q \leq 1 \} \).

So unique eqm has \( p = 3/4 \) & \( q = 1/2 \), is \( (\sigma_1, \sigma_2) = (3/4[\text{T}] + 1/4[\text{B}], 1/2[\text{L}] + 1/2[\text{R}]) \).

Expected payoffs in equilibrium?

\[ u_1(\sigma_1, \sigma_2) = (3/4)(1/2)0 + (3/4)(1/2)0 + (1/4)(1/2)1 + (1/4)(1/2)(-1) = 0 \]

\[ = u_1(\text{T}, \sigma_2) = (1/2)0 + (1/2)0 = 0 = (1/2)1 + (1/2)(-1) = u_1(\text{B}, \sigma_2). \]

\[ u_2(\sigma_1, \sigma_2) = u_2(\sigma_1, \text{[L]}) = (3/4)0 + (1/4)0 = 0 = (3/4)(-1) + (1/4)3 = u_2(\sigma_1, \text{[R]}). \]

For symmetric BoS, we have equilibria with expected payoffs \((u_1, u_2)\) as follows:

\((b_1, a_2)\) with \((2, 1)\), \((a_1, b_2)\) with \((1, 2)\), \((2/3[b_1] + 1/3[a_1], 2/3[b_2] + 1/3[a_2])\) with \((2/3, 2/3)\).

Any symmetric game must have a symmetric equilibrium, but the symmetric equilibrium of this game is Pareto-inferior to both non-symmetric equilibria \((2 > 1 > 2/3)!\)

The Stag Hunt has a third randomized equilibrium where each does \(2/3[s_i] + 1/3[h_i]\).
Example of a game on $C_1 \times C_2 = \mathbb{R}_+ \times \mathbb{R}_+$: almost-symmetric joint-project (E & A Kalai).

$c_1 \geq 0, c_2 \geq 0, \; u_1(c_1, c_2) = 25(c_1 + c_2) - (c_1 + c_2)^2 - c_1, \; u_2(c_1, c_2) = 25.2(c_1 + c_2) - (c_1 + c_2)^2 - c_2.$

1st-order conditions for $c_1 = \beta_1(c_2) > 0$: $0 = \partial u_1 / \partial c_1 = 25 - 2(c_1 + c_2) - 1 \implies \beta_1(c_2) = 12 - c_2.$

1st-order conditions for $c_2 = \beta_2(c_1) > 0$: $0 = \partial u_2 / \partial c_2 = 25.2 - 2(c_1 + c_2) - 1 \implies \beta_2(c_1) = 12.1 - c_1.$

With "$c_1 \geq 0$" constraint, the boundary condition for an optimum at $c_1 = \beta_1(c_2) = 0$
would be $0 \geq \partial u_1 / \partial c_1 = 25 - 2(0 + c_2) - 1 \implies c_2 \geq 12.$

So we get $\beta_1(c_2) = \max\{12 - c_2, 0\}$. Similarly, $\beta_2(c_1) = \max\{12.1 - c_1, 0\}$. The equilibrium is $(c_1, c_2) = (0, 12.1)$. It yields payoffs $u_1 = 156.09, \; u_2 = 146.41$. 
2×2×...×2 n-person example: reporting a crime. Given \( n \geq 2, \ v > c > 0 \). (Say \( v=100, \ c=1 \))

\( C_i = \{\text{call, don't}\}, \ u_i(a_1,...a_n) = v-c \) if \( a_i=\text{call} \), \( u_i = v \) if \( a_i=\text{don't} \) but some \( a_j=\text{call} \), else \( u_i=0 \).

Let \( p = \sigma_i(\text{call}) \) in symmetric equilibrium \( (0 \leq p \leq 1) \).

\[
\begin{array}{c|c|c|c}
   & \text{call} & \text{don't} & \vspace{0.5cm} \\
\hline
\text{call} & v-c & 0 & \vspace{0.5cm} \\
\text{don't} & v-c & v & \vspace{0.5cm} \\
\end{array}
\]

Can there be a symmetric equilibrium with \( p=0 \)? No! \((0 < v-c)\).
Can there be a symmetric equilibrium with \( p=1 \)? No (with \( n \geq 2 \)! \((v-c < v)\)

So \( 0 < p < 1 \), support =\{\text{call, don't}\}, and thus \( u_i(\sigma_i;[\text{call}]) = u_i(\sigma_i;[\text{don't}]) = \forall i \), and so \( v-c = (1-p)^{n-1} 0 + (1-(1-p)^{n-1})v \).

