1. Consider the following extensive-form game, which begins with a chance move.

*Interpretation: firm 1 has high or low costs and must decide whether to build a new factory; then firm 2 observes whether the new factory is built and decides whether to enter firm 1's market as a competitor. Firm 2 could get positive profits from entering only if firm 1 has high costs...*)

(a) Show the normal representation of this game in strategic form.
(b) Find all pure-strategy Nash equilibria of this game.
(c) For each Nash equilibrium that you found in part (b), explain whether this equilibrium corresponds to a sequential equilibrium of the extensive-form game. If so, at every information set (including sets of probability zero) you should indicate what beliefs would make this a sequential equilibrium.
(d) Apply iterative elimination of weakly dominated strategies to the normal representation. Does this analysis eliminate any of the Nash equilibria that you found in part (a)?

2. Consider a game where player 1 must choose T or B, player 2 must choose L or R, and their payoffs depend on their choices as follows.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>3, 2</td>
<td>1, 1</td>
</tr>
<tr>
<td>B</td>
<td>4, 3</td>
<td>2, 4</td>
</tr>
</tbody>
</table>

Suppose that player 1 moves first, and then player 2 makes her choice after observing 1's move.
(a) Show the extensive-form game with perfect information that describes this situation.
(b) Show the normal representation in strategic form for the extensive-form game in part (a).
(c) Find the unique sequential (subgame-perfect) equilibrium of this game.
(d) Find a Nash equilibrium of this game that is *not* sequentially rational (or subgame-perfect).

3. Consider again the game in the previous exercise 2 where player 1 moves first. But now suppose that, whatever 1 chooses, the probability that player 2 will correctly observe 1's action is 0.9, and there is probability 0.1 that player 2 will mistakenly observe the other action (which 1 did not choose). The payoffs depend on the players' actual choices according to the previous table (so, for example, if 1 chose T but 2 mistakenly observed B and chose R then 2's payoff would be 1).
(a) Show the extensive-form game that describes this situation.
(b) Show the normal representation in strategic form for the extensive-form game in part (a).
(c) Find a sequential equilibrium in which player 2 would choose [L] for sure if she observed T. *(d) Characterize the other sequential equilibria of this game.*
4. Consider the following extensive-form game, where player 1 observes the chance move, but player 2 does not observe it. If 2 gets to move, she knows only that 1 chose either x1 or z1.

(a) Find a sequential equilibrium in which the (prior) probability of player 2 getting to move is 1.
(b) Find a sequential equilibrium in which the probability of player 2 getting to move is 0. (You must describe what player 2 would believe and do if she got to move.)
(c) Find a sequential equilibrium in which the probability of player 2 getting to move is strictly between 0 and 1.
(d) Show the normal representation of this game in strategic form.
Consider a game where player 1 must choose T or B, player 2 must choose L or R, and their payoffs depend on their choices as follows.

<table>
<thead>
<tr>
<th></th>
<th>Player 2:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>L</td>
<td>R</td>
</tr>
<tr>
<td>T</td>
<td>3, 2</td>
<td>1, 1</td>
</tr>
<tr>
<td>B</td>
<td>4, 3</td>
<td>2, 4</td>
</tr>
</tbody>
</table>

Suppose that player 1 moves first, and then player 2 makes her choice after observing 1’s move.

(a) Show the extensive-form game with perfect information that describes this situation.

(b) Show the normal representation in strategic form for the extensive-form game in part (a).

(c) Find the unique subgame-perfect equilibrium of this game. Be sure to fully describe the strategies for each player in this equilibrium.

(d) Find a Nash equilibrium of this game that is not subgame-perfect (or sequentially rational) and that yields different expected payoffs from the equilibrium that you found in (c). Be sure to fully describe the strategies for each player in this equilibrium.

2. Players 1 and 2 are in a sequential all-pay-own-bid auction for a prize worth $3. First, player 1 must pay $1 or pass. When anyone passes, the other player gets the $3 prize (and game ends). Otherwise, the other player can bid next, and must either pay $2 (if he has it) or pass. A player cannot pay more than his given available funds. This game has perfect information.

(a) Find a subgame-perfect equilibrium if each player has $4 available to spend.

(b) Find a subgame-perfect equilibrium if each player has $5 available to spend.

