PPHA 41501, Autumn 2021: ASSIGNMENT 7, ***to be discussed in class Dec 1***.

1. Players 1 and 2 each must decide whether to fight for a valuable prize. If both players decide to fight, then they both lose \$1, and nobody gets the prize (it is destroyed). If one player decides to fight but the other does not, then the player who is willing to fight gets the prize. A player who does not fight is guaranteed a payoff of 0.

Everybody knows that the prize is worth $V_2 = \$2$ to player 2. But the prize may be worth more to player 1. Let V_1 denote the value of the prize to player 1.

In terms of V_1, the players' payoffs (u_1,u_2) will depend on their actions as follows:

Player 1: \ Player 2:	NotFight	Fight
NotFight	0, 0	0, 2
Fight	V_1, 0	−1, −1

Let us explore some different assumptions about this value V_1.

(a) Suppose first that the value of the prize to player 1 is $V_1 = \$3$, and everybody knows this. Find all equilibria of this game, including a mixed-strategy equilibrium in which both players have a positive probability of fighting.

(b) Suppose next that the value of the prize to player 1 is either $V_1 = \$2$ or $V_1 = \$3$.
Player 1 knows his actual value, but player 2 thinks each of these possibilities has probability 1/2.
Find a Bayesian equilibrium where player 2 randomizes between fighting and not fighting.

(c) Finally, suppose that the value of the prize to player 1 is $V_1 = \$2+\tilde{t}_1$ where $\tilde{t}_1$ can be any number between 0 and 1. Player 1 knows its actual value, but player 2 thinks of $\tilde{t}_1$ as a Uniform random variable on the interval from 0 to 1. Find a Bayesian equilibrium where player 2 randomizes between fighting and not fighting.

2. Consider a two-person game where player 1 chooses T or B, and player 2 chooses L or R.
When they play this game, player 2 also knows whether her type is A or B.
The players' utility payoffs (u_1,u_2) depend on their actions and on player 2's type as follows:

2'sType = A	L	R
T	4,0	0,2
B	0,4	2,0

2'sType = B	L	R
T	4,0	0,4
B	0,2	2,0

(a) Suppose first that player 1 thinks that 2's type is equally likely to be A or B. Find a Bayesian equilibrium.

(b) How would the Bayesian equilibrium change in a game where player 1 thinks that player 2 has probability 1/6 of being type A, and has probability 5/6 of being type B?

3. Consider the following Bayesian game, where player 2's type is her vulnerability at (T,L).
Player 2 knows the amount t_2 that she would have to pay player 1 if they play (T,L), but player 1 only knows that t_2 was drawn from a Uniform distribution on the interval from 0 to 1.

Player 1: \ Player 2:	L	R
T	$t_2, -t_2$	0, 0
B	0, 0	0.5, −0.5

(a) A student said, "player 2 should choose R if $t_2>0.5$, but 2 should choose L if $t_2<0.5$". Show that this student's analysis is not compatible with any equilibrium, by first computing player 1's best response to the strategy that the student has recommended for 2, and then computing player 2's best response to this best-response of player 1.

(b) Find a Bayesian equilibrium of this game. Be sure to fully specify the strategies for both players.

*(c) Consider the following game, where t_1 and t_2 are independent random variables drawn from the interval 0 to 1, player 1 knows t_1, and player 2 knows t_2:

Player 1: \ Player 2:	L	R
T	$t_2, -t_2$	0, 0
B	0, 0	$t_1, -t_1$

Find a Bayesian equilibrium. (Equations for cutoffs may be solved numerically with Excel goal-seek.)

PPHA 41501, Autumn 2021: ASSIGNMENT 6, *due Nov 22*.

1. Consider a repeated game where 1 and 2 repeatedly play the game below infinitely often.

	a_2	b_2
a_1	8, 8	1, 2
b_1	2, 1	0, 0

The players want to maximize their δ-discounted sum of payoffs, for some $0 \le \delta < 1$.
Consider the following state-dependent strategies: The possible states are state 1 and state 2.
In state 1, we anticipate that player 1 will play b_1 and player 2 will play a_2.
In state 2, we anticipate that player 1 will play a_1 and player 2 will play b_2.
The game begins at period 1 in state 1. The state of the game would change after any period where the outcome of play was (a_1,a_2), but otherwise the state always stays the same.
What is the lowest value of δ such that these strategies form a subgame-perfect equilibrium?