Therefore \( (1-p)^{n-1} = c/v, \ p = 1-(c/v)^{1/(n-1)} \).

Notice \( p \to 0 \) as \( n \to \infty \). \( \forall n>1: \ P(\text{no others call}) = (1-p)^{n-1} = c/v \), \( P(\text{no calls}) = (1-p)^n = (c/v)^{n/(n-1)} \to c/v \) as \( n \to \infty \) (it increases in \( n \)).
(With \( n=1 \), get \( p=1 \), \( P(\text{no calls}) = 0. \))

\( E(\#\text{calls})=np. \) As \( n \to \infty \), \( \#\text{calls} \to \text{Poisson}, \) mean \( np \to \lambda = -\text{LN}(c/v) \), so that \( e^{-\lambda} = c/v \).
Schelling's (1960) focal point effect: In a game with multiple equilibria, any salient cultural or environmental factor that focuses people's attention on one equilibrium can generate expectations that people will behave as this equilibrium predicts, so that it becomes rational for everyone to fulfill this prediction, as a self-fulfilling prophecy. (Thomas Schelling, "Bargaining, communication, and limited war," Journal of Conflict Resolution 1:19-36 (1957) <https://www.jstor.org/stable/172548> )

Consider an island where every day different matched pairs play the following rival-claimants game in various places on the island:

<table>
<thead>
<tr>
<th></th>
<th>2 claims</th>
<th>2 defers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 claims</td>
<td>-1, -1</td>
<td>9, 0</td>
</tr>
<tr>
<td>1 defers</td>
<td>0, 9</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Nash equilibria:
- (1 claims, 2 defers) with payoffs \((u_1,u_2) = (9,0)\),
- (1 defers, 2 claims) with payoffs \((u_1,u_2) = (0,9)\),
- each claims with independent probability \(9/10\), with each \(E_{u_i} = 0 = 0.9 \times -1 + 0.1 \times 9\).

Focal equilibrium: They may play the top equilibrium \((9,0)\) when player 1 is recognized as owner here. (Justice, arbitration, divination; transfer of ownership by handshake.)

Social equilibria: anarchy; traditional ownership, legislation of ownership principles; focal arbitration by a recognized leader (duly elected, with limited authority).
Computing randomized Nash equilibria. We describe here a procedure for finding Nash equilibria, from section 3.3 of Myerson (1991). We are given some game, including a given set of players $N$ and, for each $i$ in $N$, a set of feasible actions $C_i$ for player $i$ and a payoff function $u_i:C_1 \times \ldots \times C_n \to \mathbb{R}$ for player $i$. The support of a randomized equilibrium is, for each player, the set of actions that have positive probability of being chosen in this equilibrium. To find a Nash equilibrium, we can apply the following 5-step method:

1. Guess a support for all players. That is, for each player $i$, let $S_i$ be a nonempty subset of $C_i$, and guess that $S_i$ is the set of actions that player $i$ uses with positive probability.
2. Consider the smaller game where the action set for each player $i$ is reduced to $S_i$, and try to find an equilibrium where all of these actions get positive probability. To do this, we need to solve a system of equations for some unknown quantities.

The unknowns: For each player $i$ in $N$ and each action $s_i$ in $i$'s support $S_i$, let $\sigma_i(s_i)$ denote $i$'s probability of choosing $s_i$, and let $w_i$ denote player $i$'s expected payoff in the equilibrium. ($\sigma_i(a_i)=0$ if $a_i \not\in S_i$.)