3. Player 1 chooses $a_1$ between 0 and 1 ($0 \leq a_1 \leq 1$), and player 2 also chooses $a_2$ between 0 and 1 ($0 \leq a_2 \leq 1$). Their payoffs $(u_1, u_2)$ depend on the chosen numbers $(a_1, a_2)$ and a known parameter $\gamma$ as follows:

\[ u_1(a_1, a_2) = \gamma a_1 a_2 - (a_1)^2, \]
\[ u_2(a_1, a_2) = 2a_1 a_2 - a_2. \]

(a) Given $\gamma=1.5$, find all (pure) Nash equilibria of this game if players choose their $a_i$ independently.

(b) Given $\gamma=0.8$, find all (pure) Nash equilibria of this game if players choose their $a_i$ independently.

(c) Given $\gamma=1.5$, find a subgame-perfect equilibrium of this game if player 1 chooses $a_1$ first, and then player 2 chooses $a_2$ after observing $a_1$.

(d) Given $\gamma=0.8$, find a subgame-perfect equilibrium of this game if player 1 chooses $a_1$ first, and then player 2 chooses $a_2$ after observing $a_1$. 


1. Consider the following 2×3 game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0, 4</td>
<td>5, 6</td>
<td>8, 7</td>
</tr>
<tr>
<td>B</td>
<td>2, 9</td>
<td>6, 5</td>
<td>5, 1</td>
</tr>
</tbody>
</table>

Find all Nash equilibria of this game, and show the expected payoffs for each equilibrium.

2. Consider the following 3×3 games that depend on a parameter α:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>α, α</td>
<td>-1, 1</td>
<td>1,-1</td>
</tr>
<tr>
<td>C</td>
<td>1,-1</td>
<td>α, α</td>
<td>-1, 1</td>
</tr>
<tr>
<td>B</td>
<td>-1, 1</td>
<td>1,-1</td>
<td>α, α</td>
</tr>
</tbody>
</table>

(a) Suppose we are given α>1. Show that there are equilibria where the support includes two pure strategies for each player. Show also that there are pure-strategy equilibria, and show that there is an equilibrium where the support includes all three pure strategies for both players.

(b) For the support sets that you found in part (a), which of them also can be the support of an equilibrium when α<1?

(c) Suppose α=0, but now change the game by eliminating player 1's option to choose B. Find all equilibria of this 2×3 game.

3. Consider a simplified model of an election among 3 billionaires (i∈{1,2,3}) who are candidates for governor. In this model, suppose that each candidates i independently chooses an amount of money ci≥0 to allocate for spending on his or her campaign, and the candidate who spends the most money will win the election. Suppose that there is some number V>0 which each candidate considers to be the value of winning the election. For any three amounts (c₁,c₂,c₃), to account for the possibility of ties, we may let M(c₁,c₂,c₃) = {i∈{1,2,3}| ci = maxk∈{1,2,3} cₖ} denote the set of candidates who spend the most, and let m(c₁,c₂,c₃) = #M(c₁,c₂,c₃) denote the number of candidates who are spending the most.

As long as there are no exact ties then we will just have m(c₁,c₂,c₃) = 1.

Suppose that each candidate's allocated amount cᵢ will be spent regardless of what the others decide to spend. Then the payoff functions for each bidder i is

uᵢ(c₁,c₂,c₃) = V/m(c₁,c₂,c₃) - cᵢ if cᵢ∈M(c₁,c₂,c₃), else uᵢ(c₁,c₂,c₃) = -cᵢ.

(a) Find a symmetric equilibrium in which each candidate randomizes over the interval from 0 to V and the probability of any exact ties is 0.

(b) In this symmetric randomized equilibrium, what is the expected value of each candidate's spending cᵢ?

1. Consider a game where player 1 chooses an action in \{T,B\}, player 2 simultaneously chooses an action in \{L,R\}, and their payoffs \((u_1,u_2)\) depend on their actions as follows:

\[
\begin{array}{c|cc}
\text{Player 1} & \text{Player 2:} & \text{L} & \text{R} \\
\hline
T & 1, 9 & 8, 3 \\
B & 7, 2 & 4, 5 \\
\end{array}
\]

Find all Nash equilibria of this game (including equilibria with randomized strategies), and compute the players' expected payoffs in each equilibrium.

2. Consider a game where players 1 chooses an action in \{T,B\}, player 2 simultaneously chooses an action in \{L,R\}, and their payoffs \((u_1,u_2)\) depend on their actions as follows:

\[
\begin{array}{c|cc}
\text{Player 1} & \text{Player 2:} & \text{L} & \text{R} \\
\hline
T & 0, 3 & 8, 5 \\
B & 4, 6 & 7, 2 \\
\end{array}
\]

(a) Find all Nash equilibria of this game (including equilibria with randomized strategies), and compute the players' expected payoffs in each equilibrium.

