A model of decisions under uncertainty is characterized by:

- a set of alternative choices \( C \),
- a set of possible states of the world \( S \),
- a utility function \( u: C \times S \to \mathbb{R} \),
- and a probability distribution \( p \) in \( \Delta(S) \).

Suppose that \( C \) and \( S \) are nonempty finite sets.

Here we use the notation \( \Delta(S) = \{ q \in \mathbb{R}^S \mid q(s) \geq 0 \ \forall s, \ \sum_{s \in S} q(s) = 1 \} \).

The expected utility hypothesis says that an optimal decision should maximize expected utility \( \text{Eu}(c) = \text{Eu}(c \mid p) = \sum_{s \in S} p(s) u(c, s) \) over all \( c \) in \( C \), for some utility function \( u \) that is appropriate for the decision maker.

**Example 1.** Consider an example with choices \( C = \{T,M,B\} \), state \( S = \{L,R\} \), and \( u(c,s) \):

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<tr>
<td>T</td>
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To describe the probability distribution parametrically, let \( r \) be the probability of state \( R \).

So \( \text{Eu}(T) = 7(1-r)+2r \), \( \text{Eu}(M) = 2(1-r)+7r \), \( \text{Eu}(B) = 5(1-r)+6r \).

Then \( B \) is optimal when \( 5(1-r)+6r \geq 2(1-r)+7r \) and \( 5(1-r)+6r \geq 7(1-r)+2r \), which are satisfied when \( 3/4 = (5-2)/(5-6) \geq r \geq (7-5)/(7-6) = 1/3 \).

**Fact:** Given the utility function \( u: C \times S \to \mathbb{R} \) and some choice option \( d \in C \), the set of probability distributions that make \( d \) optimal is a closed convex (possibly empty) subset of \( \Delta(S) \).

This set (of probabilities that make \( d \) optimal) is empty if and only if there exists some randomized strategy \( \sigma \) in \( \Delta(C) \) such that \( u(d,s) < \sum_{c \in C} \sigma(c)u(c,s) \ \forall s \in S \).

When these inequalities hold, we say that \( d \) is strongly dominated by \( \sigma \).

[Proof: \( \{ x \in \mathbb{R}^S \mid \exists \sigma \in \Delta(C) \text{ s.t. } x_s \leq \sum_{c \in C} \sigma(c)u(c,s) \ \forall s \} \) is a convex subset of \( \mathbb{R}^S \).

d is strongly dominated iff (\( u(d,s) \))\( s \in S \) is in its interior. Use supporting-hyperplane thm, MWG p. 949.]

**Example 2:** As above, \( C = \{T,M,B\} \), \( S = \{L,R\} \), and \( u \) is same except \( u(B,R) = 3 \).

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<tr>
<td>B</td>
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As before, \( B \) would be the second-best choice in either state (if the state were known).

\( B \) would be an optimal decision under uncertainty when

\( 5(1-r)+3r \geq 7(1-r)+2r \) and \( 5(1-r)+3r \geq 2(1-r)+7r \),

which are satisfied when \( r \geq 2/3 \) and \( 3/7 \geq r \), which is impossible! So \( B \) cannot be optimal.

\( T \) is optimal when \( r \leq 1/2 \). \( M \) is optimal when \( r \geq 1/2 \).

Now consider a randomized strategy that chooses \( T \) with some probability \( \sigma(T) \)

and chooses \( M \) otherwise, with probability \( \sigma(M) = 1 - \sigma(T) \).

\( B \) would be strongly dominated by this randomized strategy \( \sigma \) if

\( 5 < \sigma(T)7 + (1-\sigma(T))2 \) (\( B \) worse than \( \sigma \) in state \( L \)), and

\( 3 < \sigma(T)2 + (1-\sigma(T))7 \) (\( B \) worse than \( \sigma \) in state \( R \)).

These inequalities are satisfied when \( 3/5 < \sigma(T) < 4/5 \). For example, \( \sigma(T) = 0.7 \) works.

That is, \( B \) is strongly dominated by \( 0.7[T]+0.3[M] \), as \( 5 < 0.7 \times 7 + 0.3 \times 2 = 5.5 \) and \( 3 < 0.7 \times 2 + 0.3 \times 7 = 3.5 \).
**Separating Hyperplane Theorem** (MWG M.G.2; or JR A2.23 with C={{w−x| x ∈ X}}):

Suppose X is a closed convex subset of \( \mathbb{R}^N \), and w is a vector in \( \mathbb{R}^N \).

Then exactly one of the following two statements is true: Either (1) \( w \in X \), or (2) there exists a vector \( y \in \mathbb{R}^N \) such that \( y'w > \max_{x \in X} y'x \)

(but not both). (Here \( y'w = y_1x_1 + \cdots + y_Nx_N \), with \( y = (y_1, \ldots, y_N) \) and \( x = (x_1, \ldots, x_N) \).)

**Supporting Hyperplane Theorem** (MWG M.G.3; or JR A2.24 with A={w} & B=interior(X)):

Suppose X is a convex subset of \( \mathbb{R}^N \), and w is a vector in \( \mathbb{R}^N \). Then exactly one of the following two statements is true: Either (1) \( w \) is in the interior of X (relative to \( \mathbb{R}^N \)),

or (2) there exists a vector \( y \in \mathbb{R}^N \) such that \( y \neq 0 \) and \( y'w \geq \max_{x \in X} y'x \)

(but not both). Here \( \mathbf{0} = (0, \ldots, 0) \).

**Fact** If X is a convex and compact (closed and bounded), then \( \max_{x \in X} y'x \) is a finite number, and this maximum must be achieved at some extreme point in X. (MWG p 946.)

**Fact** For any nonempty finite set C and any \( v \in \mathbb{R}^C \), \( \max_{c \in C} \sum_{c \in C} \sigma(c)v_c = \max_{c \in C} v_c \), and

\[ \arg \max_{\sigma \in \Delta(C)} \sum_{c \in C} \sigma(c)v_c = \{ \sigma \in \Delta(C) | \{d| \sigma(d) > 0\} \subseteq \arg \max_{c \in C} v_c \}. \]

**Strong domination Theorem.** Given the nonempty finite sets \( C = \{ \text{choices} \} \), \( S = \{ \text{states} \} \), the utility function \( u: C \times S \to \mathbb{R} \), and the choice \( d \in C \), exactly one of these two statements is true:

Either (2) \( \exists p \in \Delta(S) \) such that \( \sum_{s \in S} p(s)u(d,s) = \max_{c \in C} \sum_{s \in S} p(s)u(c,s) \), [d is optimal for some beliefs]

or (1) \( \exists \sigma \in \Delta(C) \) such that \( u(d,s) < \sum_{c \in C} \sigma(c)u(c,s) \) \( \forall s \in S \). [d is dominated by a randomized strategy]

**Proof.** Let \( X = \{ x \in \mathbb{R}^N | \exists \sigma \in \Delta(C) \text{ s.t. } x_s \leq \sum_{c \in C} \sigma(c)u(c,s) \forall s \} \). X is a convex subset of \( \mathbb{R}^N \).

Condition (1) here is equivalent to: (1') the vector \( u(d) = (u(d,s))_{s \in S} \) is in the interior of X.

By the Separating Hyperplane Thm, (1') is false iff

(2') \( \exists p \in \mathbb{R}^S \) such that \( p \neq \mathbf{0} \) and \( \sum_{s \in S} p(s)u(d,s) \geq \max_{x \in X} \sum_{s \in S} p(s)x_s \).

We must have \( p(s) \geq 0 \) for all \( s \), because \( x \) in X can have \( x_s \) approaching \(-\infty\).

So \( \sum_{s \in S} p(s) > 0 \), from \( p \neq \mathbf{0} \) and \( p \neq \mathbf{0} \). Dividing by this sum, we can make \( \sum_{s \in S} p(s) = 1 \) (wlog).

Furthermore, the maximum of the linear function \( p'x \) over \( x \in X \) must be achieved at one of the extreme points in X, which are vectors \( (u(c,s))_{s \in S} \) for the various \( c \in C \):

\[ \max_{x \in X} \sum_{s \in S} p(s)x_s = \max_{\sigma \in \Delta(C)} \sum_{s \in S} p(s) \sum_{c \in C} \sigma(c)u(c,s) = \max_{c \in C} \sum_{s \in S} p(s)u(c,s). \]

So (2') is equivalent to condition (2) in the theorem here.

**Expected Utility Theorem.** Let N be a finite set of prizes, and consider a finite sequence of pairs of lotteries \( p(i) \in \Delta(N) \) and \( q(i) \in \Delta(N) \), for \( i \in M = \{1, \ldots, m\} \). (M indexes comparisons: "p(i) preferred to q(i).")

(Here \( p(i) = (p_l(i))_{l \in N} \), and \( q(i) = (q_l(i))_{l \in N} \).) Then exactly one of these two statements is true:

Either (1) \( \exists \sigma \in \Delta(M) \) such that \( \sum_{i \in M} \sigma(i)p_i(i) = \sum_{i \in M} \sigma(i)q_i(i) \forall j \in N \), [substitution axiom is violated]

or (2) \( \exists u \in \mathbb{R}^N \) such that \( \sum_{j \in N} p_j(i)u_j > \sum_{j \in N} q_j(i)u_j \forall i \in M \). [preferences satisfy utility theory]

**Proof.** Let \( X = \{ \{i \in M | \sigma(i)(q(i)−p(i)) \in \Delta(M) \} | \sigma \in \Delta(M) \} \). Then X is a closed convex subset of \( \mathbb{R}^N \).

Condition (1) here is equivalent to: (1') the N-vector \( \mathbf{0} \) is in X.

By the Separating Hyperplane Thm, (1') is false iff

(2') \( \exists u \in \mathbb{R}^N \) such that \( 0 = u'\mathbf{0} > \max_{x \in X} u'x \).

The extreme points of X are vectors \( (q(i)−p(i))_{i \in N} \), and the linear function \( u'x = \sum_{j \in N} x_ju_j \) must achieve its maximum over \( x \in X \) at an extreme point: \( \max_{x \in X} u'x = \max_{i \in M} \sum_{j \in N} u_j(q_j(i)−p_j(i)) \).

So (2') is equivalent to (2) in the theorem here.

**Fact.** Suppose the utility-representation condition (2) is satisfied by \( u = (u_j)_{j \in N} \).

Then (2) is also satisfied by \( \tilde{u} \) if there exists \( A > 0 \) and \( B \) such that \( \tilde{u}_j = Au_j + B \forall j \in N \).
Linear Duality Theorem (Farkas's lemma, theorem of the alternatives)

Given any \( m \times n \) matrix \( A = (a_{ij})_{i \in \{1,\ldots,m\}, j \in \{1,\ldots,n\}} \) and any vector \( b = (b_i)_{i \in \{1,\ldots,m\}} \) in \( \mathbb{R}^m \), exactly one of the following two conditions is true:

1. \( \exists x \in \mathbb{R}^n \) such that \( Ax \geq b \).
2. \( \exists y \in \mathbb{R}^m \) such that \( y \geq 0 \), \( y'A = 0 \), and \( y'b > 0 \).

Here \( 0 \) denotes a vector of zeroes in some appropriate number of dimensions.

Vector inequalities denote systems of numerical inequalities:

- \( Ax \geq b \) means: \( \sum_{j=1}^{n} a_{ij} x_j \geq b_i \) \( \forall i \in \{1,\ldots,m\} \).
- \( y'A = 0 \) means \( \sum_{i=1}^{m} y_i a_{ij} = 0 \) \( \forall j \in \{1,\ldots,n\} \).
- \( y'b > 0 \) means \( \sum_{i=1}^{m} y_i b_i > 0 \).
- \( y \geq 0 \) means \( y_i \geq 0 \) \( \forall i \in \{1,\ldots,m\} \).

Vector inequalities can also be expressed in terms of orthants:

- \( y \geq 0 \) means \( y \) is in the nonnegative orthant.
- \( y'A = 0 \) means \( y \) is orthogonal to all columns of \( A \).
- \( y'b > 0 \) means \( y \) is in the orthant of positive \( b \).

We may let \( \mathbb{R}^m_+ = \{ y \in \mathbb{R}^m : y \geq 0 \} \) denote the nonnegative orthant in \( \mathbb{R}^m \).

**Proof.** Conditions (1) and (2) cannot both be true for any \( x \) and \( y \), because \( y \geq 0 \) and \( Ax \geq b \) would imply \( y'(Ax-b) \geq 0 \), while \( y'A = 0 \) and \( y'b > 0 \) would imply \( y'(Ax-b) < 0 \), a contradiction.

So (2) must be false if (1) is true.

Now suppose that (1) is false. This hypothesis means that the vector \( b \) is not in the set \( \{ Ax | x \in \mathbb{R}^n, z \in \mathbb{R}^m, z \geq 0 \} \).

This set is convex and closed. So by the separating hyperplane theorem (MWG p948), there must exist some \( y \in \mathbb{R}^m \) such that \( y'b > \max\{ y'(Ax-z) x \in \mathbb{R}^n, z \in \mathbb{R}^m, z \geq 0 \} \).

This max must be nonnegative (because \( x \) and \( z \) could be \( 0 \)), and it must be finite.

[In fact, this max must be exactly 0, because if we could achieve any \( 0 < \alpha = y'(Ax-z) \) with \( x \in \mathbb{R}^n, z \in \mathbb{R}^m, z \geq 0 \), then doubling \( x \) and \( z \) could achieve \( 2\alpha \), and so the max would be \( +\infty \).]

If the component \( y_i \) were negative, then choosing large positive \( z_i \) could drive this max to \( +\infty \).