The equations: For each player $i$, the sum of these probabilities $\sigma_i(s_i)$ must equal 1. $\forall i \in N$ and $\forall s_i \in S_i$, player $i$'s expected payoff when he chooses $s_i$ but all other players randomize independently according to their $\sigma_j$ probabilities must be equal to $w_i$. Let $u_i(\sigma_j,[a_i])$ denote player $i$'s expected payoff when he chooses action $a_i$ and all other players are expected to randomize independently according to their $\sigma_j$ probabilities. The equations can be written: $\sum_{s_i \in S_i} \sigma_i(s_i) = 1 \ \forall i \in N$; $u_i(\sigma_{-i},[s_i]) = w_i \ \forall i \in N \ \forall s_i \in S_i$. Notice that we have as many equations as unknowns ($w_i, \sigma_i(s_i)$).
(3) If the equations in step 2 have no solution, then we guessed the wrong support, and so we must return to step 1 and guess a new support.
Assuming that we have a solution from step (2), continue to (4) and (5)
(4) The solution from (2) would be nonsense if any of the "probabilities" were negative. That is, for every player \( i \) in \( N \) and every action \( s_i \) in \( i \)'s support \( S_i \), we need \( \sigma_i(s_i) \geq 0 \).
If these nonnegativity conditions are not satisfied by a solution, then we have not found an equilibrium with the guessed support. So we return to step 1, guess a new support.
If we have a solution that satisfies all these nonnegativity conditions, then it is an equilibrium of the reduced game where each player \( i \) can only choose actions in \( S_i \).
(5) A solution from (2) that satisfies the condition in (4) would still not be an equilibrium of the original game, if any player would prefer an action outside the guessed support. Recall \( u_i(\sigma_{-i},[s_i]) = w_i \ \forall s_i \in S_i \). So we must ask, for each player \( i \) and each action \( a_i \) that is in \( C_i \) but is not in the guessed support \( S_i \), could \( i \) do better than \( w_i \) by choosing \( a_i \) when all other players act according to their \( \sigma_j \) probabilities? That is, for every action \( a_i \) that is in \( C_i \) but is not in \( S_i \) (so \( \sigma_i(a_i)=0 \)), we need \( u_i(\sigma_{-i},[a_i]) \leq w_i \).
If our solution satisfies all these inequalities then it is an equilibrium of the given game. But if any of these inequalities is violated (some \( u_i(\sigma_{-i},[a_i]) > w_i \)), then we have not found an equilibrium with the guessed support, and so we must return to step 1 and guess a new support. (There are only finitely many possible supports to consider.)
Thus, an equilibrium \( \sigma = (\sigma_i(a_i))_{a_i \in C_i, i \in N} \) with payoffs \( w = (w_i)_{i \in N} \) must satisfy:
\[
\sum_{a_i \in C_i} \sigma_i(a_i) = 1 \quad \forall i \in N; \quad \text{and} \quad \sigma_i(a_i) \geq 0 \quad \text{and} \quad u_i(\sigma_{-i},[a_i]) \leq w_i \quad \text{with at least one equality} \quad \forall a_i \ \forall i
\]
(complementary slackness). The support for \( i \) is \( S_i = \{ s_i \in C_i \mid \sigma_i(s_i) > 0, \text{ so } u_i(\sigma_{-i},[a_i]) = w_i \} \).
Example. Find all Nash equilibria (pure and mixed) of the following $2 \times 3$ game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>7, 2</td>
<td>2, 7</td>
<td>3, 6</td>
</tr>
<tr>
<td>B</td>
<td>2, 7</td>
<td>7, 2</td>
<td>4, 5</td>
</tr>
</tbody>
</table>

There are $(2^2-1) \times (2^3-1) = 3 \times 7 = 21$ possible supports.

But it is easy to see that this game has no pure-strategy equilibria. (2's best response to T is M, but T is not 1's best response to M; and 2's best response to B is L, but B is not 1's best response to L). This eliminates the six cases where each player's support is just one action. Furthermore, when either player is restricted to just one action, the other player always has a unique best response, so there are no equilibria where only one player randomizes. That is, both players must have at least two actions in the support of any equilibrium. Thus, we must search for equilibria where the support of randomized strategy is \{T,B\}, and the support of 2's randomized strategy is \{L,M,R\} or \{M,R\} or \{L,M\} or \{L,R\}. 
Guess support is \{T,B\} for 1 and \{L,M,R\} for 2?

Write 1's strategy as \( \sigma_1 = p[T] + (1-p)[B] \), 2's strategy as \( \sigma_2 = q[L] + (1-q-r)[M] + r[R] \), that is \( p = \sigma_1(T) \), \( 1-p = \sigma_1(B) \), \( q = \sigma_2(L) \), \( r = \sigma_2(R) \), \( 1-q-r = \sigma_2(M) \).