(b) How would your answer change if player 1's payoff from (B,R) were increased from 7 to 9?

3. Consider a game where player 1 must choose T or M or B, player 2 must choose L or R, and their utility payoffs \((u_1,u_2)\) depend on their choices as follows:

\[
\begin{array}{c|cc}
\text{Player 1} & \text{Player 2:} & \text{L} & \text{R} \\
\hline
T & 6, 1 & 4, 9 \\
M & 5, 7 & 6, 0 \\
B & 9, 7 & 1, 8 \\
\end{array}
\]

(a) Show a randomized strategy that strongly dominates T for player 1.

(b) Find an equilibrium in randomized strategies for this game, and compute the expected payoff for each player in this equilibrium.

(c) Assuming that player 2 will act according to her equilibrium strategy that you found in part b, what would player 1's expected payoff be if he chose the action T?

4. Players 1 and 2 are involved in a joint project, and each must decide whether to work or shirk. If both work then each gets a benefit worth 1, but each also has a private effort cost \(e\) of working. So their payoffs depend on their payoffs \((u_1,u_2)\) depend on their actions as follows:

\[
\begin{array}{c|cc}
\text{Player 1} & \text{Player 2:} & 2 \text{ works} & 2 \text{ shirks} \\
\hline
1 \text{ works} & 1-e, 1-e & -e, 0 \\
1 \text{ shirks} & 0, -e & 0, 0 \\
\end{array}
\]

Suppose that \(e\) is a known parameter between 0 and 1. Find all Nash equilibria of this game.

5. Consider the penalty kick in soccer. Player 1 is the kicker, and player 2 is the goalie. Player 1 can kick to left or right. Player 2 must simultaneously decide to jump left or right. The probability that of the kick being blocked is \(\lambda\) if they both go left, but is \(\rho\) if they both go right. If they choose different directions then the probability of the kick being blocked is 0. So the players' payoffs \((u_1,u_2)\) depend on their choices as follows:

\[
\begin{array}{c|cc}
\text{Player 1} \backslash \text{Player 2} & \text{L} & \text{R} \\
\hline
L & 1-\lambda, \lambda & 1, 0 \\
R & 1, 0 & 1-\rho, \rho \\
\end{array}
\]

(a) Find a Nash equilibrium, and compute the expected payoffs to each player.

(b) If player 2 becomes more skilled at defending left then \(\lambda\) would increase in this game. How would this parametric change affect 2's probability of choosing left in equilibrium?
6. Find the nonrandomized Nash equilibria of the two-player strategic game in which each player's set of actions is the nonnegative real numbers and the players' payoff functions are 
\[ u_1(c_1, c_2) = c_1(c_2 - c_1), \quad u_2(c_1, c_2) = c_2(1 - c_1 - c_2). \]

7. Players 1 and 2 are involved in a joint project. Each player independently chooses an effort \( c_i \) that can be any number in the interval from 0 to 1; that is, \( 0 \leq c_1 \leq 1 \) and \( 0 \leq c_2 \leq 1 \).
(a) Suppose that their output will depend on their efforts by the formula \( y(c_1, c_2) = 3c_1c_2 \), and each player will get half the output, but each player must also pay an effort cost equal to \( c_i^2 \).
So \( u_1(c_1, c_2) = 1.5c_1c_2 - c_1^2 \) and \( u_2(c_1, c_2) = 1.5c_1c_2 - c_2^2 \).
Find all Nash equilibria without randomization.
(b) Now suppose that their output is worth \( y(c_1, c_2) = 4c_1c_2 \), of which each player gets half, but each player must also pay an effort cost equal to \( c_i \).
So \( u_1(c_1, c_2) = 2c_1c_2 - c_1 \) and \( u_2(c_1, c_2) = 2c_1c_2 - c_2 \).
Find all Nash equilibria without randomization.

8. There are two players numbered 1 and 2. Each player must choose a number \( c_i \) in the set \{0, 1, 2\}, which represents the number of days that player i is prepared to fight for a prize that has value \( V=9 \). A player wins the prize only if he is prepared to fight strictly longer than the other player. They will fight for as many days as both are prepared to fight, and each day of fighting costs each player \$1. Thus, the payoffs for players 1 and 2 are as follows:
Player 1's payoff is \( u_1(c_1, c_2) = 9c_2 \) if \( c_1 > c_2 \), but \( u_1(c_1, c_2) = -c_1 \) if \( c_1 \leq c_2 \).
Player 2's payoff is \( u_2(c_1, c_2) = 9c_1 \) if \( c_2 > c_1 \), but \( u_2(c_1, c_2) = -c_2 \) if \( c_2 \leq c_1 \).
(a) Show a 3x3 matrix that represents this game.
(b) What dominated strategies can you find for each player in this game?
(c) What pure-strategy (nonrandomized) equilibria can you find for this game?
(d) Find a symmetric equilibrium in randomized strategies.