If the \( j \)'th component of \( y'A \) were nonzero, then choosing \( x_j \) with the same sign and large absolute value could also drive this max to \( +\infty \).

So we must have \( y'b > 0 \), \( y'A = 0 \), and \( y \geq 0 \).

So (2) must be true if (1) is false.

Thus, (2) is true if and only if (1) is false. QED.

[This argument skips over one tricky detail: proving that the set \( \{ Ax-z | x \in \mathbb{R}^n, z \in \mathbb{R}^m, z \geq 0 \} \) is closed. Here is a sketch of the proof. Let \( J \) be a minimal set of columns of the \( A \) matrix that linearly span the rest of \( A \)’s columns. Then you can add the restriction that \( x_j = 0 \) for all \( j \notin J \), without changing the set. Let \( I \) be a maximal set of rows of the \( A \) matrix such that there exists some \( x \) such that \( Ax \geq 0 \) and \( Ax \) is strictly positive in all the \( I \) rows. Then you can also add the restriction that \( z_i = 0 \) for all \( i \in I \). With these restrictions, you can show that any sequence \( Ax(k) - z(k) \) that converges to a finite limit must have \( x(k) \) and \( z(k) \) converging to finite limits.]
First duality application: strong domination. Suppose that there is a finite set of choice alternatives $C$, a finite set of possible states $S$, and $u(c,s)$ denotes the utility payoff for the decision-maker if the state is $s$. Consider any given choice alternative $d$ in $C$. By duality, exactly one of the following two conditions is true.

The dual variable for each constraint is shown at right in red italics:

1. $\exists p \in \mathbb{R}^S$ such that $\sum_{s \in S} (u(d,s) - u(c,s)) p(s) \geq 0 \quad \forall c \in C,$
   $\sigma(c)$
   $p(s) \geq 0 \quad \forall s \in S,$
   $\delta(s)$
   and $\sum_{s \in S} p(s) \geq 1.$
   $\alpha$

2. $\exists (\sigma, \delta, \alpha) \in \mathbb{R}_+^C \times \mathbb{R}_+^S \times \mathbb{R}_+$ such that $\sum_{c \in C} \sigma(c)(u(d,s) - u(c,s)) + \delta(s) + \alpha = 0 \quad \forall s \in S,$
   $p(s)$
   and $\alpha > 0.$

Condition (1) holds iff there is some probability distribution on the states such that $d$ maximizes expected utility (because we could divide $p$ by its sum to make it a probability distribution). Condition (2) holds iff there exists some probability distribution $\sigma$ on the set of choice alternatives (a randomized strategy) such that $u(d,s) < \sum_{c \in C} \sigma(c) u(c,s) \quad \forall s \in S.$ (When (2) holds, the $\sigma(c)$ cannot all be zero, and so we could divide $(\sigma, \delta, \alpha)$ by $\sum_c \sigma(c)$ to make $\sigma$ a probability distribution.) So $d$ is optimal for some beliefs iff it is not strongly dominated by some randomized strategy. (Myerson, 1991, Theorem 1.6)

Second application: Utility theory. Suppose there are $n$ possible prizes numbered $1,...,n.$ We ask a decision maker $m$ questions. In the $i$th question, we ask whether he prefers a lottery $p(i)$, which offers each prize $j$ with probability $p_j(i)$, or a lottery $q(i)$ which offers each prize $j$ with probability $q_j(i).$ In each case, suppose $p(i)$ denotes the lottery that he strictly prefers.

Let $\varepsilon$ be any positive number. By duality, exactly one of the following two conditions is true:

1. $\exists u \in \mathbb{R}^n$ such that $\sum_{i=1}^n (p_{j}(i) - q_{j}(i)) u_j \geq \varepsilon \quad \forall i \in \{1,...,m\}.$
   $\sigma_i$

2. $\exists \sigma \in \mathbb{R}_+^m$ such that $\varepsilon \sum_{i=1}^m \sigma_i > 0,$ and $\sum_{i=1}^m \sigma_i (p_{j}(i) - q_{j}(i)) = 0 \quad \forall j \in \{1,...,n\}.$
   $u_j$

Condition (1) holds iff we can find a utility function for which his revealed preferences are compatible with expected utility maximization. Condition (2) holds iff we can find a violation of the substitution axiom in compound lotteries. (Renormalize so that $\sum \sigma_i = 1$, then consider the first compound lottery that gives each $p(i)$ lottery with probability $\sigma_i$, and the second compound lottery that gives each $q(i)$ lottery with probability $\sigma_i$. After the first stage of the compound lotteries, the first would always seem better than the second, but the expected ante probability of each prize $j$ is equal in the two compound lotteries.) So the revealed preferences are compatible with expected utility maximization iff there is no violation of the substitution axiom.

(Conjecture of Daniel Bernoulli, 1738, verified by Von Neumann and Morgenstern, 1947.)

Third application: weak domination. Let $S$, $C$, $u$, and $d$ be as in the previous example. Let $\varepsilon$ be any small positive number. By duality, exactly one of the following two conditions is true:

1. $\exists p \in \mathbb{R}^S$ such that $\sum_{s \in S} (u(d,s) - u(c,s)) p(s) \geq 0 \quad \forall c \in C,$
   $\sigma(c)$
   and $p(s) \geq \varepsilon \quad \forall s \in S.$
   $\delta(s)$

2. $\exists (\sigma, \delta) \in \mathbb{R}_+^C \times \mathbb{R}_+^S$ such that $\sum_{c \in C} \sigma(c)(u(d,s) - u(c,s)) + \delta(s) = 0 \quad \forall s \in S,$
   $p(s)$
   and $\varepsilon \sum_{s \in S} \delta(s) > 0.$

Condition (1) holds iff there is some probability distribution $p$ on the set of states such that every state has strictly positive probability and $d$ maximizes expected utility over all choices in $C$.

Condition (2) holds iff there exists some probability distribution $\sigma$ on the set of choice alternatives such that $u(d,s) \leq \sum_{c \in C} \sigma(c) u(c,s) \quad \forall s \in S,$ with at least one strict inequality ($<)$ for some $s$.

So $d$ is optimal for some beliefs where all states have positive probability iff $d$ is not weakly dominated by some randomized strategy. (Myerson, 1991, Theorem 1.7)
**Duality in linear programming (LP)** (MWG appendix M.M, Myerson 1991 pp 125-127.)

Suppose that we are given $m \times n$ matrix $A = (a_{ij})_{i=1}^{m},j=1,...,n,$ and vectors $b = (b_i)_{i=1}^{m} \in \mathbb{R}^m$ and $c = (c_j)_{i=1}^{n} \in \mathbb{R}^n.$

Consider the primal linear-programming problem:

choose $x$ in $\mathbb{R}^n$ to minimize $c'x$ subject to $Ax \geq b.$

This problem is equivalent to: minimize $\max \{c'x + y'(b-Ax) \mid y \in \mathbb{R}^m, y \geq 0\},$

because if $x$ violated the constraints then $b-Ax$ would have some positive components and so the max here would become $+\infty$ (very bad when we are minimizing).

If we reversed the order of min and max, we would get

choose $y$ in $\mathbb{R}^m$ to maximize $\min \{y'b + (c' - y'A)x \mid x \in \mathbb{R}^n\}$ subject to $y \geq 0.$

This problem is (similarly) equivalent to the dual linear-programming problem:

choose $y$ in $\mathbb{R}^m$ to maximize $y'b$ subject to $y'A = c' \text{ and } y \geq 0.$

**Duality Theorem of Linear Programming.** Suppose that the constraints of the primal and dual LP problems both have feasible solutions. Then these problems have optimal solutions $x$ and $y$ such that $c'x = y'b$ (equal values) and $y'(b-Ax) = 0$ (complementary slackness).

**Proof** If $x$ and $y$ satisfy the primal and dual constraints then we must have

$$c'x \geq c'x + y'(b-Ax) = y'b + (c' - y'A)x = y'b.$$

So both the primal and dual problems must have bounded optimal values.

Dual boundedness implies that we cannot find any $\hat{y}$ such that $\hat{y}'A=0'$, $\hat{y} \geq 0$, and $\hat{y}'b>0$, because otherwise we could infinitely improve any dual solution by adding multiples of $\hat{y}$.

Now for any number $\theta$, linear duality implies that exactly one of the following is true:

1. $\exists x \in \mathbb{R}^n$ such that $Ax \geq b$ and $-c'x \geq -\theta$.
2. $\exists (y,\omega) \in \mathbb{R}^m \times \mathbb{R}$ such that $y \geq 0$, $\omega \geq 0$, $y'A - \omega c' = 0$, and $y'b - \omega \theta > 0$.

But when (2) is true, we must have $\omega > 0$, or else we would have the vector $\hat{y}$ described above.

So we could divide any solution of (2) through by $\omega > 0$ to get a solution of (2) with $\omega=1$.

So whenever (2) holds, we must also have

2' $\exists y \in \mathbb{R}^m$ such that $y \geq 0$, $y'A = c'$, and $y'b > \theta$.

So the dual maximization can do better than any value $\theta$ that is below the minimal value of the primal.

Thus, the optimal values of the primal and dual problems must be equal.

It can be shown that the set of feasible values for each problem is a closed set, and so optimal solutions actually exist. At optimal solutions $x$ and $y$, we have $y'b=c'x$, which (by (2) above) implies the complementary slackness equation $y'(b-Ax) = 0.$ QED.

[Our basic linear duality thm can also be seen as a special case of duality in linear programming. Given the matrix $A$ and the vector $b$ from basic linear duality, suppose we let $c=0$ in $\mathbb{R}^n.$

Then the constraints of the dual LP problem always have a feasible solution $y=0$.

So LP duality implies that this primal is feasible ($\exists x \text{ s.t. } Ax \geq b$) if and only if its and its dual share the same optimal value, which must be $c'x = 0'x = 0$, and so any $y \geq 0$ with $y'A=0$ must have $y'b \leq 0.$]
In game theory we assume that players are rational and intelligent. Here rational means that each player acts to maximize his own expected utility, and intelligent means that the players know everything that we know about their situation when we analyze it game-theoretically. Intelligence implies that game model that we analyze must be common knowledge among the players, that is, all players know (that all players know) the model, \( \forall k=\{0,1,2,\ldots\} \).

A strategic-form game is characterized by \((N, (C_i)_{i \in N}, (u_i)_{i \in N})\) where \( N = \{1,2,\ldots,n\} \) is the set of players, and, for each player \( i \):

- \( C_i \) is the set of alternative actions or (pure) strategies that are feasible for \( i \) in the game, and
- \( u_i: C_1 \times C_2 \times \ldots \times C_n \to \mathbb{R} \) is player \( i \)'s utility function in the game.

We generally assume that each player \( i \) independently chooses an action in \( C_i \).

If \( c = (c_1,c_2,\ldots,c_n) \) is the combination (or profile) of actions chosen by the players then each player \( i \) will get the expected utility payoff \( u_i(c_1,c_2,\ldots,c_n) \).

We let \( C = C_1 \times C_2 \times \ldots \times C_n = \times_{i \in N} C_i \) denote the set of all combinations or profiles of actions that the players could choose. Let \( C_{-i} \) denote the set of all profiles of actions that can be chosen by players other than \( i \).

When \( c \in C \) is a profile of actions for the players, \( c_i \) denotes the action of each player \( i \), \( c_{-i} \) denotes the profile of actions for players other than \( i \) where they act as in \( c \), and \((c_{-i};d_i)\) denotes the profile of actions in which \( i \)'s action is changed to \( d_i \) but all others choose the same action as in \( c \). (We may use this notation even if player \( i \) is not the "last" player.) So \( c = (c_{-i};c_i) \).

A randomized strategy (or mixed strategy) for player \( i \) is a probability distribution over \( C_i \), so \( \Delta(C_i) \) denotes the set of all randomized strategies for player \( i \). (pure = nonrandomized.)

An action \( d_i \) for player \( i \) is strongly dominated by a randomized strategy \( \sigma_i \in \Delta(C_i) \) if
\[
 u_i(c_i;d_i) < \sum_{c_i \in C_i} \sigma_i(c_i) u_i(c_i;c_i) \quad \forall c_i \in C_i.
\]

An action \( d_i \) for player \( i \) is weakly dominated by a randomized strategy \( \sigma_i \in \Delta(C_i) \) if
\[
 u_i(c_i;d_i) \leq \sum_{c_i \in C_i} \sigma_i(c_i) u_i(c_i;c_i) \quad \forall c_i \in C_i, \text{ with strict inequality (<) for at least one } c_i.
\]

The set of player \( i \)'s best responses to any profile of opponents' actions \( c_{-i} \) is
\[
 \beta(c_{-i}) = \arg\max_{d_i \in C_i} u_i(c_i;d_i) = \left\{ d_i \in C_i \mid u_i(c_i;d_i) = \max_{c_i \in C_i} u_i(c_i;c_i) \right\}.
\]

Similarly, if \( i \)'s beliefs about the other players' actions can be described by a probability distribution \( \mu \) in \( \Delta(C_{-i}) \), then the set of player \( i \)'s best responses to the beliefs \( \mu \) is
\[
 \beta(\mu) = \arg\max_{d_i \in C_i} \sum_{d_{-i} \in C_{-i}} \mu(c_{-i}) u_i(c_i;d_i).
\]

Fact. If we iteratively eliminate strongly dominated actions for all players until no strongly dominated actions remain, then we get a reduced game in which each remaining action for each player is a best response to some beliefs about the other players' actions. These remaining actions are rationalizable.