Player 1 randomizing over \{T,B\} requires \( w_1 = u_1(T,\sigma_2) = u_1(B,\sigma_2) \), and so
\[
w_1 = 7q + 2(1-q-r) + 3r = 2q + 7(1-q-r) + 4r.
\]

Player 2 randomizing over \{L,M,R\} requires \( w_2 = u_2(\sigma_1,L) = u_2(\sigma_1,M) = u_2(\sigma_1,R) \), and so
\[
w_2 = 2p + 7(1-p) = 7p + 2(1-p) = 6p + 5(1-p).
\]

We have three equations for three unknowns \( p,q,r \), but they have no solution (as the two indifference equations for player 2 imply both \( p=1/2 \) and \( p = 3/4 \), which is impossible).

Thus, there is no equilibrium with this support.
Guess support is \{T,B\} for 1 and \{M,R\} for 2?

We write 1's strategy as $p[T] + (1-p)[B]$ , 2's strategy as $(1-r)[M] + r[R]$. (q=0)

Player 1 randomizing over \{T,B\} requires $w_1 = u_1(T,\sigma_2) = u_1(B,\sigma_2)$,
so $w_1 = 2(1-r)+3r = 7(1-r)+4r$.

Player 2 randomizing over \{M,R\} requires $w_2 = u_2(\sigma_1,M) = u_2(\sigma_1,R)$,
so $w_2 = 7p+2(1-p) = 6p+5(1-p)$.

The solution for these equations is $p = \frac{3}{4}$ and $r = \frac{5}{4}$, with $w_1 = \frac{13}{4}$, $w_2 = \frac{23}{4}$.

But this would yield $\sigma_2(M) = 1-r = -\frac{1}{4} < 0$, and so no equilibrium has this support.

(Notice: if player 2 never chose L then T would be dominated by B for player 1.)
Player 2

<table>
<thead>
<tr>
<th></th>
<th>L (q)</th>
<th>M (1-q)</th>
<th>R (0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p) T</td>
<td>7, 2</td>
<td>2, 7</td>
<td>3, 6</td>
</tr>
<tr>
<td>(1-p) B</td>
<td>2, 7</td>
<td>7, 2</td>
<td>4, 5</td>
</tr>
</tbody>
</table>

Guess support is \{T,B\} for 1 and \{L,M\} for 2?

We write 1's strategy as \(p[T] + (1-p)[B]\), 2's strategy as \(q[L] + (1-q)[M]\). \((r=0)\)

Player 1 randomizing over \{T,B\} requires \(w_1 = u_1(T, \sigma_2) = u_1(B, \sigma_2)\),
so \(w_1 = 7q + 2(1-q) = 2q + 7(1-q)\).

Player 2 randomizing over \{L,M\} requires \(w_2 = u_2(\sigma_1, L) = u_2(\sigma_1, M)\),
so \(w_2 = 2p + 7(1-p) = 7p + 2(1-p)\).

The solution for these equations is \(p = 1/2\) and \(q = 1/2\), with \(w_1 = 9/2, w_2 = 9/2\).

This solution yields nonnegative probabilities for all actions.

We must check that player 2 would not prefer deviating outside her support to R. But \(u_2(\sigma_1, R) = 6p + 5(1-p) = 6\times 1/2 + 5\times 1/2 = 11/2 > w_2 = u_2(\sigma_1, L) = 2\times 1/2 + 7\times 1/2 = 9/2\).

So there is no equilibrium with this support.
Guess support is \{T,B\} for 1 and \{L,R\} for 2? (Nash says some support must work!)

We write 1's strategy as \( p[T] + (1-p)[B] \), 2's strategy as \( q[L] + (1-q)[R] \). (\( r=1-q \))

Player 1 randomizing over \{T,B\} requires \( w_1 = u_1(T,\sigma_2) = u_1(B,\sigma_2) \),
so \( w_1 = 7q + 3(1-q) = 2q + 4(1-q) \).

Player 2 randomizing over \{L,R\} requires \( w_2 = u_2(\sigma_1,L) = u_2(\sigma_1,R) \),
so \( w_2 = 2p + 7(1-p) = 6p + 5(1-p) \).

The solution for these equations is \( p = 1/3 \) and \( q = 1/6 \), with \( w_1 = 11/3, w_2 = 16/3 \).

This solution yields nonnegative probabilities for all actions.