2. Consider a repeated game where 1 and 2 repeatedly play the game below infinitely often.

	a_2	b_2
a_1	3, 3	0, 5
b_1	5, 0	−4,−4

The players want to maximize their δ-discounted sum of payoffs, for some $0 \le \delta < 1$.
(a) Find the lowest value of δ such that you can construct an equilibrium in which the players will actually choose (a_1,a_2) forever, but if any player i ever chose b_i at any period then they would play the symmetric randomized equilibrium of the one-stage game forever afterwards.
(b) What is the lowest value of δ such that you can construct a subgame-perfect equilibrium in which the players will actually choose (a_1,a_2) forever, but if some player i unilaterally deviated to b_i at any period then that player i would get payoff 0 at every round thereafter? Be sure to precisely describe state-dependent strategies that form this equilibrium.

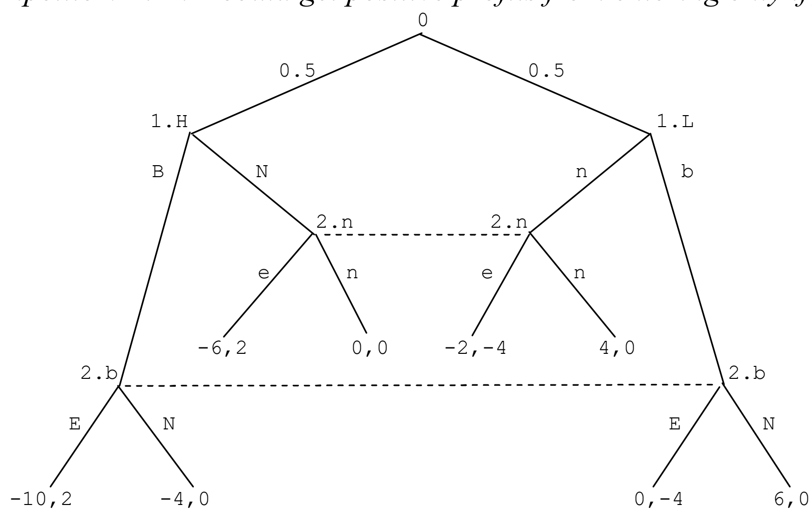
3. Consider a repeated game where 1 and 2 repeatedly play the game below infinitely often.

	a_2	b_2
a_1	0, 8	2, 0
b_1	8, 0	0, 2

Each player i wants to maximize his or her δ_i-discounted sum of payoffs, for some δ_1 and δ_2, where each $0 \le \delta_i < 1$.
Find the lowest values of δ_1 and δ_2 such that you can construct an equilibrium in which the players will actually alternate between (a_1,a_2) and (b_1,a_2) forever, but if any player ever deviated then they would play the randomized equilibrium of the one-stage game forever afterwards.

PPHA 41501, Autumn 2021: ASSIGNMENT 5, *due Nov 15*.

1. Consider the following extensive-form game, which begins with a chance move.
(*Interpretation: firm 1 has high or low costs and must decide whether to build a new factory; then firm 2 observes whether the new factory is built and decides whether to enter firm 1's market as a competitor. Firm 2 could get positive profits from entering only if firm 1 has high costs...*)



(a) Show the normal representation of this game in strategic form.

(b) Find all pure-strategy Nash equilibria of this game.

(c) For each Nash equilibrium that you found in part (b), explain whether this equilibrium corresponds to a sequential equilibrium of the extensive-form game. If so, at every information set (including sets of probability zero) you should indicate what beliefs would make this a sequential equilibrium.

(d) Apply iterative elimination of weakly dominated strategies to the normal representation. Does this analysis eliminate any of the Nash equilibria that you found in part (a)?

2. Consider again the game from exercise 1 of Assignment 4, where player 1 first chooses T or B, then player 2 chooses L or R after observing 1's move, and their payoffs depend on their choices as follows:

Player 1 \ Player 2:	L	R
T	3, 2	1, 1
B	4, 3	2, 4

But now suppose that, whatever 1 chooses, the probability that player 2 will correctly observe 1's action is 0.9, and there is probability 0.1 that player 2 will mistakenly observe the other action (which 1 did not choose). The payoffs depend on the players' **actual** choices according to the previous table (so, for example, if 1 chose T but 2 mistakenly observed B and chose R then 2's payoff would be 1).