If each player \( j \) independently uses a strategy \( \sigma_j \in \Delta(C_j) \), then player \( i \)'s expected payoff is
\[
 u_i(\sigma) = u_i(\sigma_1;\sigma_2;\ldots;\sigma_n) = \sum_{c \in C} \left( \prod_{j \in N} \sigma_j(c_j) \right) u_i(c).
\]

Here \( |c| \in \Delta(C) \) with \( |c|(c_i) = 1, |c|(d_i) = 0 \) if \( d_i \neq c_i \). Notice \( \sum_{c_i \in C_i} \sigma_i(c_i)(u_i(\sigma_1;[c_i]) - u_i(\sigma)) = 0 \).

Fact. \( \sigma_i \in \arg\max_{\tau_i \in \Delta(C_i)} u_i(\sigma_i;\tau_i) \) if and only if \( \{ c_i \in C_i \mid \sigma_i(c_i) > 0 \} \subseteq \arg\max_{d_i \in C_i} u_i(\sigma_i;[d_i]) \).

The set \( \{ c_i \mid \sigma_i(c_i) > 0 \} \) of actions that have positive probability under \( \sigma_i \) is called the support of \( \sigma_i \).

A Nash equilibrium is a profile of actions or randomized strategies such that each player is using a best response to the others. That is \( \sigma = (\sigma_1,\ldots,\sigma_n) \) is a Nash equilibrium in randomized strategies iff \( \sigma_i \in \arg\max_{\tau_i \in \Delta(C_i)} u_i(\sigma_i;\tau_i) \) for every player \( i \) in \( N \).

Fact. Any finite strategic-form game has at least one Nash equilibrium in randomized strategies.
Computing randomized Nash equilibria for games that are larger than 2×2 can be difficult, but working a few examples can help you better understand Nash’s subtle concept of equilibrium. We describe here a procedure for finding Nash equilibria, from section 3.3 of Myerson (1991).

We are given some game, including a given set of players N and, for each i in N, a given set of feasible actions Ci for player i and a given payoff function ui:C1×⋯×Cn→ℝ for player i. The support of a randomized equilibrium is, for each player, the set of actions that have positive probability of being chosen in this equilibrium.

To find a Nash equilibrium, we can apply the following 5-step method:

1. Guess a support for all players. That is, for each player i, let Si be a subset of i’s actions Ci, and let us guess that Si is the set of actions that player i will use with positive probability.

2. Consider the smaller game where the action set for each player i is reduced to Si, and try to find an equilibrium where all of these actions get positive probability.

To do this, we need to solve a system of equations for some unknown quantities. The unknowns: For each player i in N and each action si in i’s support Si, let σi(si) denote i’s probability of choosing si, and let wi denote player i’s expected payoff in the equilibrium. (σi(ai)=0 if ai∉Si.) The equations: For each player i, the sum of these probabilities must equal 1.

For each player i and each action si in Si, player i’s expected payoff when he chooses si but all other players randomize independently according to their σj probabilities must be equal to wi. Let uσ,a = Eu(a|σ−i) denote player i’s expected payoff when he chooses action ai and all other players are expected to randomize independently according to their σj probabilities.

Then the equations can be written:

\[ \sum_{a_i \in S_i} \sigma_i(a_i) = 1 \quad \forall i \in N; \quad \text{and} \quad u_i(\sigma_{-i},[s_i]) = w_i \quad \forall i \in N \quad \forall S_i \in S.
\]

(Here \(\forall\) means "for all", \(\in\) means "in"). We have as many equations as unknowns \((w_i, \sigma_i(s_i))\).

3. If the equations in step 2 have no solution, then we guessed the wrong support, and so we must return to step 1 and guess a new support.

Assuming that we have a solution from step (2), continue to (4) and (5)

4. The solution from (2) would be nonsense if any of the "probabilities" were negative. That is, for every player i in N and every action si in i’s support Si, we need σi(si) ≥ 0.

If these nonnegativity conditions are not satisfied by a solution, then we have not found an equilibrium with the guessed support, and so we must return to step 1 and guess a new support.

If we have a solution that satisfies all these nonnegativity conditions, then it is a randomized equilibrium of the reduced game where each player must can only choose actions in Si.

5. A solution from (2) that satisfies the condition in (4) would still not be an equilibrium of the original game, however, if any player would prefer an action outside the guessed support. So next we must ask, for each player i and for each action a that is in Ci but is not in the guessed support Si, could player i do better than wi by choosing a when all other players randomize independently according to their σj probabilities? Recall uσ,a = Eu(a|σ−i) = wi for all si in Si.

Now, for every action ai that is in Ci but is not in Si (so σi(ai)=0), we need uσ,a = w_i. If our solution satisfies all these inequalities then it is an equilibrium of the given game. But if any of these inequalities is violated (some uσ,a > wi), then we have not found an equilibrium with the guessed support, and so we must return to step 1 and guess a new support.

In a finite game, there are only a finite number of possible supports to consider.

Thus, an equilibrium \(\sigma = (\sigma_i(a_i))_{a_i \in C_i \in N}\) with payoffs \(w = (w_i)_{i \in N}\) must satisfy:

\[ \sum_{a_i \in C_i} \sigma_i(a_i) = 1 \quad \forall i \in N; \quad \text{and} \quad \sigma_i(a_i) \geq 0 \quad \text{and} \quad u_i(\sigma_{-i},[a_i]) \leq w_i \quad \forall a_i \in C_i, \forall i \in N.
\]

The support for each player i is the set of actions si in Ci for which \(\sigma_i(s_i) > 0\), so that \(u_i(\sigma_{-i},[s_i]) = w_i\).
Example. Find all Nash equilibria (pure and mixed) of the following 2×3 game:

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>T</td>
<td>7, 2</td>
<td>2, 7</td>
</tr>
<tr>
<td>B</td>
<td>2, 7</td>
<td>7, 2</td>
</tr>
</tbody>
</table>

There are 3×7=21 possible supports. But it is easy to see that this game has no pure-strategy equilibria (2's best response to T is M, but T is not 1's best response to M; and 2's best response to B is L, but B is not 1's best response to L). This eliminates the six cases where each player's support is just one action. Furthermore, when either player is restricted to just one action, the other player always has a unique best response, and so there are no equilibria where only one player randomizes. That is, both players must have at least two actions in the support of any equilibrium. Thus, we must search for equilibria where the support of player 1’s randomized strategy is {T,B}, and the support of player 2's randomized strategy is {L,M,R} or {M,R} or {L,M} or {L,R}. We consider these alternative supports in this order.

Guess support is {T,B} for 1 and {L,M,R} for 2?

We may denote 1’s strategy by \( \sigma_1 = p\text{T}+(1-p)\text{B} \) and 2’s strategy by \( \sigma_2 = q\text{L}+(1-q)\text{M}+r\text{R} \), that is \( p = \sigma_1(\text{T}) \), \( 1-p = \sigma_1(\text{B}) \), \( q = \sigma_2(\text{L}) \), \( r = \sigma_2(\text{R}) \), \( 1-q-r = \sigma_2(\text{M}) \).

Player 1 randomizing over \{T,B\} requires \( w_1 = u_1(T, \sigma_2) = u_1(B, \sigma_2) \), and so \( w_1 = 7q+2(1-q-r)+3r = 2q+7(1-q-r)+4r \).

Player 2 randomizing over \{L,M,R\} requires \( w_2 = u_2(\sigma_1, \text{L}) = u_2(\sigma_1, \text{M}) = u_2(\sigma_1, \text{R}) \), and so \( w_2 = 2p+7(1-p) = 7p+2(1-p) = 6p+5(1-p) \).

We have three equations for three unknowns \((p,q,r)\), but they have no solution (as the two indifference equations for player 2 imply both \(p=1/2\) and \(p=3/4\), which is impossible).

Thus there is no equilibrium with this support.

Guess support is \{T,B\} for 1 and \{M,R\} for 2?

We may denote 1’s strategy by \( \sigma_1 = p\text{T}+(1-p)\text{B} \) and 2’s strategy by \( (1-r)\text{M}+r\text{R} \), \( q=0 \).

Player 1 randomizing over \{T,B\} requires \( w_1 = u_1(T, \sigma_2) = u_1(B, \sigma_2) \), so \( w_1 = 7q+2(1-q)-3r = 7(1-r)+4r \).

Player 2 randomizing over \{M,R\} requires \( w_2 = u_2(\sigma_1, \text{M}) = u_2(\sigma_1, \text{R}) \), so \( w_2 = 7p+2(1-p) = 6p+5(1-p) \).

These solution for these two equations in two unknowns is \( p = 1/2 \) and \( r = 5/4 \). But this solution would yield \( \sigma_2(\text{M}) = 1-r = -1/4 < 0 \), and so there is no equilibrium with this support. (Notice: if player 2 never chose L then T would be dominated by B for player 1.)

Guess support is \{T,B\} for 1 and \{L,M\} for 2?

We may denote 1’s strategy by \( p\text{T}+(1-p)\text{B} \) and 2’s strategy by \( q\text{L}+(1-q)\text{M} \), \( r=0 \).

Player 1 randomizing over \{T,B\} requires \( w_1 = u_1(T, \sigma_2) = u_1(B, \sigma_2) \), so \( w_1 = 7q+2(1-q) = 2q+7(1-q) \).

Player 2 randomizing over \{L,M\} requires \( w_2 = u_2(\sigma_1, \text{L}) = u_2(\sigma_1, \text{M}) \), so \( w_2 = 2p+7(1-p) = 7p+2(1-p) = 6p+5(1-p) \).

These solution for these two equations in two unknowns is \( p = 1/2 \) and \( q = 1/2 \), with \( w_1 = 4.5 = w_2 \). This solution yields nonnegative probabilities for all actions. But we also need to check that player 2 would not prefer deviating outside her support to R. However \( u_2(\sigma_1, \text{R}) = 6p+5(1-p) = 6\times1/2+5\times1/2 = 5.5 > w_2 = u_2(\sigma_1, \text{L}) = 2\times1/2+7\times1/2 = 4.5 \). So there is no equilibrium with this support.

Guess support is \{T,B\} for 1 and \{L,R\} for 2?

We may denote 1’s strategy by \( p\text{T}+(1-p)\text{B} \) and 2’s strategy by \( q\text{L}+(1-q)\text{R} \), \( r=1-q \).

Player 1 randomizing over \{T,B\} requires \( w_1 = u_1(T, \sigma_2) = u_1(B, \sigma_2) \), so \( w_1 = 7q+3(1-q) = 2q+4(1-q) \).

Player 2 randomizing over \{L,R\} requires \( w_2 = u_2(\sigma_1, \text{L}) = u_2(\sigma_1, \text{R}) \), so \( w_2 = 2p+7(1-p) = 6p+5(1-p) \).

These solution for these two equations in two unknowns is \( p = 1/3 \) and \( q = 1/6 \). This solution yields nonnegative probabilities for all actions. We also need to check that player 2 would not prefer deviating outside her support to M; \( u_2(\sigma_1, \text{M}) = 7p+2(1-p) = 7\times1/3+2\times2/3 = 11/3 < w_2 = u_2(\sigma_1, \text{L}) = 2\times1/3+7\times2/3 = 16/3 \). Thus, we have an equilibrium with this support: \( ((1/3)\text{T}+(2/3)\text{B}, (1/6)\text{L}+(5/6)\text{R}) \).

The expected payoffs in this equilibrium are \( w_1 = Eu_1 = 7\times1/6+3\times5/6 = 2\times1/6+4\times5/6 = 11/3 = 3.667 \) and \( w_2 = Eu_2 = 2\times1/3+7\times2/3 = 6\times1/3+5\times2/3 = 16/3 = 5.333 \).
A First Bayesian Game  Bayesian games are models of one-stage games where players choose actions simultaneously, but where each player may have private information, called his type.

Let us consider an example where player 2 is uncertain about one of player 1's payoffs.

Each player must independently decide whether to act with generosity (g_i) or hostility (h_i).

Player 1 might be the kind of person who would be contented (type 1c) or envious (type 1e) of player 2.

Player 2 thinks that each of 1's possible types has probability 0.5.

The players' payoffs (u_1,u_2) depend on their actions and 1's type as follows:

<table>
<thead>
<tr>
<th></th>
<th>g_1</th>
<th>h_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>g_2</td>
<td>7,7</td>
<td>0,4</td>
</tr>
<tr>
<td>h_2</td>
<td>4,0</td>
<td>4,4</td>
</tr>
</tbody>
</table>

If 1's type is 1c:

p(1c) = 0.5

If 1's type is 1e:

p(1e) = 0.5

How shall we analyze about this game? Let me first sketch a common mistake.

To deal with the uncertainty about 1's payoff from (g_1,g_2), some students try to analyze the game where player 1's payoff from (g_1,g_2) is the expected utility 0.5(7)+0.5(3) = 5. So these students consider a 2x2 payoff matrix that differs from the second (1e) case only in that the payoff 3 would be replaced by 5, and then they find an "equilibrium" at (g_1,g_2) (as 5>4 for player 1 and 7>4 for player 2).

Such analysis would be nonsense, however. This "equilibrium" would correspond to a theory that each player is sure to choose generosity. But player 2 knows that if player 1 is type 1e then he will not choose g_1, because g_1 would be dominated by h_1 for player 1 when his type is 1e. Thus, player 2 must believe that there is at least a probability 0.5 of player 1 having the envious type 1e and thus choosing hostility h_1.

A correct analysis must recognize this fact.

To find a correct approach, we may consider the situation before the players learns any private information, but when they know that each will learn his private type information before he acts in the game.

A strategy for a player is a complete plan that specifies a feasible action for the player in every possible contingency that the player could find.