We also need to check that player 2 would not prefer deviating outside her support to M;
\( u_2(\sigma_1,M) = 7p + 2(1-p) = 7\times1/3 + 2\times2/3 = 11/3 < w_2 = u_2(\sigma_1,L) = 2\times1/3 + 7\times2/3 = 16/3 \).

Thus, the equilibrium with this support is \(((1/3)[T] + (2/3)[B], (1/6)[L] + (5/6)[R])\).
**Introduction to extensive games with perfect information.**

An extensive game with perfect information is a tree diagram with: nodes, branches, a root node, terminal nodes, decision nodes, player label, terminal payoffs, chance nodes, chance probabilities.

Analytical concepts: the path of play, a strategy for a player, the normal representation in strategic form, subgame, subgame-perfect equilibrium.

![Extensive Game Diagram]

**Ex1**

Normal representation in strategic form:

<table>
<thead>
<tr>
<th></th>
<th>Player 1: \ Player 2:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acquiesce</td>
<td>Acquiesce</td>
</tr>
<tr>
<td>In</td>
<td>2, 1</td>
</tr>
<tr>
<td>Out</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

sgp-eqm (I,A); other nash eqms (O,q[F]+(1−q)[A]) for q≥1/2.
Normal representation in strategic form:

<table>
<thead>
<tr>
<th></th>
<th>Player 2:</th>
<th>E_CE_D</th>
<th>E_CF_D</th>
<th>F_CE_D</th>
<th>F_CF_D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1: C</td>
<td>1, 3</td>
<td>1, 3</td>
<td>3, 2</td>
<td>3, 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>0, 0</td>
<td>2, 1</td>
<td>0, 0</td>
<td>2, 1</td>
</tr>
</tbody>
</table>

sgp (D,E_CF_D); nash (C,p[E_CE_D]+(1−p)[E_CF_D]) for p≥1/2, (D,p[E_CF_D]+(1−p)[F_CF_D]) for 1>p≥1/2.

**Fact:** The normal representation in strategic form of an extensive game with perfect information must have at least one equilibrium in pure strategies. A pure-strategy subgame-perfect eqm can be found by backward induction, starting with the analysis of subgames of shortest length. In strategic form, Nash eqms that do not correspond to subgame-perfect eqms can be eliminated by iterative elimination of weakly dominated (or equivalent) strategies.
The Holdup Problem

Player 1 can invest to improve an asset which he may later sell player 2. First player 1 chooses an amount $e \geq 0$ to invest on improving the asset. Given $e$, the asset is worth $v_1(e) = e^{0.5}$ to player 1, but is worth $v_2(e) = 2e^{0.5}$ to player 2. We consider two versions of this game, which differ in how they bargain over the price.

Buyer-offer game  First player 1 chooses the amount $e \geq 0$ to invest in the asset. Player 2 observes this investment $e$. Then player 2 chooses a price $p \geq 0$ at which she offers to buy the asset from player 1. Player 1 observes this offer, and then can choose to accept or reject it. Final payoffs are: $u_1(e, p, \text{accept}) = p - e$, $u_2(e, p, \text{accept}) = v_2(e) - p$, $u_1(e, p, \text{reject}) = v_1(e) - e$, $u_2(e, p, \text{reject}) = 0$.

There is a unique subgame-perfect equilibrium. At the last stage, player 1 accepts if $p > v_1(e)$ and rejects if $p < v_1(e)$. So player 2's optimal offer, given $e$, must be to offer $p = v_1(e)$, which player 1 accepts. (Note: Player 1 is actually indifferent between accepting and rejecting, but there would be no optimal offer for 2 if player 1 rejected in this case of indifference!) So player 1 expects that his payoff from $e$ will be $v_1(e) - e = e^{0.5} - e$, maximized by $e = 0.25$.

So the equilibrium outcome is: 1 chooses $e = 0.25$, 2 offers $p = 0.25^{0.5} = 0.5$, and payoffs are $u_1 = 0.5 - 0.25 = 0.25$, $u_2 = 2 \times 0.25^{0.5} - 0.5 = 1 - 0.5 = 0.5$. 
Seller-offer game. First player 1 chooses his investment $e \geq 0$. Then player 1 chooses the price $p \geq 0$ at which he offers to sell the asset. Player 2 observes $e$ and $p$, and then can choose to accept or reject 1's offer. Payoffs are still $u_1(e, p, \text{accept}) = p - e$, $u_2(e, p, \text{accept}) = v_2(e) - p$, $u_1(e, p, \text{reject}) = v_1(e) - e$, $u_2(e, p, \text{reject}) = 0$.