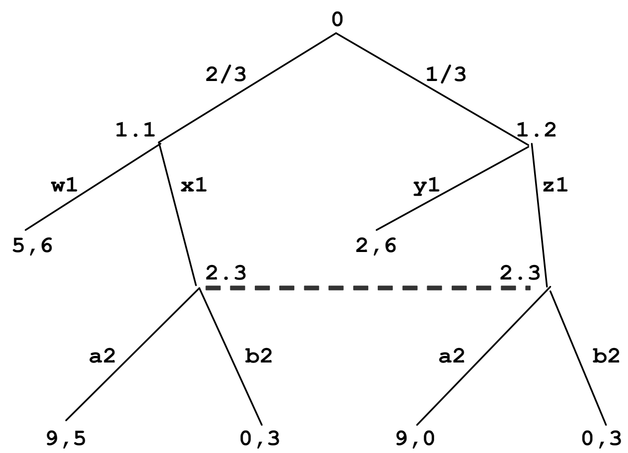
(a) Show the extensive-form game that describes this situation.

(b) Show the normal representation in strategic form for the extensive-form game in part (a).

(c) Find a sequential equilibrium in which player 2 would choose [L] for sure if she observed T.

*(d) Characterize the other sequential equilibria of this game.

3. Consider the following extensive-form game, where player 1 observes the chance move, but player 2 does not observe it. If 2 gets to move, she knows only that 1 chose either x1 or z1.



(a) Find a sequential equilibrium in which the (prior) probability of player 2 getting to move is 1.

(b) Find a sequential equilibrium in which the probability of player 2 getting to move is 0. (You must describe what player 2 would believe and do if she got to move.)

(c) Find a sequential equilibrium in which the probability of player 2 getting to move is strictly between 0 and 1.

(d) Show the normal representation of this game in strategic form.

PPHA 41501, Autumn 2021: ASSIGNMENT 4, *due Nov 1*.

1. Consider a game where player 1 must choose T or B, player 2 must choose L or R, and their payoffs depend on their choices as follows.

Player 1 \ Player 2:	L	R
T	3, 2	1, 1
B	4, 3	2, 4

Suppose that player 1 moves first, and then player 2 makes her choice after observing 1's move.

(a) Show the extensive-form game with perfect information that describes this situation.

(b) Show the normal representation in strategic form for the extensive-form game in part (a).

(c) Find the unique subgame-perfect equilibrium of this game. Be sure to fully describe the strategies for each player in this equilibrium.

(d) Find a Nash equilibrium of this game that is **not** subgame-perfect (or sequentially rational) and that yields different expected payoffs from the equilibrium that you found in (c). Be sure to fully describe the strategies for each player in this equilibrium.

2. Players 1 and 2 are in a sequential all-pay-own-bid auction for a prize worth \$3. First, player 1 must pay \$1 or pass. When anyone passes, the other player gets the \$3 prize (and game ends). Otherwise, the other player can bid next, and must either pay \$2 (if he has it) or pass.
A player cannot pay more than his given available funds. This game has perfect information.

(a) Find a subgame-perfect equilibrium if each player has \$4 available to spend.

(b) Find a subgame-perfect equilibrium if each player has \$5 available to spend.

3. Player 1 chooses a_1 between 0 and 1 ($0 \le a_1 \le 1$), and player 2 also chooses a_2 between 0 and 1 ($0 \le a_2 \le 1$). Their payoffs (u_1,u_2) depend on the chosen numbers (a_1,a_2) and a known parameter γ as follows:

$$u_1(a_1,a_2) = \gamma\, a_1 a_2 - (a_1)^2\,,$$
$$u_2(a_1,a_2) = 2a_1 a_2 - a_2.$$

(a) Given $\gamma=1.5$, find all (pure) Nash equilibria of this game if players choose their a_i independently.

(b) Given $\gamma=0.8$, find all (pure) Nash equilibria of this game if players choose their a_i independently.

(c) Given $\gamma=1.5$, find a subgame-perfect equilibrium of this game if player 1 chooses a_1 first, and then player 2 chooses a_2 after observing a_1.

(d) Given $\gamma=0.8$, find a subgame-perfect equilibrium of this game if player 1 chooses a_1 first, and then player 2 chooses a_2 after observing a_1.

PPHA 41501, Autumn 2021: ASSIGNMENT 3, ***due Oct 25*****.**

1. Consider the following 2×3 game:

Player 1: \ Player 2:	L	M	R
T	0, 4	5, 6	8, 7
B	2, 9	6, 5	5, 1

Find all Nash equilibria of this game, and show the expected payoffs for each equilibrium.

2. Consider the following 3×3 games that depend on a parameter α:

Player 1: \ Player 2:	L	M	R
T	α, α	-1, 1	1, -1
C	1, -1	α, α	-1, 1
B	-1, 1	1, -1	α, α

(a) Suppose we are given $\alpha>1$. Show that there are equilibria where the support includes two pure strategies for each player. Show also that there are pure-strategy equilibria, and show that there is an equilibrium where the support includes all three pure strategies for both players.
(b) For the support sets that you found in part (a), which of them also can be the support of an equilibrium when $\alpha<1$?
(c) Suppose $\alpha=0$, but now change the game by eliminating player 1's option to choose B.
Find all equilibria of this 2×3 game.