Before player 1 learned his type, he would have 4 strategies {g_cge, g_che, h_cge, h_che} because he will learn his type before acting. (For example, g_che denotes the strategy "act generous if type 1c, act hostile if type 1e." )

Player 2 would have only two strategies {g_2, h_2}, because she must act without learning 1's type.

For each pair of strategies, we can compute the expected payoffs to each player, given that each of 1's types has probability 1/2. So the normal representation in strategic form of this Bayesian game is:

<table>
<thead>
<tr>
<th></th>
<th>g_2</th>
<th>h_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>g_cge</td>
<td>5,7</td>
<td>0,4</td>
</tr>
<tr>
<td>g_che</td>
<td>5.5, 3.5</td>
<td>2, 4</td>
</tr>
<tr>
<td>h_cge</td>
<td>3.5, 3.5</td>
<td>2, 4</td>
</tr>
<tr>
<td>h_che</td>
<td>4, 0</td>
<td>4, 4</td>
</tr>
</tbody>
</table>

This strategic game has one equilibrium: (h_che, h_2), where both are hostile and get payoffs (4,4).

In this strategic game, g_cge and h_che are strictly dominated for 1 (by g_che and h_che respectively).

When we eliminate these dominated strategies, then g_2 becomes dominated (by h_2) for player 2, and then h_che is the unique best response for player 1 against 2's remaining strategy h_2.

(The students' mistake above was to consider only the strategies g_cge and h_che here.)

A Bayesian game is defined by a set of players N; a set of actions C_i, a set of types T_i, and a utility function u_i: (\times_{j \in N} C_j) \times (\times_{j \in T_i} T_j) \rightarrow \mathbb{R}, for each i in N; and a probability distribution p \in \Delta(\times_{j \in N} T_j)

The Bayesian game (N,(C_i,T_i,u_i)_{i \in N},p) is assumed to be common knowledge among the players in the game, but each player i also privately knows his own actual type t_i \in T_i, which is a random variable in the model.

Mixed strategies for player i are probability distributions over functions from T_i to C_i, in \Delta(C_i^{T_i}).

But nobody cares about correlations among plans of i's different types; so we can instead analyze behavioral strategies, which are functions from T_i to probability distributions over C_i, in \Delta(C_i^{T_i}).

A behavioral strategy \sigma_i specifies conditional probabilities \sigma_i(c_i|t_i) = \text{Prob}(i \text{ does } c_i | t_i = t_i), \forall c_i \in C_i, \forall t_i \in T_i,
Increasing differences and increasing strategies in Bayesian games

We may consider Bayesian games where each player $i$ first learns his type $\tilde{t}_i$, and then each player $i$ chooses his action $a_i$. We assume here that each player $i$’s type is drawn from some probability distribution $p_i$, independently of all other players’ types, and so the joint probability distribution of the players’ types can be written $p((t_i)_{i\in N}) = \prod_{i\in N} p_i(t_i)$, where $p_i(t_i) = \text{Prob}(\tilde{t}_i = t_i)$.

The payoffs of each player $i$ may depend on all players’ types and actions according to some utility payoff function $u_i(c_1,\ldots,c_n,\tilde{t}_1,\ldots,\tilde{t}_n)$. Suppose that types and actions are ordered as numbers ($c_i \in C_i \subseteq \mathbb{R}$, $\tilde{t}_i \in T_i \subseteq \mathbb{R}$). A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is increasing (in the weak sense) iff, for all $x$ and $\hat{x}$, $\hat{x} \geq x$ implies $f(\hat{x}) \geq f(x)$. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing iff, for all $x$ and $\hat{x}$, $\hat{x} > x$ implies $f(\hat{x}) > f(x)$.

Consider a two-player Bayesian game where player 1 has two possible actions, T and B. Player 1 has several possible types, and each possible type is represented by a number $t_1$. Player 2 may have many possible actions $c_2$ and many possible types $t_2$. Suppose that player 2’s type $t_2$ is independent of player 1’s type $t_1$. The difference in player 1’s payoff in switching from B to T is $u_1(T,c_2,t_1,t_2) - u_1(B,c_2,t_1,t_2)$. This difference depends on player 1’s type $t_1$, player 2’s action $c_2$, and player 2’s type $t_2$.

We say that player 1’s payoffs satisfy (weakly or strictly) increasing differences if this difference $u_1(T,c_2,t_1,t_2) - u_1(B,c_2,t_1,t_2)$ is a (weakly or strictly) increasing function of $t_1$, no matter what player 2’s action $c_2$ and type $t_2$ may be.

That is, increasing differences (in the weak sense) means that, for every $r_1$, $t_1$, $c_2$, and $t_2$:
if $r_1 \geq t_1$ then $u_1(T,c_2,r_1,t_2) - u_1(B,c_2,r_1,t_2) \geq u_1(T,c_2,t_1,t_2) - u_1(B,c_2,t_1,t_2)$.
Strictly increasing differences means that, for every $r_1$, $t_1$, $c_2$, and $t_2$:
if $r_1 > t_1$ then $u_1(T,c_2,r_1,t_2) - u_1(B,c_2,r_1,t_2) > u_1(T,c_2,t_1,t_2) - u_1(B,c_2,t_1,t_2)$.

With increasing differences, 1’s higher types find T relatively more attractive than lower types do.

Player 1 is using a cutoff strategy if there is some number $\theta$ (the cutoff) such that, for each possible type $t_1$ of player 1: if $t_1 > \theta$ then type $t_1$ would choose $[T]$ for sure in this strategy,
if $t_1 < \theta$ then type $t_1$ would choose $[B]$ for sure in this strategy,
if $t_1 = \theta$ then type $t_1$ may choose T or B or may randomize in this strategy.

Comparing cutoff strategies, the probability of 1 choosing T decreases as the cutoff $\theta$ increases.

Fact. If player 1’s payoffs satisfy increasing differences then, no matter what strategy player 2 may use, player 1 will always want to use a cutoff strategy. Thus, when we are looking for equilibria, the increasing-differences property assures us that player 1 must be using a cutoff strategy.

More generally, in games where player 1’s action can be any number in some range, we say that player 1’s payoffs satisfy (weakly or strictly) increasing differences if, for every pair of possible actions $c_1$ and $d_1$ such that $c_1 > d_1$, the difference $u_1(c_1,c_2,t_1,t_2) - u_1(d_1,c_2,t_1,t_2)$ is a (weakly or strictly) increasing function of player 1’s type $t_1$, no matter what player 2’s action $c_2$ and type $t_2$ may be.

If $u_i$ is differentiable then the condition for increasing differences is $\frac{\partial^2 u_i}{\partial c_1 \partial t_1} \geq 0$.

Fact. If 1’s payoffs satisfy weakly increasing differences, then, against any strategy of player 2, player 1 will have some best-response strategy $s_1:T_1 \rightarrow C_1$ that is weakly increasing ($r_1 \geq t_1 \Rightarrow s_1(r_1) \geq s_1(t_1)$).

Fact. When 1’s payoffs have strictly increasing differences then all player 1’s best-response strategies must be weakly increasing: if $r_1 > t_1$ and, against some strategy $s_2$ for player 2, action $c_1$ is optimal for type $t_1$ and action $d_1$ is optimal for type $r_1$, then $d_1 \geq c_1$. So in equilibrium, if type $t_1$ would choose $c_1$ with positive probability, and type $r_1 > t_1$ would choose $d_1$ with positive probability, then $d_1 \geq c_1$.

(By optimality, $E u_i(c_1,s_2,t_1,\tilde{t}_2) - E u_i(d_1,s_2,t_1,\tilde{t}_2) \geq 0$ and $0 \geq E u_i(c_1,s_2,r_1,\tilde{t}_2) - E u_i(d_1,s_2,r_1,\tilde{t}_2)$, but this would contradict strictly increasing differences if we had $c_1 > d_1$ with $r_1 > t_1$.)
For a cutoff strategy with player 1’s possible types \( t_1 \) is \{0, .1, .2, .3\}, and each has probability \( p_1(t_1) = 1/4 \). Player 2 has no private information. 1’s actions are \{T,B\}, 2’s actions are \{L,R\}.

Given 1’s type \( t_1 \), the payoff matrix is

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T, t_1</td>
<td>0</td>
<td>t_1 -1</td>
</tr>
<tr>
<td>B, t_1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

If 1 chooses \( t_1 = 0 \), then \( T \) is the best response. For 1 choosing \( T \) with probability \( 3/5 \) player 1 chooses \( T \). The remaining \( 3/4 \) probability of \( T \) must come from \( T \) when \( t_1 = .1 \), \( t_1 = .2 \), \( t_1 = .3 \). For that to occur in an increasing cutoff strategy, the cutoff must be at \( .1 \).

For that to occur in an increasing cutoff strategy, the cutoff must be at \( .1 \). To make player 2 willing to randomize, player 1 must use a strategy such that \( P(T) = 3/4 \).

To make type \( t_1 = .1 \) willing to randomize, player 2’s probability of choosing \( L \) must be \( q = (.1+1)/2 = .11 / 20 \).

Notice that these differences increase in \( t_1 \). So higher types \( t_1 \) always find \( T \) relatively more attractive than lower types, and player 1 will use a cutoff strategy. Thus, although player 1 has \( 2^4 = 16 \) pure strategies in this Bayesian game, we only need to consider 1’s cutoff strategies with the following 9 possible supports:

- \( \theta = .3 \) every type would choose \( B \), so 2 thinks the probability of \( T \) is \( 0 \);
- \( \theta < .3 \) \{0, .1, .2\} would choose \( B \), .3 would randomize in some way, so 2 thinks \( P(T) \leq 1/4 \);
- \( \theta > .3 \) every type would choose \( T \), and so 2 thinks \( P(T) = 1 \).

If player 2 uses \( \sigma_2 = q[L]+(1-q)[R] \), then player 1’s optimal cutoff \( \theta \) would have the property:

- \( \theta = .3 \) \{0, .1, .2\} would choose \( B \), .3 would randomize in some way, so 2 thinks \( 0 \leq P(T) \leq 1/4 \);
- \( \theta < .3 \) \{0, .1, .2\} would choose \( B \), .3 would randomize in some way, so 2 thinks \( P(T) \leq 1/4 \);
- \( \theta > .3 \) every type would choose \( T \), so 2 thinks \( P(T) = 1 \).

There is obviously no equilibrium in which player 2 chooses \( L \) for sure or \( R \) for sure. (check!)

Now let \( q \) denote the probability of 2 choosing \( L \). To make 1’s cutoff strategy optimal for him, 2’s randomized strategy \( q[L]+(1-q)[R] \) must make player 1 prefer \( B \) when \( t_1 = 0 \), but must make player 1 prefer \( T \) when \( t_1 = .1 \).

Now suppose instead player 1 has five possible types \{0, .1, .2, .3, .4\}, each with probability \( p_1(t_1) = 1/5 \).

To make player 2 willing to randomize, player 1 must use a strategy such that \( P(T) = 3/4 \).

For that to occur in an increasing cutoff strategy, the cutoff must be at \( \theta = .1 \). So \( t_1 = 0 \) chooses \( B \); and when \( t_1 > .1 \) (which has probability \( 3/5 \)) player 1 chooses \( T \). The remaining \( 3/4 - 3/5 = 0.15 \) probability of \( T \) must come from player 1 choosing \( T \) with probability \( \sigma_1[T] = 0.15/p_1(.1) = 0.15/0.2 = 0.75 \) when \( t_1 = 1 \).

To make type \( t_1 = 0 \) willing to randomize, player 2’s probability of choosing \( L \) must be \( q = (.1+1)/2 = 11/20 \).
Example. Player 1’s type $t_1$ is drawn from a Uniform distribution on the interval from 0 to 1, and payoffs $(u_1, u_2)$ depend on 1’s type as follows, where $\varepsilon$ is a number between 0 and 1 (say $\varepsilon=0.1$):

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$\varepsilon t_2, 0$</td>
<td>$\varepsilon t_1, -1$</td>
</tr>
<tr>
<td>B</td>
<td>1, 0</td>
<td>$-1, 3$</td>
</tr>
</tbody>
</table>

Player 1’s payoffs satisfy increasing differences, so player 1 should use a cutoff strategy, doing T if $t_1>\theta_1$, doing B if $t_1 < \theta_1$, where $\theta_1$ is some number between 0 and 1.

Then player 2 would think that the probability of 1 doing T is $\text{Prob}(t_1 > \theta) = 1-\theta$.

You can easily verify that there is no equilibrium where player 2 is sure to choose either L or R.

For player 2 to be willing to randomize between L and R, both L and R must give her the same expected payoff, so $0 = (-1)(1-\theta_1) + (3)\theta_1$, and so $\theta_1 = 0.25$.

So in equilibrium, player 1 must use the strategy: do T if $t_1 > 0.25$, do B if $t_1 < 0.25$.

Now consider a game with two-sided incomplete information from Myerson (1991) section 3.10.

Suppose player 1’s type $t_1$ is drawn from a Uniform distribution on the interval from 0 to 1, player 2’s type $t_2$ is drawn independently from a Uniform distribution on the interval from 0 to 1, and the payoffs depend on 1’s type as follows, for some given number $\varepsilon$ between 0 and 1:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$\varepsilon t_1, \varepsilon t_2$</td>
<td>$\varepsilon t_1, -1$</td>
</tr>
<tr>
<td>B</td>
<td>1, $\varepsilon t_2$</td>
<td>$-1, 3$</td>
</tr>
</tbody>
</table>

With increasing differences, the action T becomes more attractive to higher types of player 1. Similarly, the action L becomes more attractive to higher types of player 2.