In the unique subgame-perfect equilibrium:
player 2 accepts if $p \leq v_2(e)$ but rejects if $p > v_2(e)$, so given $e$, player 1 offers $p = v_2(e)$.

So player 1 chooses $e = 1$ to maximize $2e^{0.5} - e$. Thus the equilibrium outcome is:
1 chooses $e = 1$ and offers $p = 2 \times 1^{0.5} = 2$, payoffs are $u_1 = 2 - 1 = 1$, $u_2 = 2 \times 1^{0.5} - 2 = 0$.

This seller-offer game also has many other Nash equilibria that are not subgame perfect.

The equilibrium sum of payoffs $u_1 + u_2$ is greater in the seller-offer game.

For an efficient outcome, the person who made the first-period investment should have more control in the process of bargaining over the price.

If they were about to play the buyer-offer game, the buyer would be willing to sell her right to set the price for any payment more than 0.5, and the seller would be willing to pay up to 0.75 for the right to set the price.

Both games have many other Nash equilibria that are not subgame-perfect. Consider any $(\hat{e}, \hat{p})$ such that $v_2(\hat{e}) \geq \hat{p} \geq \hat{e} + \max_e (v_1(e) - e) = \hat{e} + 0.25$ (such as $\hat{e} = 1$, $\hat{p} = 1.625$), so that each does better than he could alone. With either player offering the price, there is a Nash equilibrium in which 1 invests this $\hat{e}$, and then this price $\hat{p}$ is offered and accepted, but rejection would follow any other investment $e \neq \hat{e}$ or any other price-offer $p \neq \hat{p}$. These Nash equilibria violate sequential rationality, however, as threats to reject prices between $v_1(e)$ and $v_2(e)$ would not be credible.
Introduction to general finite extensive games (with imperfect information).
See sections 2.1-2.2 in my book.
Games in extensive form are shown by tree diagrams with: nodes, branches, root node, terminal nodes, decision nodes, player labels, terminal payoffs, chance nodes, chance probabilities, information labels, move labels.
I write "PlayerLabel.InformationLabel" at decision nodes.

An information set is a set of decision nodes with the same player & information labels. A player's (pure) strategy is a complete plan specifying a feasible move for the player at every possible information set of the player.

Eqm (C,F) $\rightarrow$ (3,2).

Sgp-eqm (D,Ef) $\rightarrow$ (2,1).
Example (from Figure 2.2 in my book):

Normal representation in strategic form:

<table>
<thead>
<tr>
<th></th>
<th>2:</th>
<th>M</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.a/1.b:</td>
<td>0, 0</td>
<td>1,-1</td>
<td></td>
</tr>
<tr>
<td>Rr</td>
<td>.5,-.5</td>
<td>0, 0</td>
<td></td>
</tr>
<tr>
<td>Fr</td>
<td>-.5,.5</td>
<td>1,-1</td>
<td></td>
</tr>
<tr>
<td>Ff</td>
<td>0, 0</td>
<td>0, 0</td>
<td></td>
</tr>
</tbody>
</table>

Notice that 1's mixed strategies 0.5[Rr]+0.5[Ff] and 0.5[Rf]+0.5[Fr] are both equivalent to the same behavioral strategy (0.5[R]+0.5[F], 0.5[r]+0.5[f]).

The mixed-strategy equilibrium $\tau = ((1/3)[Rr]+(2/3)[Rf], (2/3)[M]+(1/3)[P])$ is equivalent to behavioral strategy $\sigma = ([R], (1/3)[r]+(2/3)[f]; (2/3)[M]+(1/3)[P])$, yields expected payoffs $Eu_1 = 1/3$, $Eu_2 = -1/3$.

The sequential equilibrium consists of the behavioral strategy $\sigma$ and belief probabilities $\mu_{2.r}(1.a,R) = (1/2)(1)/((1/2)(1)+(1/2)(1/3)) = 3/4$, $\mu_{2.r}(1.b,r) = 1-3/4 = 1/4$. 