3. Consider a simplified model of an election among 3 billionaires ($i\in\{1,2,3\}$) who are candidates for governor. In this model, suppose that each candidates i independently chooses an amount of money $c_i\geq0$ to allocate for spending on his or her campaign, and the candidate who spends the most money will win the election. Suppose that there is some number $V>0$ which each candidate considers to be the value of winning the election. For any three amounts (c_1,c_2,c_3), to account for the possibility of ties, we may let
$M(c_1,c_2,c_3) = \{i\in\{1,2,3\}\,|\, c_i = \max_{k\in\{1,2,3\}} c_k\}$ denote the set of candidates who spend the most,
and let $m(c_1,c_2,c_3) = \#M(c_1,c_2,c_3)$ denote the number of candidates who are spending the most.
As long as there are no exact ties then we will just have $m(c_1,c_2,c_3) = 1$.
Suppose that each candidate's allocated amount c_i will be spent regardless of what the others decide to spend.
Then the payoff functions for each bidder i is
$u_i(c_1,c_2,c_3) = V/m(c_1,c_2,c_3) - c_i$ if $c_i\in M(c_1,c_2,c_3)$, else $u_i(c_1,c_2,c_3) = -c_i$.
(a) Find a symmetric equilibrium in which each candidate randomizes over the interval from 0 to V and the probability of any exact ties is 0.
(b) In this symmetric randomized equilibrium, what is the expected value of each candidate's spending c_i?

PPHA 41501, Autumn 2021: ASSIGNMENT 2, *due Oct 13*.

1. Consider a game where player 1 chooses an action in {T,B}, player 2 simultaneously chooses an action in {L,R}, and their payoffs (u_1,u_2) depend on their actions as follows:

Player 1 \ Player 2:	L	R
T	1, 9	8, 3
B	7, 2	4, 5

Find all Nash equilibria of this game (including equilibria with randomized strategies), and compute the players' expected payoffs in each equilibrium.

2. Consider a game where players 1 chooses an action in {T,B}, player 2 simultaneously chooses an action in {L,R}, and their payoffs (u_1,u_2) depend on their actions as follows:

Player 1 \ Player 2:	L	R
T	0, 3	8, 5
B	4, 6	7, 2

(a) Find all Nash equilibria of this game (including equilibria with randomized strategies), and compute the players' expected payoffs in each equilibrium.
(b) How would your answer change if player 1's payoff from (B,R) were increased from 7 to 9?

3. Consider a game where player 1 must choose T or M or B, player 2 must choose L or R, and their utility payoffs (u_1,u_2) depend on their choices as follows:

Player 1 \ Player 2:	L	R
T	6, 1	4, 9
M	5, 7	6, 0
B	9, 7	1, 8

(a) Show a randomized strategy that strongly dominates T for player 1.
(b) Find an equilibrium in randomized strategies for this game, and compute the expected payoff for each player in this equilibrium.
(c) Assuming that player 2 will act according to her equilibrium strategy that you found in part b, what would player 1's expected payoff be if he chose the action T?

4. Players 1 and 2 are involved in a joint project, and each must decide whether to work or shirk. If both work then each gets a benefit worth 1, but each also has a private effort cost e of working. So their payoffs depend on their payoffs (u_1,u_2) depend on their actions as follows:

Player 1 \ Player 2:	2 works	2 shirks
1 works	$1-e$, $1-e$	$-e$, 0
1 shirks	0, $-e$	0, 0

Suppose that e is a known parameter between 0 and 1. Find all Nash equilibria of this game.

5. Consider the penalty kick in soccer. Player 1 is the kicker, and player 2 is the goalie.
Player 1 can kick to left or right. Player 2 must simultaneously decide to jump left or right.
The probability that of the kick being blocked is λ if they both go left, but is ρ if they both go right.
If they choose different directions then the probability of the kick being blocked is 0.
So the players' payoffs (u_1,u_2) depend on their choices as follows:

Player 1 \ Player 2:	L	R
L	$1-\lambda$, λ	1, 0
R	1, 0	$1-\rho$, ρ

(a) Find a Nash equilibrium, and compute the expected payoffs to each player.
(b) If player 2 becomes more skilled at defending left then λ would increase in this game. How would this parametric change affect 2's probability of choosing left in equilibrium?