So we should look for an equilibrium where each uses a cutoff strategy of the form

- player 1 does T if $t_1 > \theta_1$, player 1 does B if $t_1 < \theta_1$,
- player 2 does L if $t_2 > \theta_2$, player 2 does R if $t_2 < \theta_2$,

for some pair of cutoffs $\theta_1$ and $\theta_2$.

It is easy to check that neither player’s action can be certain to the other, and so these cutoffs $\theta_1$ and $\theta_2$ must be strictly between 0 and 1.

With $t_1$ Uniform on 0 to 1, the probability of player 1 doing T ($t_1>\theta_1$) is $1-\theta_1$.

Similarly, the probability of player 2 doing L ($t_2>\theta_2$) is $1-\theta_2$.

The cutoff types must be indifferent between the two actions. So we have the equations

\[ \varepsilon \theta_1 = (1)(1-\theta_2) + (-1)\theta_2, \quad \varepsilon \theta_2 = (-1)(1-\theta_1) + (3)\theta_1. \]

The unique solution to these equations is $\theta_1 = (2+\varepsilon)/(8+\varepsilon^2)$, $\theta_2 = (4-\varepsilon)/(8+\varepsilon^2)$.

Unless a player’s type exactly equals the cutoff (which has zero probability), he is not indifferent between his two actions, and he uses the action yielding a higher expected payoff given his type.

As $\varepsilon \rightarrow 0$, these equilibria approach the randomized strategies (.75[T]+.25[B], .5[L]+.5[R]).

These examples show how randomized equilibria can become pure-strategy equilibria in Bayesian games where each player has minor private information that determines his optimal action in equilibrium.

This is called purification of randomized equilibria by Bayesian games (Harsanyi, IJGT, 1973.)
**Introduction to auctions.** Consider $n=2$ bidders in a first-price auction to buy an object for which they have **independent private values** drawn from a Uniform distribution on $[0,M]$. The set of players is $N=\{1,2\}$. Each player $i$’s type set is $T_i = [0,M]$, where the type $t_i$ is $i$’s value of the object being sold. Player $i$’s action is a bid $c_i$ which must be a nonegative number in $\mathbb{R}_+$. Lower bids don’t reduce what $i$ pays and may lose profit opportunities when $c_i < c_{−i}$. The high bidder gets the object, which is worth his type to him, but the winner must pay the amount that he bid. Losers pay nothing. So the utility function for each player $i$ is $u_i(c_1,c_2,t_1,t_2) = t_i − c_i$ if $c_i > c_{−i}$; $u_i(c_1,c_2,t_1,t_2) = 0$ if $c_i < c_{−i}$. A strategy for player $i$ specifies $i$’s bid as some function of $i$’s type, say $c_i = b_i(t_i)$.

Let us try to find a symmetric equilibrium of this game, and let us guess that the equilibrium strategy is linear, of the form $b_i(t_i) = \alpha t_i$, for some $\alpha > 0$. Can this be an equilibrium, for some $\alpha$?

Consider the problem of player 1’s best response, when player 2 uses such a strategy, so $\tilde{c}_2 = \alpha \tilde{t}_2$. Then the first-order condition for an optimal bid $b(s)$ is $0 = \frac{\partial EU_1(c_1|t_1)}{\partial c_1} = (t_1−2c_1)/(\alpha M)$, which implies $c_1 = t_1/2$. So player 1’s best-response strategy is the same linear function as 2’s strategy if $\alpha = 1/2$.

Thus, bidders 1 and 2 each bidding half of his/her type-value ($c_i = b_i(t_i) = t_i/2$) is an equilibrium in this auction. $E(\text{payment from } i | \tilde{t}_i = t_i) = (t_i/2)/2 < t_i/2) = (t_i/2)(t_i/M) = t_i^2/(2M)$.

Now let us change the game to an all-pay-own-bid auction, with the same bidders and types. As before, there are two bidders with independent private values drawn from Uniform $[0,M]$, and the high bidder gets the object. But now each pays his own bid whether he wins or loses. So now $i$’s payoff is: $u_i = t_i$ for some $\alpha > 0$. Can this be an equilibrium, for some $\alpha$?

With increasing differences, we can look more generally for some increasing strategy $b(\bullet)$ such that each player bidding $c_i = b(t_i)$ is a symmetric equilibrium. Let us guess that $b(\bullet)$ is continuous and differentiable. Type 0 must bid $b(0) = 0$. No one should bid more than the highest possible opposing bid, so player 1 should only consider bids in the range of 2’s possible bids, that is, bids $c_1$ such that $c_1 = b(s)$ for some $s \in [0,M]$.

When player 1 knows his type is $t_1$, player 1’s expected payoff from choosing bid $c_1$ would be $EU_1(c_1|t_1) = (t_1−c_1) P(\tilde{c}_2 < c_1) = (t_1−c_1) P(\tilde{c}_2 < c_1) = (t_1−c_1)(c_1/\alpha)/M$, (assuming that $c_i$ is between 0 and $\alpha M$). So the first-order optimality conditions are $0 = \frac{\partial EU_1(c_1|t_1)}{\partial c_1} = (t_1−2c_1)/(\alpha M)$, which implies $c_1 = t_1/2$. Thus, bidders 1 and 2 each bidding half of his/her type-value ($c_i = b(t_i) = t_i/2$) is an equilibrium in this auction. $E(\text{payment from } i | \tilde{t}_i = t_i) = (t_i/2)/2 < t_i/2) = (t_i/2)(t_i/M) = t_i^2/(2M)$.

Finally, let us change the game to a second-price auction. As in the first-price auction, the high bidder wins the object and is the only bidder to pay anything (the loser pays nothing), but now the amount that the high bidder pays is the second-highest bid, submitted by the other bidder. So now $i$’s payoff is: $u_i = t_i−c_i$ if $c_i > c_{−i}$; $u_i = 0$ if $c_i < c_{−i}$, $u_i = (t_i−c_i)/2$ if $c_i = c_{−i}$. For any cumulative distribution $F(c_2) = P(b_2(t_2) \leq c_2)$, $EU_1(c_1|t_1) = \int_0^{c_1} (t_1−c_2)dF(c_2)$ is maximized by $c_1 = t_1$. So in this auction, there is an equilibrium in which each honest bidder honestly bids his value $c_i = b(t_i) = t_i$. In fact bidding $c_i = t_i$ weakly dominates any other strategy. Higher bids only add unprofitable wins with $c_{−i} > t_i$. Lower bids don’t reduce what $i$ pays and may lose profit opportunities when $c_{−i} < t_i$. $E(\text{payment from } i | \tilde{t}_i = t_i) = E(\tilde{t}_i | \tilde{t}_i < t_i) P(\tilde{t}_i < t_i) = (t_i/2)(t_i/M) = t_i^2/(2M)$. $\text{Fact.}$ Given any type $t_i$, $i$’s expected payment is $t_i^2/(2M)$ in these equilibria of different auctions. Given $i$’s type $t_i$, $i$’s expected profit in each eqm is $EU_i(t_i) = t_i P(\tilde{t}_i > t_i) − t_i^2/(2M) = t_i^2/(2M)$. The seller’s expected revenue from the bidders is $2E(t_i^2/(2M)) = 2 \int_0^M (t_i^2/(2M)) \, dt/M = M/3$. 

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A first-price auction with common values (from Myerson 1991, section 3.11).
Consider n=2 bidders in a first-price auction to buy an object which would have the same common value \( \tilde{V} \) to either bidder, if he were to win it. Each bidder has different private information about this common value.

Each bidder i observes an independent signal \( t_i \) drawn from a Uniform distribution on the interval 0 to 1, and the object's common value depends on these signals according to the formula \( \tilde{V} = A_1 t_1 + A_2 t_2 \).

So each player i's payoff \( u_i(c_1,c_2,t_1,t_2) \) depends on the bids \( (c_1,c_2) \) and the types \( (t_1,t_2) \) as follows:

\[
u_i = A_1 t_1 + A_2 t_2 - c_i \quad \text{if} \quad c_i > c_{-i}, \quad u_i = 0 \quad \text{if} \quad c_i < c_{-i}, \quad u_i = (A_1 t_1 + A_2 t_2 - c_i)/2 \quad \text{if} \quad c_i = c_{-i}.
\]

A strategy for player i specifies i's bid as some function of i's type, say \( c_i = b_i(t_i) \).

Let us guess that the equilibrium strategies are linear, of the form \( b_i(t_i) = \alpha t_i \) for some \( \alpha > 0 \).

Nobody would want to bid more than his opponent's highest possible bid, and so we must have \( \alpha_1 = \alpha_2 = \alpha \), so that both bidders have the same range \([0, \alpha]\) of possible bids in equilibrium.

Now suppose that bidder 1, believing that 2 will bid \( b_2(t_2) = \alpha t_2 \), knows his type \( t_1 \) and is thinking of bidding some other \( c \). Then player 1 will win if \( \alpha t_2 < c \), that is, \( \tilde{t}_2 < c/\alpha \).

Then 1’s expected payoff would be

\[
EU_1(c | t_1) = \int_0^{c/\alpha} (A_1 t_1 + A_2 t_2 - c) dt_2 = A_1 t_1 (c/\alpha) + 0.5 A_2 (c/\alpha)^2 - c (c/\alpha) = c A_1 t_1/\alpha - c^2 (1-0.5 A_2/\alpha) / \alpha
\]

First-order conditions for \( c \) to be an optimal bid are then

\[ 0 = \alpha EU_1(c | t_1)/\partial c = (A_1 t_1 + A_2 c/\alpha - 2 c)/\alpha, \quad \text{so} \quad c = A_1 t_1/(2 - A_2 / \alpha).
\]

Thus, for 1’s optimal bid here to be \( c = \alpha t_1 \), we need \( \alpha = A_1/(2-A_2/\alpha) \), and so our equilibrium must have \( \alpha = 0.5(A_1 + A_2) \).

With this \( \alpha \) and \( A_1 > 0 \), we get \( (1-0.5 A_2/\alpha) = A_1/(A_1 + A_2) > 0 \), so \( c = \alpha t_1 \) uniquely maximizes \( EU_1(c | t_1) \).

This symmetric formula for \( \alpha \) also works for player 2, who wants to bid \( \alpha t_2 \) when 1 is expected to bid \( \alpha t_1 \).

So the expected profit for type \( t_1 \) of player 1 when he bids \( b_1(t_1) = \alpha t_1 = 0.5(A_1 + A_2) t_1 \) in this equilibrium is

\[ EU_1(t_1) = [A_1 t_1 + 0.5 A_2 t_1 - 0.5 (A_1 + A_2) t_1] t_1 = 0.5 A_1 t_1^2.
\]

Let \( \tilde{V}_i = A_i \tilde{t}_i \) denote the value that player i has privately seen going into the object here.

So in this model, each \( \tilde{V}_i \) is an independent Uniform random variable on the interval from 0 to \( A_i \).

In terms of his privately observed value \( v_i \), player i’s equilibrium bid is \( 0.5(1 + A_i / A_0) v_i \), and player i’s conditional expected profit given his type is \( 0.5 v_i^2 / A_i \).

**Example:** Suppose \( A_1 = A_2 = 100 \). If \( \tilde{t}_1 = 0.01 \) then 1 bids \( 100 t_1 = 1 \) and gets \( P(\text{win}) = 0.01 \). Notice \( E(\tilde{V} | \tilde{t}_1 = 0.01) = 51 \), but \( EU_1(c_1 | \tilde{t}_1 = 0.01) = (1 + 0.5 c_1 - c_1)(c_1 / 100) < 0 \) if \( c_1 > 2 \).

**Example:** Suppose \( A_1 = \varepsilon \), \( A_2 = 100 - \varepsilon \), for some small \( \varepsilon > 0 \). In equilibrium, each bids \( b_i(t_i) = 50 \tilde{t}_i \).

Both bids are Uniform random variables on \([0,50]\), but 2’s bid is much more highly correlated with \( \tilde{V} \).

In the limit as \( \varepsilon \to 0 \), we get an equilibrium where 2’s bid is \( b_2(t_2) = 50 t_2 = \tilde{V}/2 \), which is perfectly correlated with the value \( \tilde{V} \), but the uninformed bidder 1’s bid is independent of \( \tilde{V} \).

Now let \( \tilde{V}_0 \) be another Uniform \([0,1]\) random variable that is observed by both bidders.

If we increased the common value by the commonly known amount \( A_0 \tilde{t}_0 \), then the equilibrium bid for each type of each bidder would increase by this commonly known amount \( A_0 \tilde{t}_0 \).

That is, if the common value were \( \tilde{V} = A_0 \tilde{t}_0 + A_1 \tilde{t}_1 + A_2 \tilde{t}_2 \), where bidder 1 observes \( \tilde{t}_0 \) and \( \tilde{t}_1 \) and bidder 2 observes \( \tilde{t}_0 \) and \( \tilde{t}_2 \), then the equilibrium bidding strategies would be

\[ b_1(t_0, \tilde{t}_1) = A_0 \tilde{t}_0 + 0.5 (A_1 + A_2) \tilde{t}_1 \text{ and } b_2(t_0, \tilde{t}_2) = A_0 \tilde{t}_0 + 0.5 (A_1 + A_2) \tilde{t}_2.
\]

**Example:** \( A_0 = \varepsilon = A_1 \), \( A_2 = (100-\varepsilon) \). Then equilibrium strategies are \( b_i(t_0, \tilde{t}_i) = \varepsilon \tilde{t}_0 + 50 \tilde{t}_i \) for \( i = 1,2 \).