9. Consider a symmetric three-player game where each player must choose L or R. If all three players choose L, then each of them gets payoff 1. If all three players choose R, then each of them gets payoff 4. Otherwise, if the players do not all choose the same action, then they all get payoff 0. Find a symmetric randomized equilibrium in which both actions get positive probability.

10. Players 1 and 2 are bidding to buy an object in a sealed-bid auction. The object would be worth \( V_1 = 53.40 \) to player 1 if he could get it, but it would be worth \( V_2 = 67.90 \) to player 2 if she could get it. These values are commonly known by both players. Each player chooses a bid \( c_i \) that must be a nonnegative multiple of \( \varepsilon \), the smallest monetary unit. (\( \varepsilon > 0 \) is given.)
The high bidder wins the object, paying the price that he or she bid, and the loser pays nothing. If their bids are equal, then they each have probability 1/2 of buying the object for the bid price.
So \( u_i(c_1, c_2) = V_i - c_i \) if \( c_i > c_{-i} \), but \( u_i(c_1, c_2) = 0 \) if \( c_i < c_{-i} \), and \( u_i(c_1, c_2) = 0.5(V_i - c_i) \) if \( c_i = c_{-i} \).
(a) Show that, for each player \( i \), bidding more than \( V_i \) is a weakly dominated action.
(b) Suppose that \( \varepsilon = 1 \). Show that there is a unique nonrandomized equilibrium of this game after weakly dominated actions are eliminated, and compute the players' payoffs in this equilibrium.
(c) If we considered a sequence of games as \( \varepsilon \to 0 \), what would be a limit of undominated equilibrium strategies and payoffs in this game? Characterize the limit of each player's bid and the limit of each player's probability of winning the object in this auction.
PPHA 41501, Autumn 2021: ASSIGNMENT 1, for discussion in class on Sept 29
[The problems in this first homework assignment are not to be handed in. You should know how to solve problem 1, but problems 2 and 3 are for discussion only.]

1. A decision-maker must choose between three alternative decisions \{d1,d2,d3\}. Her utility payoff will depend as follows on her decision and on an uncertain state of the world in \{s1,s2\}:

<table>
<thead>
<tr>
<th>Decision</th>
<th>State s1</th>
<th>State s2</th>
</tr>
</thead>
<tbody>
<tr>
<td>d1</td>
<td>15</td>
<td>90</td>
</tr>
<tr>
<td>d2</td>
<td>B</td>
<td>75</td>
</tr>
<tr>
<td>d3</td>
<td>55</td>
<td>40</td>
</tr>
</tbody>
</table>

Let \( p \) denote the decision-maker's subjective probability of state s2.

(a) Suppose first that \( B=35 \). For what range of values of \( p \) is decision d1 optimal? For what range is decision d2 optimal? For what range is decision d3 optimal? Is any decision strongly dominated? If so, by what randomized strategies?

(b) Suppose now the \( B=20 \). For what range of values of \( p \) is decision d1 optimal? For what range is decision d2 optimal? For what range is decision d3 optimal? Is any decision strongly dominated? If so, by what randomized strategies?

(c) For what range of values for the parameter \( B \) is decision d2 strongly dominated?

2. A decision-maker has expressed the following preferences:
   Getting $1000 for sure is as good as a lottery offering 0.27 probability of $5000 or else $0.
   Getting $2000 for sure is as good as a lottery offering 0.50 probability of $5000 or else $0.

   That is: \[ \$1000 \sim 0.27[\$5000]+0.73[\$0], \quad \$2000 \sim 0.50[\$5000]+0.50[\$0]. \]

   If this person is logically consistent, which should he prefer among the following:
   a lottery offering a 0.5 probability of $2000 or else $1000 \( (0.5[\$2000]+0.5[\$1000]) \),
   a lottery offering a 0.4 probability of $5000 or else $0 \( (0.4[\$5000]+0.6[\$0]) \).

   Justify your answer as fundamentally as you can.