6. Find the nonrandomized Nash equilibria of the two-player strategic game in which each player's set of actions is the nonnegative real numbers and the players' payoff functions are
$u_1(c_1,c_2) = c_1(c_2-c_1)$, $u_2(c_1,c_2) = c_2(1-c_1-c_2)$.

7. Players 1 and 2 are involved in a joint project. Each player i independently chooses an effort c_i that can be any number in the interval from 0 to 1; that is, $0 \leq c_1 \leq 1$ and $0 \leq c_2 \leq 1$.
(a) Suppose that their output will depend on their efforts by the formula $y(c_1,c_2) = 3c_1c_2$, and each player will get half the output, but each player i must also pay an effort cost equal to c_i^2.
So $u_1(c_1,c_2) = 1.5c_1c_2 - c_1^2$ and $u_2(c_1,c_2) = 1.5c_1c_2 - c_2^2$.
Find all Nash equilibria without randomization.
(b) Now suppose that their output is worth $y(c_1,c_2) = 4c_1c_2$, of which each player gets half, but each player i must also pay an effort cost equal to c_i.
So $u_1(c_1,c_2) = 2c_1c_2 - c_1$ and $u_2(c_1,c_2) = 2c_1c_2 - c_2$.
Find all Nash equilibria without randomization.

8. There are two players numbered 1 and 2. Each player i must choose a number c_i in the set $\{0,1,2\}$, which represents the number of days that player i is prepared to fight for a prize that has value V=$9. A player wins the prize only if he is prepared to fight strictly longer than the other player. They will fight for as many days as both are prepared to fight, and each day of fighting costs each player $1. Thus, the payoffs for players 1 and 2 are as follows:
Player 1's payoff is $u_1(c_1,c_2) = 9-c_2$ if $c_1 > c_2$, but $u_1(c_1,c_2) = -c_1$ if $c_1 \leq c_2$.
Player 2's payoff is $u_2(c_1,c_2) = 9-c_1$ if $c_2 > c_1$, but $u_2(c_1,c_2) = -c_2$ if $c_2 \leq c_1$.
(a) Show a 3×3 matrix that represents this game.
(b) What dominated strategies can you find for each player in this game?
(c) What pure-strategy (nonrandomized) equilibria can you find for this game?
(d) Find a symmetric equilibrium in randomized strategies.

9. Consider a symmetric three-player game where each player must choose L or R.
If all three players choose L, then each of them gets payoff 1.
If all three players choose R, then each of them gets payoff 4.
Otherwise, if the players do not all choose the same action, then they all get payoff 0.
Find a symmetric randomized equilibrium in which both actions get positive probability.

10. Players 1 and 2 are bidding to buy an object in a sealed-bid auction. The object would be worth $V_1 = 53.40$ to player 1 if he could get it, but it would be worth $V_2 = 67.90$ to player 2 if she could get it. These values are commonly known by both players. Each player i chooses a bid c_i that must be a nonnegative multiple of ε, the smallest monetary unit. ($\varepsilon>0$ is given.)
The high bidder wins the object, paying the price that he or she bid, and the loser pays nothing. If their bids are equal, then they each have probability 1/2 of buying the object for the bid price.
So $u_i(c_1,c_2) = V_i-c_i$ if $c_i > c_{-i}$, but $u_i(c_1,c_2) = 0$ if $c_i < c_{-i}$, and $u_i(c_1,c_2) = 0.5(V_i-c_i)$ if $c_1=c_2$.
(a) Show that, for each player i, bidding more than V_i is a weakly dominated action.
(b) Suppose that $\varepsilon=1$. Show that there is a unique nonrandomized equilibrium of this game after weakly dominated actions are eliminated, and compute the players' payoffs in this equilibrium.
(c) If we considered a sequence of games as $\varepsilon\to0$, what would be a limit of undominated equilibrium strategies and payoffs in this game? Characterize the limit of each player's bid and the limit of each player's probability of winning the object in this auction.

PPHA 41501, Autumn 2021: ASSIGNMENT 1, ***for discussion in class on Sept 29***
[The problems in this first homework assignment are not to be handed in. You should know how to solve problem 1, but problems 2 and 3 are for discussion only.]

1. A decision-maker must choose between three alternative decisions {d1,d2,d3}. Her utility payoff will depend as follows on her decision and on an uncertain state of the world in {s1,s2}:

	State s1	State s2
Decision d1	15	90
Decision d2	B	75
Decision d3	55	40

Let p denote the decision-maker's subjective probability of state s2.