Notice that 1’s two signals are equally minor in their impact on the value of the object, but 1’s bid depends much more on his private information \( \tilde{t}_1 \) than his shared information \( \tilde{t}_0 \).
Move probabilities, belief probabilities and sequential equilibria

Suppose that we are given some extensive game with imperfect information. Given any randomized strategy for any player $i$, at any information set $t_i$ of player $i$ that could occur with positive probability when he plays this strategy, we can compute a probability distribution over the set of possible actions $\{d_i\}$ for player $i$ at this information set. These probabilities $\sigma_i(d_i|t_i)$ are called move probabilities (or action probabilities). That is, the move-probability for any move $d_i$ at any information state $t_i$ of any player $i$ denotes the probability that player $i$ will choose move $d_i$ if information set $t_i$ occurs in the game.

A behavioral strategy $\sigma_i$ for player $i$ is a vector that specifies a move-probability distribution for each of player $i$’s information sets.

A behavioral-strategy profile $\sigma$ is a vector that specifies a behavioral strategy $\sigma_i$ for each player $i$, and so it must specify an move probability $\sigma_i(d_i|t_i)$ for every possible move $d_i$ at every possible information set $t_i$ of every player $i$ in the game.

Given $\sigma$, a profile of behavioral or randomized strategies for all players in the game, the prior probability $P(x|\sigma)$ of any node $x$ in the tree is the multiplicative product of all chance-probabilities and move-probabilities on the path that leads to this node from the starting node. (Here the chance probabilities on all branches that follow chance nodes are part of the given structure of the extensive game. We assume that these chance probabilities are all positive.)

A full-support behavioral strategy profile assigns strictly positive probability ($\sigma_i(d_i|t_i)>0$) to every possible move $d_i$ at every information set $t_i$ of every player $i$, so that every node $x$ in the tree has positive probability. When player $i$ moves at his information set $t_i$, the belief probability that player $i$ should assign to any node $x$ in this information set should be, by Bayes’s formula,

$$\mu_i(x|t_i) = \frac{P(x|\sigma)}{\sum_{y \in t_i} P(y|\sigma)}.$$ 

That is, the belief probability $\mu_i(x|t_i)$ should equal the prior probability of $x$ divided by the sum of prior probabilities of all nodes in the information set $t_i$, whenever this formula is well-defined (not 0/0).

A belief system $\mu$ is a vector that specifies a belief-probability distribution $\mu_i(x|t_i)$ over the nodes of each information set $t_i$ of each player $i$ in the game. Bayes’s formula yields a unique belief system for any full-support behavioral strategy profile. But for strategy profiles that do not have full support, Bayes’s formula may leave some belief probabilities undefined, at any information set where all nodes have zero prior probabilities. A beliefs system $\mu$ is consistent with a behavioral strategy profile $\sigma$ iff there exists a sequence of full-support behavioral strategies $\sigma^k$ that converge to $\sigma$ (all $\sigma^k(d_i|t_i) \to \sigma_i(d_i|t_i)$) and yield Bayesian beliefs $\mu^k$ that converge to $\mu$ as $k \to \infty$ (all $\mu^k(x|t_i) \to \mu_i(x|t_i)$).

A behavioral-strategy profile $\sigma$ is sequentially rational given a beliefs system $\mu$ iff, at each information set $t_i$ of each player $i$, $\sigma_i(x|t_i)$ assigns positive move-probabilities only to moves that maximize $i$’s expected payoff at $t_i$, given $i$’s beliefs $\mu_i(x|t_i)$ about the current node in the information set $t_i$ and given what the behavioral-strategy profile $\sigma$ specifies about players’ behavior after this information set.

A sequential equilibrium is a pair $(\sigma, \mu)$, where $\sigma$ is a behavioral strategy profile and $\mu$ is a belief system, such that $\sigma$ is sequentially rational given the beliefs system $\mu$, and the beliefs system $\mu$ is consistent with the behavioral-strategy profile $\sigma$.

A game has perfect information if every information set consists of just one node. A game with perfect information can have only one possible beliefs system, which trivially assigns belief probability 1 to every decision node. For a game with perfect information, a behavioral strategy profile $\sigma$ is a subgame-perfect equilibrium if it would form a sequential equilibrium together with this (trivial) beliefs system $\mu$. 


The Holdup Problem  Player 1 can invest to improve an asset which he may later sell player 2. First player 1 chooses an amount $e \geq 0$ to spend on improving the asset. With this investment, the asset will be worth $v_1(e) = e^{0.5}$ to player 1, but it will be worth $v_2(e) = 2e^{0.5}$ to player 2. We consider two different versions of this game, which differ in how they bargain over the price.

Buyer-offer game  First player 1 chooses the amount $e \geq 0$ to spend on improving the asset. Player 2 observes this investment $e$. Then player 2 chooses a price $p \geq 0$ at which she offers to buy the asset from player 1. Player 1 observes this offer, and then can choose to accept or reject it. Final payoffs are: $u_1(e, p, \text{accept}) = p - e, \quad u_2(e, p, \text{accept}) = v_2(e) - p, \quad u_1(e, p, \text{reject}) = v_1(e) - e, \quad u_2(e, p, \text{reject}) = 0$.

There is a unique subgame-perfect equilibrium. At the last stage, player 1 accepts if $p > v_1(e)$ and rejects if $p < v_1(e)$. So player 2's optimal offer, given $e$, must be to offer $p = v_1(e)$, which player 1 must accept. (Note: Player 1 is actually indifferent between accepting and rejecting, but there would be no optimal offer for 2 if player 1 rejected in this case of indifference!)

So player 1 knows that his payoff from $e$ will be $v_1(e) - e = e^{0.5} - e$, which is maximized by $e = 0.25$. So the equilibrium outcome is: 1 chooses $e = 0.25, \quad$ 2 offers $p = 0.25^{0.5} = 0.5$, and payoffs are: $u_1 = 0.5 - 0.25 = 0.25, \quad u_2 = 2 \times 0.25^{0.5} - 0.5 = 1 - 0.5 = 0.5$.

Seller-offer game. First player 1 chooses his investment $e \geq 0$. Then player 1 chooses the price $p \geq 0$ at which he offers to sell the asset. Player 2 observes $e$ and $p$, and then can choose to accept or reject 1's offer. Payoffs are still:

$u_1(e, p, \text{accept}) = p - e, \quad u_2(e, p, \text{accept}) = v_2(e) - p, \quad u_1(e, p, \text{reject}) = v_1(e) - e, \quad u_2(e, p, \text{reject}) = 0$.

In the unique subgame-perfect equilibrium, player 2 accepts if $p \leq v_2(e)$ but rejects if $p > v_2(e)$, so player 1 offers $p = v_2(e)$. So player 1 chooses $e = 1$ to maximize $2e^{0.5} - e$.

So the equilibrium outcome is: 1 chooses $e = 1$ and offers $p = 2 \times 1^{0.5} = 2$, and payoffs are: $u_1 = 2 - 1 = 1, \quad u_2 = 2 \times 1^{0.5} - 2 = 0$.

This seller-offer game also has many other Nash equilibria that are not subgame perfect.

Notice that the equilibrium sum of payoffs $u_1 + u_2$ is greater in the seller-offer game. That is, for an efficient outcome, the person who made the first-period investment should have more control in the process of bargaining over the price. If they were about to play the buyer-offer game, the buyer would be willing to sell her right to set the price for any payment more than 0.5, and the seller would be willing to pay up to 0.75 for the right to set the price.

Both of these games have many other Nash equilibria that are not subgame-perfect. Consider any $(\hat{e}, \hat{p})$ such that $v_2(\hat{e}) \geq \hat{p} \geq \hat{e} + \max_e (v_1(e) - e) = \hat{e} + 0.25$ (such as $\hat{e} = 1, \hat{p} = 1.625$), so that each does better than he could alone. With either player offering the price, there is a Nash equilibrium in which 1 invests this $\hat{e}$, and then this price $\hat{p}$ is offered and accepted, but rejection would follow any other investment $e \neq \hat{e}$ or any other price-offer $p \neq \hat{p}$. These Nash equilibria violate sequential rationality, however, as threats to reject prices between $v_1(e)$ and $v_2(e)$ would not be credible.
**Introduction to repeated games** Players 1 and 2 will meet on \(\tau+1\) days, numbered 0, 1, 2, \ldots, \(\tau\).

On each day, each player \(i\) must choose to be generous (\(g_i\)) or selfish (\(f_i\)).

On each day \(k\), they get payoffs \((u_{1k}, u_{2k})\) that depend on their actions \((c_{1k}, c_{2k})\) as follows:

<table>
<thead>
<tr>
<th>Player 1: (g_1)</th>
<th>Player 2: (f_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g_1) 3, 3</td>
<td>(f_2) 0, 5</td>
</tr>
<tr>
<td>(f_1) 5, 0</td>
<td>(f_2) 2, 2</td>
</tr>
</tbody>
</table>

except on the last day \(\tau\) their payoffs will be:

<table>
<thead>
<tr>
<th>Player 1: (g_1)</th>
<th>Player 2: (f_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g_1) 5, 5</td>
<td>(f_2) 0, 4</td>
</tr>
<tr>
<td>(f_1) 0, 4</td>
<td>(f_2) 2, 2</td>
</tr>
</tbody>
</table>

On each day, each player knows what both players did on all previous days.

Each player wants to maximize the expected discounted sum of his payoffs \(V_i = \sum_{k=0}^{\tau} \delta^k u_{ik}\) for some given discount factor \(\delta\) between 0 and 1.

If the first payoff matrix (the prisoners' dilemma) were played once, \((f_1, f_2)\) would be the unique equilibrium, yielding the Pareto-dominated payoff allocation \((2,2)\).

But in multi-period games, opportunities to respond later can enlarge the set of equilibria.

Consider the strategy for each player \(i\) to choose \(g_i\) until \(f_i\) or \(f_2\) is chosen, but thereafter choose \(f_i\).

We can show that it is an equilibrium here for both players to choose this strategy, if \(\delta > 2/3\).

Consider first the case of \(\tau = 1\), where the prisoners' dilemma is played once, followed by one play of the trust game at the end. Under the strategies described here, on the last day, they will play the good \((g_1, g_2)\) equilibrium of the "trust game" if both were previously generous, but they will play the bad \((f_1, f_2)\) equilibrium if either player was previously selfish.

So the overall payoffs will depend on their first-day choices as follows:

<table>
<thead>
<tr>
<th>Player 1: (g_1)</th>
<th>Player 2: (f_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g_1) 3+5\delta, 3+5\delta</td>
<td>(f_2) 0+5\delta, 5+5\delta</td>
</tr>
<tr>
<td>(f_1) 5+5\delta, 0+5\delta</td>
<td>(f_2) 2+5\delta, 2+5\delta</td>
</tr>
</tbody>
</table>

Then \((g_1, g_2)\) is an equilibrium at the first day if \(3+5\delta > 5+2\delta\), that is, if \(\delta > 2/3\).

A similar calculation can be made for any number \(\tau \geq 1\) of repetitions of the prisoners' dilemma.

The discounted value of payoffs from \((f_1, f_2)\)-always would be \(F(\tau) = (1-\delta^\tau)/(1-\delta) = 2+\delta F(\tau-1)\).

The discounted value of payoffs from \((g_1, g_2)\)-always would be \(G(\tau) = 3(1-\delta^\tau)/(1-\delta)+5\delta^\tau = 3+5\delta G(\tau-1)\).

(We use \(w+w\delta+w\delta^2+\cdots+w\delta^{\tau-1} = w/(1-\delta^\tau)/(1-\delta)\).)

**Lemma:** If \(1>\delta \geq 2/3\) then \(G(\tau)-F(\tau) \geq 3\) for all \(\tau\). (Proof by induction: \(G(0)-F(0) = 5-2 = 3\), and then for any \(\tau \geq 1\) we get inductively \(G(\tau)-F(\tau) = 3-2 + \delta(G(\tau-1)-F(\tau-1)) \geq 1 + (2/3)(3) = 3\).)

Now assuming that the strategies described above will be played after the first stage, the players' overall payoffs will depend on their first-day choices as follows:

<table>
<thead>
<tr>
<th>Player 1: (g_1)</th>
<th>Player 2: (f_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g_1) 3+5\delta/(1-\delta), 3+5\delta/(1-\delta)</td>
<td>(f_2) 0+5\delta/(1-\delta), 5+5\delta/(1-\delta)</td>
</tr>
<tr>
<td>(f_1) 5+5\delta/(1-\delta), 0+5\delta/(1-\delta)</td>
<td>(f_2) 2+5\delta/(1-\delta), 2+5\delta/(1-\delta)</td>
</tr>
</tbody>
</table>

With \(1>\delta \geq 2/3\), for any \(\tau\), it is an equilibrium for both to start doing \(g_1\), as these strategies specify, because \(3+5\delta G(\tau-1) \geq 5+2\delta F(\tau-1)\). (Proof: \(3+5\delta G(\tau-1) - (5+5\delta F(\tau-1)) = -2+\delta(G(\tau-1)-F(\tau-1)) \geq -2+(2/3)(3) = 0\).)

As \(\tau \to \infty\), overall payoffs in this good equilibrium depend on first-day actions as follows:

<table>
<thead>
<tr>
<th>Player 1: (g_1)</th>
<th>Player 2: (f_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g_1) 3+3\delta/(1-\delta), 3+3\delta/(1-\delta)</td>
<td>(f_2) 0+2\delta/(1-\delta), 5+2\delta/(1-\delta)</td>
</tr>
<tr>
<td>(f_1) 5+2\delta/(1-\delta), 0+2\delta/(1-\delta)</td>
<td>(f_2) 2+2\delta/(1-\delta), 2+2\delta/(1-\delta)</td>
</tr>
</tbody>
</table>

The equilibrium condition \(3+3\delta/(1-\delta) \geq 5+2\delta/(1-\delta)\) is satisfied when \(1>\delta \geq 2/3\).
The War-of-Attrition game. There are two players, numbered 1 and 2, who can meet on day 0, day 1, ..., through day $\tau$, to try to get a valuable prize that is worth $V$.