3. Members of a primitive tribe may own bundles of various goods, which anthropologists have numbered \{1,...,m\}. The tribe has various ritual exchange activities, numbered \{1,...,n\}.
   In each activity \( j \), there is a "host" and a "guest", and the host gives the guest some net quantity \( \theta_{ij} \) of each good i (where a negative \( \theta_{ij} \) denotes the guest giving \( -\theta_{ij} \) units of i to the host).
   Any tribesman may do each activity any number of times, as guest or host.

   Prove a theorem of the following form: "Given any such matrix of parameters \( \theta_{ij} \), exactly one of the following two conditions is true: (1) There is a way to use some combination of these exchange activities to increase one's holdings of every good by at least one unit. (2) ...."

   [You may assume that people can also do any activity \( j \) at any level \( x_j \) in \( \mathbb{R} \), which would then yield a net transfer \( \theta_{ij}x_j \) of each good i, but the results would not change if the \( x_j \) had to be integers.]

   (If you cannot do the proof here, at least try to formulate a conjecture as to what condition (2) might be.)
Assignment 1 answers:

1(a) With $B=35$, $d_1$ is optimal for $p \geq 4/7$, $d_2$ is optimal for $4/11 \leq p \leq 4/7$, $d_3$ is optimal for $p \leq 4/11$.
(b) With $B=20$, $d_1$ is optimal for $p \geq 4/9$, $d_3$ is optimal for $p \leq 4/9$, $d_2$ is never optimal and is strongly dominated by $q[d_1]+(1-q)[d_3]$ for $7/10 = (75-40)/(90-40) < q < (55-B)/(55-15) = 7/8$.
(If you want to include strongly dominating strategies that include positive probability of $d_2$ itself, you would have $q[d_1]+r[d_3]+(1-q-r)[d_2]$ such that $r>0$ and $7 > q/r > 7/3$.)
(c) $d_2$ is strongly dominated when $7/10 = (75-40)/(90-40) < (55-B)/(55-15)$, that is, $B < 27$.

2 The decision-maker should prefer $0.4[5000]+0.6[0]$ over $0.5[2000]+0.5[1000]$ because by substitution and reduction:

$0.5[2000]+0.5[1000] \sim 0.5(0.27[5000]+0.73[0])+0.5(0.50[5000]+0.50[0])$

$\sim (0.5 \times 0.27+0.5 \times 0.50)[5000]+(0.5 \times 0.73+0.5 \times 0.50)[0] \sim 0.385[5000]+0.615[0]$.

So the decision-maker should prefer $q[5000]+(1-q)[0]$ over $0.5[2000]+0.5[1000]$ for any $q > 0.385$. In particular, $0.4 > 0.385$.

3 Given any such matrix of parameters $\theta_{ij}$, exactly one of the following two conditions is true:

(1) $\exists x \in \mathbb{R}^n$ such that $\sum_{j \in \{1,...,n\}} \theta_{ij} x_j \geq 1 \ \forall i \in \{1,...,m\}$.

(2) $\exists p \in \mathbb{R}^m$ such that $p_i \geq 0 \ \forall i \in \{1,...,m\}$, $\sum_{i \in \{1,...,m\}} p_i > 0$, $\sum_{i \in \{1,...,m\}} p_i \theta_{ij} = 0 \ \forall j \in \{1,...,n\}$.

For the proof, consider the closed convex set

$B = \{b \in \mathbb{R}^m | \exists x \in \mathbb{R}^n$ such that $b_i \leq \sum_{j \in \{1,...,n\}} \theta_{ij} x_j \ \forall i \in \{1,...,m\}\}$.

Statement (1) is equivalent to saying that the vector $(1,...,1)$ is in the set $B$.

By the Separating Hyperplane Theorem, $(1,...,1)$ is not in $B$ if and only

(2') there exists some $p$ in $\mathbb{R}^m$ such that $\max_{b \in B} \sum_{i \in \{1,...,m\}} p_i b_i < \sum_{i \in \{1,...,m\}} p_i$.

But $\max_{b \in B} \sum_{i \in \{1,...,m\}} p_i b_i$ would be $+\infty$ if $\sum_{i \in \{1,...,m\}} p_i \theta_{ij}$ were not 0 for any $j$ in $\{1,...,n\}$
(consider $b$ such that $b_i=\theta_{ij} x_j \ \forall i$, and take $x_j$ to $+\infty$ or to $-\infty$) or if we had any $p_i<0$ (take $b_i$ to $-\infty$).

So the statement (2') is equivalent to (2) above.

Here (2) says that the goods can be assigned prices, which are nonnegative and not all 0, such that the net exchanged value is 0 for each participant in each ritual exchange activity.