(a) Suppose first that B=35. For what range of values of p is decision d1 optimal? For what range is decision d2 optimal? For what range is decision d3 optimal? Is any decision strongly dominated? If so, by what randomized strategies?
(b) Suppose now the B=20. For what range of values of p is decision d1 optimal? For what range is decision d2 optimal? For what range is decision d3 optimal? Is any decision strongly dominated? If so, by what randomized strategies?
(c) For what range of values for the parameter B is decision d2 strongly dominated?

2. A decision-maker has expressed the following preferences:
Getting \$1000 for sure is as good as a lottery offering 0.27 probability of \$5000 or else \$0.
Getting \$2000 for sure is as good as a lottery offering 0.50 probability of \$5000 or else \$0.
That is: [\$1000] ~ 0.27[\$5000]+0.73[\$0], [\$2000] ~ 0.50[\$5000]+0.50[\$0].
If this person is logically consistent, which should he prefer among the following:
a lottery offering a 0.5 probability of \$2000 or else \$1000 (0.5[\$2000]+0.5[\$1000]),
a lottery offering a 0.4 probability of \$5000 or else \$0 (0.4[\$5000]+0.6[\$0]).
Justify your answer as fundamentally as you can.

3. Members of a primitive tribe may own bundles of various goods, which anthropologists have numbered {1,..,m}. The tribe has various ritual exchange activities, numbered {1,...,n}.
In each activity j, there is a "host" and a "guest", and the host gives the guest some net quantity θ_{ij} of each good i (where a negative θ_{ij} denotes the guest giving $-\theta_{ij}$ units of i to the host).
Any tribesman may do each activity any number of times, as guest or host.
Prove a theorem of the following form: "Given any such matrix of parameters θ_{ij}, exactly one of the following two conditions is true: (1) There is a way to use some combination of these exchange activities to increase one's holdings of every good by at least one unit. (2) ...."

[You may assume that people can also do any activity j at any level x_j in $\mathbb{R}$, which would then yield a net transfer $\theta_{ij}x_j$ of each good i, but the results would not change if the x_j had to be integers.]
(*If you cannot do the proof here, at least try to formulate a conjecture as to what condition (2) might be.*)

Assignment 1 answers:

1(a) With B=35, d1 is optimal for $p \geq 4/7$, d2 is optimal for $4/11 \leq p \leq 4/7$, d3 is optimal for $p \leq 4/11$.
(b) With B=20, d1 is optimal for $p \geq 4/9$, d3 is optimal for $p \leq 4/9$, d2 is never optimal and is strongly dominated by q[d1]+(1−q)[d3] for $7/10 = (75-40)/(90-40) < q < (55-B)/(55-15) = 7/8$.
(*If you want to include strongly dominating strategies that include positive probability of d2 itself, you would have q[d1]+r[d3]+(1-q-r)[d2] such that r>0 and 7 > q/r > 7/3.*)
(c) d2 is strongly dominated when $(75-40)/(90-40) < (55-B)/(55-15)$, that is, $B < 27$.

2 The decision-maker should prefer 0.4[$5000]+0.6[$0] over 0.5[$2000]+0.5[$1000] because by substitution and reduction:
0.5[$2000]+0.5[$1000] ~ 0.5(0.27[$5000]+0.73[$0])+0.5(0.50[$5000]+0.50[$0])
~ (0.5×0.27+0.5×0.50)[$5000]+(0.5×0.73+0.5×0.50)[$0] ~ 0.385[$5000]+0.615[$0].
So the decision-maker should prefer q[$5000]+(1−q)[$0] over 0.5[$2000]+0.5[$1000] for any $q > 0.385$. In particular, $0.4 > 0.385$.

3 Given any such matrix of parameters θ_{ij}, exactly one of the following two conditions is true:
(1) $\exists x \in \mathbb{R}^n$ such that $\sum_{j\in\{1,\dots,n\}} \theta_{ij} x_j \geq 1 \ \forall i \in \{1,\dots,m\}$.
(2) $\exists p \in \mathbb{R}^m$ such that $p_i \geq 0 \ \forall i \in \{1,\dots,m\}$, $\sum_{i\in\{1,\dots,m\}} p_i > 0$, $\sum_{i\in\{1,\dots,m\}} p_i\theta_{ij} = 0 \ \ \forall j \in \{1,\dots,n\}$.
For the proof, consider the closed convex set
$B = \{b \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \text{ such that } b_i \leq \sum_{j\in\{1,\dots,n\}} \theta_{ij} x_j \ \forall i \in \{1,\dots,m\}\}$.
Statement (1) is equivalent to saying that the vector (1,...,1) is in the set B.
By the Separating Hyperplane Theorem, (1,...,1) is not in B if and only if
(2′) there exists some p in $\mathbb{R}^m$ such that $\max_{b\in B} \sum_{i\in\{1,\dots,m\}} p_i b_i < \sum_{i\in\{1,\dots,m\}} p_i$.
But $\max_{b\in B} \sum_{i\in\{1,\dots,m\}} p_i b_i$ would be $+\infty$ if $\sum_{i\in\{1,\dots,m\}} p_i\theta_{ij}$ were not 0 for any j in {1,...,n}
(consider b such that $b_i = \theta_{ij} x_j \ \forall i$, and take x_j to $+\infty$ or to $-\infty$) or if we had any $p_i < 0$ (take b_i to $-\infty$).
So the statement (2′) is equivalent to (2) above.