On each day, if the game has not yet ended, each player can choose to fight or quit. The game ends as soon as somebody quits, or it ends after day $\tau$ if nobody quits.

On each day when both choose to fight, they both lose $1$. On any day when one player fights and the other player quits, the fighter gets the prize worth $V$ (and the game ends). If they both quit on the same day, or if they both fight on all days (0 through $\tau$), then nobody gets the prize.

The normal-form strategy for each player $i$ can be described by a number $c_i$ chosen from the set $\{0,1,\ldots,\tau+1\}$, where $c_i$ represents the day when player $i$ would quit, if the other player does not quit first, except that $c_i=\tau+1$ represents the strategy "never quit". So the payoffs functions are

$u_1(c_1,c_2) = V-c_2$ if $c_1 > c_2$, but $u_1(c_1,c_2) = -c_1$ if $c_1 \leq c_2$;

$u_2(c_1,c_2) = V-c_1$ if $c_2 > c_1$, but $u_2(c_1,c_2) = -c_2$ if $c_2 \leq c_1$.

Suppose player 2 chooses $\tilde{c}_2$ randomly, according to a probability distribution $\sigma_2(t) = P(\tilde{c}_2=t)$.

Player 1's expected payoff from choosing $c_1=d$ is $E[u_1(d,\tilde{c}_2) = \sum_{t<d} (V-t)\sigma_2(t) + \sum_{t=d} (-d)\sigma_2(t)$.

So $E[u_1(0,\tilde{c}_2) = 0$, and $E[u_1(d+1,\tilde{c}_2) = E[u_1(d,\tilde{c}_2) = V\sigma_2(d) - \sum_{t=d} \sigma_2(t) = (V+1)\sigma_2(d) - \sum_{t=d} \sigma_2(t)$.

(After $d$ days, 1's willingness to fight one more day would earn $V$ if $\tilde{c}_2=d$, or lose $1$ if $\tilde{c}_2>d$.)

There is a symmetric full-support randomized equilibrium in which each player $i$ chooses $\tilde{c}_i$ randomly according to a probability distribution $\sigma_i = \sigma_2$. We can find this $\sigma_2$ by solving the equations

$0 = E[u_1(d+1,\tilde{c}_2) - E[u_1(d,\tilde{c}_2) = (V+1)\sigma_2(d) - \sum_{t=d} \sigma_2(t)$ for all $d$ in $\{0,1,\ldots,\tau\}$.

First, using $\sum_{t=0} \sigma_2(t) = 1$, we get $\sigma_2(0) = 1/(V+1)$.

Then $\sigma_2(1) = \left[\sum_{t=1} \sigma_2(t)/(V+1) = [1-\sigma_2(0)]/(V+1) = [1 - 1/(V+1)]/(V+1)\right.$

Then for all $d=1,\ldots,\tau$, we can recursively compute $\sigma_2(d) = \left[\sum_{t=d} \sigma_2(t)/(V+1) = [1-\sum_{t=d} \sigma_2(t)]/(V+1) = [1 - 1/(V+1)]^d/(V+1)\right.$.

At the end, we have $\sigma_2(\tau+1) = 1 - \sum_{t=\tau+1} \sigma_2(t)$ (which goes to 0 as $\tau \to \infty$).

On each day $d\leq\tau$, we have $\sigma_2(d)/\sum_{t=d} \sigma_2(t) = 1/(V+1)$. This ratio is the conditional probability of player 2 quitting on day $d$, given that she has not quit earlier.

So this mixed strategy corresponds to a behavioral strategy in which, on any given day, if nobody has quit earlier, then the probability of player $i$ quitting today is $q = 1/(V+1)$.

This conditional probability $q$ satisfies the equation $qV+(1-q)(-1) = 0$, which makes the other player just indifferent between quitting immediately and fighting one more day.

In this symmetric randomized equilibrium, each player is willing to quit on day 0, and so each player's expected payoff is $0 = E[u_1] = E[u_2]$.

There is also a nonsymmetric equilibrium in which player 1 is always expected to fight and player 2 is expected to quit immediately, so that $c_1=\tau+1$, $c_2=0$, $u_1=V$, and $u_2=0$.

There is also a nonsymmetric equilibrium in which player 2 is always expected to fight and player 1 is expected to quit immediately, so that $c_1=0$, $c_2=\tau+1$, $u_1=0$, and $u_2=V$.

These nonsymmetric equilibria can be interpreted as a model of property rights.
Infinitely Repeated games

Infinitely repeated games can be used as simple models of long-term relationships. The game will be played at an infinite sequence of time periods numbered 1,2,3,...

Suppose that the set of players is \{1,2\}. In each period \(k\), each player \(i\) must choose an action \(c_{ik}\) in some set \(C_i\). In period \(k\), each player \(i\)'s payoff \(u_{ik}\) will depend on both players' actions according to some utility function \(u_i: C_1 \times C_2 \rightarrow \mathbb{R}\); that is, \(u_{ik} = u_i(c_{1k}, c_{2k})\).

We assume here that the actions at each period are publicly observable, and so each player's action in each period may depend on the history of actions by both players at all past periods.

Given any discount factor \(\delta\) such that \(0 \leq \delta < 1\), the \(\delta\)-discounted sum of player \(i\)'s payoffs is:

\[
V(u_{i1},u_{i2},u_{i3},...) = u_{i1} + \delta u_{i2} + \delta^2 u_{i3} + \ldots + \delta^{k-1} u_{ik} + \ldots
\]

For a constant payoff \(x\) each period, the \(\delta\)-discounted sum would be \(x / (1 - \delta)\).

The objective of each player \(i\) in the repeated game is to maximize the expected discounted sum of his payoffs, with respect to some discount factor \(\delta\), where \(0 \leq \delta < 1\).

Fact. (Recursion formula) \(V(u_{i1},u_{i2},u_{i3},...) = u_{i1} + \delta V(u_{i2},u_{i3},u_{i4},...)\).

We may describe equilibria of repeated games in terms of various social states. At each period of the game, the players will understand that their current relationship is described by one of these social states, and their expectations about each others' behavior will be determined by this state.

This state may be called the state of play in the game at this period.

(These social states are a characteristic of the equilibrium, not of the game, as they describe the different kinds of expectations that the players may have about each others' future behavior.)

To describe an equilibrium or scenario in terms of social states, we must specify the following:

1. Social states: We must list the set of social states in this equilibrium. (States may be denoted by numbers or may be named for the kinds of interpersonal relationships that they represent.)
2. State-dependent strategies: For each social state \(\theta\), we must specify a profile of (possibly randomized) actions \((\hat{s}_1(\theta), \hat{s}_2(\theta))\) describing the predicted behavior of the players in any period when this \(\theta\) is the state of play.
3. Transitions: For each social state \(\theta\), we must specify the profiles of the players' actions that would cause the state of play in the next period to change from this state to another state. We may let \(\Theta(a_1,a_2;\theta)\) denote the state of play in the next period after a period when the state of play was \(\theta\) and the players chose actions \((a_1,a_2)\) (possibly deviating from the prediction \((\hat{s}_1(\theta), \hat{s}_2(\theta))\)).
4. Initial state: We must specify which social state is initial state of play in the first period of the game.

Here we will generally let state "0" denote this initial state.

Given any scenario as in (1)-(3) above, and given any discount factor \(\delta\), let \(V_i(\theta)\) denote the expected \(\delta\)-discounted sum of player \(i\)'s payoffs in this scenario when (ignoring (4)) the state of play begins in state \(\theta\).

Given \(\delta < 1\), these numbers \(V_i(\theta)\) can be computed (with algebra) from the equations:

\[
V_i(\theta) = E[u_i(\hat{s}_1(\theta), \hat{s}_2(\theta))] + \delta \cdot V_i(\Theta(\hat{s}_1(\theta), \hat{s}_2(\theta);\theta))
\]

Fact. A scenario as in (1)-(3) above is a subgame-perfect equilibrium if, for every player \(i\) and every social state \(\theta\), player \(i\) could not expect to gain by unilaterally deviating from the prediction \(\hat{s}_i(\theta)\) in a period when the state of play is \(\theta\). That is, we have an equilibrium if, for every state \(\theta\),

\[
V_1(\theta) \geq E[u_i(a_1, \hat{s}_2(\theta))] + \delta \cdot V_i(\Theta(c_1, \hat{s}_2(\theta);\theta)), \quad \text{for all } c_1 \in C_1,
\]

\[
V_2(\theta) \geq E[u_i(\hat{s}_1(\theta), c_2)] + \delta \cdot V_i(\Theta(\hat{s}_1(\theta), c_2;\theta)), \quad \text{for all } c_2 \in C_2.
\]

(This is the one-deviation condition for a subgame-perfect equilibrium in a repeated game: if nobody could ever gain in any state by a one-round deviation, then longer strategic deviations are also not profitable.)

(The Folk Theorem of Repeated Games says that, when \(\delta\) is close to 1, subgame-perfect equilibria can be constructed to achieve, in some state, almost any feasible payoff allocation which gives each player more than the maxmin security level that he could guarantee himself against punitive actions by other players.)
Example 1. Consider a repeated game where, in each period, the players play the following "Prisoners' dilemma" game in which each must decide whether to "cooperate" or "defect".

<table>
<thead>
<tr>
<th></th>
<th>c2</th>
<th>d2</th>
</tr>
</thead>
<tbody>
<tr>
<td>c1</td>
<td>5, 5</td>
<td>0, 6</td>
</tr>
<tr>
<td>d1</td>
<td>6, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

Each player wants to maximize his or her $\delta$-discounted sum of payoffs, for some $0 \leq \delta < 1$.

We first consider a version of the "grim trigger" equilibrium:

The states are $\{0, 1\}$. (State 0 represents "trust" or "friendship"; state 1 represents "distrust".)

The predicted behavior in state 0 is $(c_1,c_2)$. The predicted behavior in state 1 is $(d_1,d_2)$.

In any period when the current state of play is 0, if the players' action profile is $(c_1,d_2)$ or $(d_1,c_2)$ then the state of play next period will switch to state 1, otherwise it will remain state 0.

When the state of play is 1, the future state of play always remains state 1.

The expected discounted values for the players in the states satisfy the equations:

$$V_1(0) = u_1(c_1,c_2) + \delta V_1(0), \quad V_1(1) = u_1(d_1,d_2) + \delta V_1(1),$$

$$V_2(0) = u_2(c_1,c_2) + \delta V_2(0), \quad V_2(1) = u_2(d_1,d_2) + \delta V_2(1).$$

So $V_1(0) = 5 + \delta V_1(0)$, $V_1(1) = 1 + \delta V_1(1)$, and so $V_1(0) = 5/(1 - \delta)$, $V_1(1) = 1/(1 - \delta)$.

Similarly, $V_2(0) = 5/(1 - \delta)$, $V_2(1) = 1/(1 - \delta)$.

For this scenario to be an equilibrium, we need:

$$V_1(0) \geq u_1(c_1,c_2) + \delta V_1(1), \quad V_1(1) \geq u_1(d_1,d_2) + \delta V_1(1),$$

$$V_2(0) \geq u_2(c_1,c_2) + \delta V_2(1), \quad V_2(1) \geq u_2(d_1,d_2) + \delta V_2(1).$$

That is, we need: $5/(1 - \delta) \geq 6 + \delta 1/(1 - \delta)$ and $1/(1 - \delta) \geq 0 + \delta 1/(1 - \delta)$,

or equivalently (with $\delta < 1$), $5 \geq 6(1 - \delta) + \delta 1$ and $1 \geq 0(1 - \delta) + \delta 1$, which are satisfied when $1 > \delta \geq 1/5$.

Now let's consider another (more forgiving) equilibrium:

The states are $\{0, 1, 2\}$. (State 0 is "friendship"; state 1 is "punishing 1"; state 2 is "punishing 2".)

The predicted behavior in state 0 is $(c_1,c_2)$. The predicted behavior in state 1 is $(c_1,d_2)$.

The predicted behavior in state 2 is $(d_1,c_2)$.

When the state of play is 0, if the players choose $(d_1,c_2)$ then the next state of play will be 1, if the players choose $(c_1,d_2)$ then the state of play next period will be 2, and otherwise the state will remain 0.

When the state of play is 1, if the players choose $(c_1,d_2)$ then the next state of play will be 0, otherwise it will remain 1. When the state of play is 2, if the players choose $(d_1,c_2)$ then the next state of play will be 0, otherwise it will remain 2.

The expected discounted values $V_i(\theta)$ for player 1 in each state $\theta$ satisfy the equations:

$$V_1(0) = u_1(c_1,c_2) + \delta V_1(0), \quad V_1(1) = u_1(c_1,d_2) + \delta V_1(0), \quad V_1(2) = u_1(d_1,c_2) + \delta V_1(0).$$

Thus $V_1(0) = 5 + \delta V_1(0)$, and so $V_1(0) = 5/(1 - \delta)$;

$$V_1(1) = 0 + \delta 5/(1 - \delta), \quad \text{and so} \quad V_1(1) = 5\delta/(1 - \delta);$$

and $V_1(2) = 6 + \delta 5/(1 - \delta) = (6 - \delta)/(1 - \delta)$.

Similarly, $V_2(0) = 5/(1 - \delta)$, $V_2(1) = (6 - \delta)/(1 - \delta)$, $V_2(2) = 5\delta/(1 - \delta)$.