0, 0 0, 0 0, 0
Move probabilities, belief probabilities and sequential equilibria

Suppose that we are given some extensive game with imperfect information. Given a randomized strategy for player i, at any information set $t_i$ of player i that could occur with positive probability when i plays this strategy, we can compute a probability distribution over the set of possible actions $\{d_i\}$ for player i at this information set. These probabilities $\sigma_i(d_i|t_i)$ are called move probabilities (or action probabilities). That is, the move-probability for any move $d_i$ at any information state $t_i$ of player i denotes the probability that player i will choose move $d_i$ if information set $t_i$ occurs.

A behavioral strategy $\sigma_i$ for player i is a vector that specifies a move-probability distribution for each of player i’s information sets.

A behavioral-strategy profile $\sigma$ is a vector that specifies a behavioral strategy $\sigma_i$ for each player i, and so it must specify an move probability $\sigma_i(d_i|t_i)$ for every possible move $d_i$ at every possible information set $t_i$ of every player i in the game. When $D_{i,s}$ denotes the set of moves at for each player i at each information set, then mixed strategies are $\tau \in \times_{i \in N} \Delta(\times_{s \in S_i} D_{i,s})$, behavioral strategies are $\sigma \in \times_{i \in N} \times_{s \in S_i} \Delta(D_{i,s})$.

The chance probabilities on all branches that follow chance nodes are given parameters of the extensive game. We assume here that the chance probabilities are all positive. Given $\sigma$, a profile of behavioral strategies for all players, the prior probability $P(x|\sigma)$ of any node x in the tree is the multiplicative product of all chance-probabilities and move-probabilities on the path that leads to this node from the root node.
A full-support behavioral strategy profile assigns strictly positive probability \((\sigma_i(d_i|t_i)>0)\) to every possible move \(d_i\) at every information set \(t_i\) of every player \(i\), so that every node \(x\) in the tree has positive probability.

When player \(i\) moves at his information set \(t_i\), the belief probability that player \(i\) should assign to any node \(x\) in this information set should be, by Bayes's formula,

\[
\mu_i(x|t_i) = \frac{P(x|\sigma)}{\sum_{y \in t_i} P(y|\sigma)}.
\]

That is, the belief probability \(\mu_i(x|t_i)\) should equal the prior probability of \(x\) divided by the sum of prior probabilities of all nodes in the information set \(t_i\), whenever this formula is well-defined (not \(0/0\)).

A belief system \(\mu\) is a vector that specifies a belief-probability distribution \(\mu_i(\bullet|t_i)\) over the nodes of each information set \(t_i\) of each player \(i\) in the game.

Bayes's formula yields one belief system for any full-support behavioral strategy profile. But for strategies that do not have full support, Bayes's formula may leave some belief probabilities undefined, at information sets where all nodes have prior probability 0. A beliefs system \(\mu\) is consistent with a behavioral strategy profile \(\sigma\) iff there exists a sequence of full-support behavioral strategies \(\tilde{\sigma}^k\) that converge to \(\sigma\) and yield Bayesian beliefs \(\tilde{\mu}^k\) that converge to \(\mu\) as \(k \to \infty\). (All \(\tilde{\sigma}^k_i(d_i|t_i) \to \sigma_i(d_i|t_i) \& \tilde{\mu}^k_i(x|t_i) \to \mu_i(x|t_i)\).)
A behavioral-strategy profile $\sigma$ is sequentially rational given a beliefs system $\mu$ iff, at each information set $t_i$ of each player $i$, $\sigma_i(\bullet | t_i)$ assigns positive move-probabilities only to moves that maximize $i$'s expected payoff at $t_i$, given $i$'s beliefs $\mu_i(\bullet | t_i)$ about the current node in the information set $t_i$ and given what the behavioral-strategy profile $\sigma$ specifies about players' behavior after this information set.

A sequential equilibrium is a pair $(\sigma, \mu)$, where $\sigma$ is a behavioral strategy profile and $\mu$ is a belief system, such that $\sigma$ is sequentially rational given the beliefs system $\mu$, and the beliefs system $\mu$ is consistent with the behavioral-strategy profile $\sigma$.

A game has perfect information if every information set consists of just one node. A game with perfect information can have only one possible beliefs system, which trivially assigns belief probability 1 to every decision node. For a game with perfect information, a behavioral strategy profile $\sigma$ is a subgame-perfect equilibrium if it would form a sequential equilibrium together with this (trivial) beliefs system $\mu$. 