Here (2) says that the goods can be assigned prices, which are nonnegative and not all 0, such that the net exchanged value is 0 for each participant in each ritual exchange activity.

Answers to Assignment 7

1.

2'sType = A

	L	R
T	4,0	0,2
B	0,4	2,0

2'sType = B

	L	R
T	4,0	0,4
B	0,2	2,0

It is easy to see that there is no equilibrium where 1 uses T for sure or where 1 uses B for sure.
To make player 1 willing to randomize, 1 must think that 2's probability choosing R satisfies
$4\times(1-P(R)) + 0\times P(R) = EU_1(T) = EU_1(B) = 0\times(1-P(R)) + 2\times P(R)$, and so $P(R) = 2/3$.
Let $\alpha_2(R|t_2)$ denote the probability of player 2 choosing R given that her type is t_2.
So $P(R) = p_2(A)\times\alpha_2(R|A) + p_2(B)\times\alpha_2(R|B)$, where $p_2(t_2)$ is the probability of 2's type being t_2.
With increasing differences we can see that, among player 2's types, type A is more inclined toward choosing L, and type B is more inclined toward choosing R.

(a) We assume that $p_2(A) = p_2(B) = 0.5$. So we need $2/3 = 0.5\times\alpha_2(R|A) + 0.5\times\alpha_2(R|B)$.
If 2's type A was not willing to choose R, then the probability of 2 choosing R could not be more than $p_2(B) = 0.5 < 2/3$.
So there must be a positive probability of 2's type A choosing R. But by increasing differences, if 2's type A would be willing to choose R, then 2's type B must strictly prefer R over L.
So $\alpha_2(R|B) = 1$. So we need $2/3 = 0.5\times\alpha_2(R|A) + 0.5\times 1$. So $\alpha_2(R|A) = 1/3$.
That is, 2's type A would do (2/3)[L]+(1/3)[R], but 2's type B would to [R].
To make 2's type A willing to randomize, player 1's probability of choosing B must satisfy
$0\times(1-P(B)) + 4\times P(B) = EU_2(L|A) = EU_2(R|A) = 2\times(1-P(B)) + 0\times P(B)$.
So we need $P(B) = 1/3$. That is player 1 must do (2/3)[T]+(1/3)[B].

(b) Now we assume that $p_2(A) = 1/6$ and $p_2(B) = 5/6$. So we need
$2/3 = P(R) = (1/6)\,\alpha_2(R|A) + (5/6)\,\alpha_2(R|B)$ and $1/3 = P(L) = (1/6)\,\alpha_2(L|A) + (5/6)\,\alpha_2(L|B)$.
So there must be a positive probability of 2's type B choosing L. But by increasing differences, if 2's type B would be willing to choose L, then 2's type A must strictly prefer L over R.
So $\alpha_2(L|A) = 1$. So we need $1/3 = (1/6)(1) + (5/6)\,\alpha_2(L|B)$. So $\alpha_2(L|B) = 1/5$.
That is, 2's type A would do [L], but 2's type B would do (1/5)[L]+(4/5)[R].
To make 2's type B willing to randomize, player 1's probability of choosing B must satisfy
$0\times(1-P(B)) + 2\times P(B) = EU_2(L|B) = EU_2(R|B) = 4\times(1-P(B)) + 0\times P(B)$.
So we need $P(B) = 2/3$. That is player 1 must do (1/3)[T]+(2/3)[B].

2.

	2 NotFights	2 Fights
1 NotFights	0, 0	0, 2
1 Fights	V_1, 0	−1, −1

(a) We assume first $V_1=3$. ([1Fights], [2NotFights]) is an equilibrium.
Also ([1NotFights], [2Fights]) is an equilibrium.
Also ((1/3)[1NotFights]+(2/3)[1Fights], (1/4)[2NotFights]+(3/4)[2Fights]) is an equilibrium.
(Notice that, in the randomized equilibrium, 2 fights with higher probability even though she has the lower value for the prize.)

(b) Let $P_1(F)$ denote the probability that player 1 will fight, which depends on 1's strategy α_1 according to: $P_1(F) = (1/2)\ \alpha_1(F|V_1=2) + (1/2)\ \alpha_1(F|V_1=3)$.
To make player 2 willing to randomize, player 1's probability of fighting must satisfy
$0 = EU_2(\text{2NotFights}) = EU_2(\text{2Fights}) = 2\times(1-P_1(F)) + -1\times P_1(F)$, and so $P_1(F) = 2/3$.
So 1's type $V_1=2$ must be willing to fight. Then by increasing differences, 1's type $V_1=3$ strictly prefers to fight. So $\alpha_1(F|V_1=3) = 1$. So $2/3 = (1/2)\alpha_1(F|V_1=2) + (1/2)(1)$, $\alpha_1(F|V_1=2) = 1/3$.
That is, 1's type $V_1=2$ does (1/3)[1Fights]+(2/3)[1NotFights], but 1's type $V_1=3$ does [1Fights].
To make 1's type $V_1=2$ willing to randomize, player 2's probability of fighting $P_2(F)$ must satisfy
$-1\times P_2(F) + 2\times(1-P_2(F)) = EU_1(F|V_1=2) = EU_1(NF|V_1=2) = 0$.
So we need $P_2(F) = 2/3$. That is, player 2 must do (1/3)[2NotFights]+(2/3)[2Fights].

(c) By increasing differences, player 1 should use a cutoff strategy, Fighting if $\tilde{t}_1 > \theta$ and NotFighting if $\tilde{t}_1 < \theta$, for some cutoff θ.
To make player 2 willing to randomize, player 1's probability of fighting must be 2/3, and so $P(\tilde{t}_1>\theta) = 2/3$, which implies that $\theta = 1/3$ (since $\tilde{t}_1$ is Uniform on the interval 0 to 1).
To make 1's type θ indifferent, player 2's probability of fighting $P_2(F)$ must satisfy
$-1\times P_2(F) + (2+\theta)\times(1-P_2(F)) = EU_1(F|\tilde{t}_1=\theta) = EU_1(NF|\tilde{t}_1=\theta) = 0$.
With $\theta = 1/3$, this implies that player 2's $P_2(F) = 0.7$. (Note the difference from part (b).)
That is, player 2 must do 0.3[2NotFights]+0.7[2Fights].

3. (a) If player 2 would do L when $t_2<0.5$ but would do R when $t_2>0.5$, then player 1 could earn a positive payoff with equal probability 1/2 by choosing T or B, but the positive payoff from B would be 0.5 for sure, whereas the positive payoff from T would be somewhere between 0 and 0.5.
So 1 should want to choose B against this strategy, but 2's best response to B is L for any t_2!

(b) We look for an equilibrium where player 1 randomizes $q[T]+(1-q)[B]$, with $0<q<1$.
As a function of her type, player 2 expects $-t_2q$ from L, $-0.5(1-q)$ from R. So player 2 should prefer L when $t_2 < 0.5(1-q)/q$, R when $t_2 > 0.5(1-q)/q$.
This is a cutoff strategy with "L<R" and a cutoff $\theta = 0.5(1-q)/q$. This implies $q = 1/(2\theta+1)$.
Against such a θ-cutoff strategy for player 2, player 1's expected payoff is
$\int_0^\theta t_2\, dt_2 = \theta^2/2$ from choosing T, or $\int_\theta^1 0.5\, dt_2 = (1-\theta)0.5$ from choosing B.
So for 1 to randomize, we must have $\theta^2/2 = (1-\theta)0.5$, and so $\theta^2 + \theta - 1 = 0$,
which has the positive solution $(-1+(1+4)0.5)/2 = 0.6180...$
Then $q = 1/(2\theta+1) = 0.4472...$

*(c) There is an equilibrium where player 1 does T if $t_1<\theta_1$, B if $t_1>\theta_1$,
player 2 does L if $t_2<\theta_2$, R if $t_2>\theta_2$, where $\theta_1=0.5499...$, and $\theta_2 = 0.6342...$
These satisfy the equations $\theta_2\theta_1 = (1-\theta_1)(1+\theta_1)/2$ and $\theta_1(1-\theta_2) = \theta_2(\theta_2/2)$.