To have a subgame-perfect equilibrium, we need:

$$V_1(0) \geq u_1(d_1,c_2) + \delta V_1(1), \quad V_1(1) \geq u_1(d_1,d_2) + \delta V_1(1), \quad V_1(2) \geq u_1(c_1,c_2) + \delta V_1(2),$$

and similar conditions for player 2. These inequalities (for both players) become:

$$5/(1 - \delta) \geq 6 + \delta 5\delta/(1 - \delta), \quad 5\delta/(1 - \delta) \geq 1 + \delta 5\delta/(1 - \delta), \quad (6 - \delta)/(1 - \delta) \geq 5 + \delta(6 - \delta)/(1 - \delta).$$

With $\delta < 1$, these inequalities are equivalent to: $5(1 - \delta^2)/(1 - \delta) \geq 6$, $5\delta \geq 1$, $6 - \delta \geq 5$.

With $(1 - \delta^2) = (1 - \delta)(1 + \delta)$ (and $\delta < 1$), the first inequality further simplifies to $5(1 + \delta) \geq 6$, and so these conditions for a subgame-perfect equilibrium are all satisfied when $1 > \delta \geq 1/5$.
Example 2. Consider a repeated game where players 1 and 2 play the game below infinitely often. In each round, each player i must decide whether to fight \((f_i)\) or not \((n_i)\).

\[
\begin{array}{c|cc}
  & f_2 & n_2 \\
\hline
f_1 & -1, -1 & 9, 0 \\
n_1 & 0, 9 & 0, 0 \\
\end{array}
\]

Each player i wants to maximize his or her \(\delta_i\)-discounted sum of payoffs, for some \(0 \leq \delta_i < 1\).

A subgame-perfect equilibrium:

States: there are three states, numbered 0,1,2. The initial state in period 1 is state 0. (State 1 may be interpreted as "1 has ownership", state 2 may be interpreted as "2 has ownership" and state 0 may be interpreted as "fighting for ownership" or war of attrition.)

Strategies: Let \(s_i(\theta)\) denote the move that player i would choose in state \(\theta\).

Player 1’s strategy is \(s_1(1) = f_1, s_1(2) = n_1, s_1(0) = q_1[f_1] + (1- q_1)[n_1]\) for some \(0 < q_1 < 1\).

Player 2’s strategy is \(s_2(1) = n_1, s_2(2) = f_1, s_2(0) = q_2[f_2] + (1- q_2)[n_2]\) for some \(0 < q_2 < 1\).

We will need to find what \((q_1, q_2)\) makes this a subgame-perfect equilibrium.

Transitions: When the current state is state 0, the state next period would be:

- state 1 if \((f_1, n_2)\) is played now,
- state 2 if \((n_1, f_2)\) is played now, and
- state 0 if \((f_1, f_2)\) or \((n_1, n_2)\) is played now.

Values: Let \(V_i(\theta)\) denote the expected discounted sum of payoffs for player i in state \(\theta\).

The recursion equations for states 1 and 2 are

\[
\begin{align*}
V_1(1) &= U_1(f_1, n_2) + \delta_1 V_1(1), & \text{for } i=1,2, \text{ and so } V_1(1) = 9/(1-\delta_1) \text{ and } V_2(1) = 0/(1-\delta_2) = 0; \\
V_1(2) &= U_1(n_1, f_2) + \delta_1 V_1(2), & \text{for } i=1,2, \text{ and so } V_1(2) = 0 \text{ and } V_2(2) = 9/(1-\delta_2). \\
\end{align*}
\]

To check the equilibrium condition in state 1, notice that

\[
9/(1-\delta_1) = V_1(1) \geq U_1(n_1, n_2) + \delta_1 V_1(1) = 0 + \delta_1 9/(1-\delta_1) = \delta_1 9/(1-\delta_1), \text{ which is true when } 0 \leq \delta_1 < 1;
\]

\[
0 = V_2(1) \geq U_2(f_1, f_2) + \delta_2 V_2(1) = -1 + \delta_2 0 = -1, \text{ which is true when } 0 \leq \delta_2 < 1.
\]

The equilibrium conditions in state 2 are similarly

\[
9/(1-\delta_2) = V_2(2) \geq U_2(n_1, n_2) + \delta_2 V_2(2) = 0 + \delta_2 9/(1-\delta_2) = \delta_2 9/(1-\delta_2), \text{ which is true when } 0 \leq \delta_2 < 1;
\]

\[
0 = V_1(2) \geq U_2(f_1, f_2) + \delta_1 V_1(1) = -1 + \delta_1 0 = -1, \text{ which is true when } 0 \leq \delta_1 < 1.
\]

In state 0, for player 1 to be willing to randomize between \(f_1\) and \(n_1\), he must expect the same discounted value \(V_1(0)\) from choosing \(f_1\) or \(n_1\) this period, and so we must have

\[
\begin{align*}
V_1(0) &= q_2(U_1(f_1, f_2) + \delta_1 V_1(0)) + (1- q_2)(U_1(f_1, n_2)) + \delta_1 V_1(1), & \text{and} \\
V_1(0) &= q_2(U_1(n_1, f_2) + \delta_1 V_1(2)) + (1- q_2)(U_1(n_1, n_2)) + \delta_1 V_1(0). \\
\end{align*}
\]

The latter is \(V_1(0) = q_2(-1) + q_2 \delta_1 V_1(0) + (1- q_2)9 + (1- q_2)\delta_1 V_1(1)\), implying \(V_1(0) = 0\).

Then \(V_1(0) = q_2(-1) + q_2 \delta_1 V_1(0) + (1- q_2)9 + (1- q_2)\delta_1 9/(1-\delta_1)\) implies \(q_2 = 9/(10-\delta_1)\).

For player 2 to be willing to randomize between \(f_2\) and \(n_2\) in state 0, we must have

\[
\begin{align*}
V_2(0) &= q_1(U_2(f_1, f_2) + \delta_2 V_2(0)) + (1- q_1)(U_2(f_1, n_2)) + \delta_2 V_2(2), & \text{and} \\
V_2(0) &= q_1(U_2(f_1, n_2) + \delta_2 V_2(1)) + (1- q_1)(U_2(n_1, n_2)) + \delta_2 V_2(0); \\
\end{align*}
\]

and these equations similarly imply that \(V_2(0) = 0\) and \(q_1 = 9/(10-\delta_2)\).

When \(\delta_1\) and \(\delta_2\) are close to 1 (very patient players), each player’s probability of continuing to fight each round in state 0 is just close enough to 1 that the expected costs of conflict exactly cancel out the expected benefits of winning the prize for the other player.
Facts about Uniform distributions. Suppose that $\tilde{X}$ is a random variable drawn from a Uniform distribution on the interval from $A$ to $B$, given $A < B$. Then $E(\tilde{X}) = (A+B)/2$, and $\forall \theta \in [A,B]$: $F(\theta) = P(\tilde{X} \leq \theta) = P(\tilde{X} < \theta) = (\theta - A)/(B - A)$, $f(\theta) = F'(\theta) = 1/(B - A)$, $1 - F(\theta) = (B - \theta)/(B - A)$, $E(\tilde{X} | \tilde{X} \leq \theta) = E(\tilde{X} | \tilde{X} < \theta) = (A + \theta)/2$, and $E(\tilde{X} | \tilde{X} \geq \theta) = E(\tilde{X} | \tilde{X} > \theta) = (\theta + B)/2$.

Fact: Suppose that $\tilde{X}$ is Uniform on $[t-\epsilon, t+\epsilon]$ and, conditional on $\tilde{X}$, $\tilde{S}$ is Uniform on $[\tilde{X} - \epsilon, \tilde{X} + \epsilon]$. $\tilde{S}$ has a continuous (triangular) distribution on the interval $[t-2\epsilon, t+2\epsilon]$, symmetric around $t$, and so $P(\tilde{S} > t + 2\epsilon) = 0 = P(\tilde{S} < t - 2\epsilon)$. For any $\delta \in [0, 2\epsilon]$, we get:

$P(\tilde{S} > t + \delta) = 0.5(1 - 0.5\delta/\epsilon)^2$,

$E(\tilde{X} | \tilde{S} > t + \delta) = t + (\epsilon + \delta)/3$,

$P(\tilde{S} < t + \delta) = 1 - 0.5(1 - 0.5\delta/\epsilon)^2$, and

$E(\tilde{X} | \tilde{S} < t + \delta) = [t - 0.5(1 - 0.5\delta/\epsilon)^2(t + (\epsilon + \delta)/3)]/[1 - 0.5(1 - 0.5\delta/\epsilon)^2]$

In the case of $\delta = 0$, these formulas simplify to:

$P(\tilde{S} > t) = 1/2 = P(\tilde{S} < t)$, $E(\tilde{X} | \tilde{S} > t) = t + \epsilon/3$, $E(\tilde{X} | \tilde{S} < t) = t - \epsilon/3$.

Proof: Notice first that we cannot get $\tilde{S} > t + \delta$ unless $\tilde{X} > t + \delta - \epsilon$.

Integrals below are transformed using a substitution of $y = (x + \epsilon - t - \delta)/\epsilon$, with $dy = dx/\epsilon$.

$P(\tilde{S} > t + \delta) = \int_{x \in [t+\delta-\epsilon, t+\epsilon]} \big( \int_{y \in [t+\delta-\epsilon, x+\epsilon]} ds/(2\epsilon) \big) dx/(2\epsilon) = \int_{x \in [t+\delta-\epsilon, t+\epsilon]} (x + \epsilon - t - \delta) dx/(4\epsilon^2)$

$= \int_{y \in [0,2-\delta/\epsilon]} y dy/4 = (2-\delta/\epsilon)^2/8 = 0.5(1 - 0.5\delta/\epsilon)^2$.

$E(\tilde{X} | \tilde{S} > t + \delta) = \left( \int_{x \in [t+\delta-\epsilon, t+\epsilon]} \frac{x}{(2\epsilon)} dx / (2\epsilon) \right) / (0.5(1 - 0.5\delta/\epsilon)^2)$

$= \int_{x \in [t+\delta-\epsilon, t+\epsilon]} x(x + \epsilon - t - \delta) dx / (0.5\epsilon^2(2 - \delta/\epsilon)^2) = \int_{y \in [0,2-\delta/\epsilon]} (\epsilon y + t + \delta - \epsilon) y dy / (0.5(2 - \epsilon/\delta)^2)$

$= [\epsilon (2 - \delta/\epsilon)^3/3 + (t + \delta - \epsilon)(2 - \delta/\epsilon)^2/2] / (0.5(2 - \epsilon/\delta)^2) = t + (\epsilon + \delta)/3$.

We get $P(\tilde{S} < t + \delta) = 1 - P(\tilde{S} > t + \delta)$ because the continuous distribution has $P(\tilde{S} = t + \delta) = 0$.

The expected value for $\tilde{S} < t + \delta$ is computed from the fact $t = E(\tilde{X}) = P(\tilde{S} > t + \delta) E(\tilde{X} | \tilde{S} > t + \delta) + P(\tilde{S} < t + \delta) E(\tilde{X} | \tilde{S} < t + \delta)$.

By a symmetric argument, it can also be shown in this model that $P(\tilde{S} < t - \delta) = 0.5(1 - 0.5\delta/\epsilon)^2$ and $E(\tilde{X} | \tilde{S} < t + \delta) = t - (\epsilon + \delta)/3$. 

22
Comparing symmetric equilibria of a symmetric game: risk dominance and global games

A symmetric equilibrium $\sigma$ risk-dominates another symmetric equilibrium $\tau$ if each player $i$ would strictly prefer to play $\sigma_i$ over $\tau_i$ when the other players were equally likely to play according to $\sigma_i$ or $\tau_i$.

Consider the following example:

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X, X, X, 0</td>
<td></td>
<td>0, X, 4, 4</td>
<td></td>
</tr>
</tbody>
</table>

When $2 < X < 4$, the $\alpha$ equilibrium risk-dominates the $\beta$ equilibrium, even though $\beta$ Pareto-dominates $\alpha$.

Following Carlsson and van Damme (1993), we can find a rationale for risk-dominance here by perturbing the game to a Bayesian "global game" where players have small uncertainty about $X$ in a wide range.

Suppose that $X$ is drawn from a Uniform distribution on $[-1, 5]$; and, given $X$, each player $i$'s type $t_i$ is independently drawn from a Uniform distribution on $[X-\epsilon, X+\epsilon]$, where $\epsilon$ is known and satisfies $0 < \epsilon < 0.5$.

Thus, given any type $t_i$ with $-1+\epsilon < t_i < 5-\epsilon$, player $i$'s belief about $X$ should be Uniform on $[t_i-\epsilon, t_i+\epsilon]$, and so $i$'s expected value of $X$ with type $t_i$ would be $E(X|t_i) = t_i$; and player $i$ would also know that the other player's type $t_i$ was between $t_i-2\epsilon$ and $t_i+2\epsilon$.

Given any type $t_i > 4$, player $i$ would expect more than 4 from $\alpha_i$ and so would choose $\alpha_i$.

Given any type $t_i < 0$, player $i$ would expect less than 0 from $\alpha_i$ and so would choose $\beta_i$.

Let $A$ denote the smallest number such that each player $i$ would always play $\alpha_i$ whenever $t_i > A$, in every Bayesian equilibrium of this game.

Let $B$ denote the greatest number such that each player $i$ would always play $\beta_i$ whenever $t_i < B$, in every Bayesian equilibrium of this game. Obviously $0 \leq B \leq A \leq 4$. We will show that $A = B = 2$.

Given the type $t_i$, player $i$'s expected payoff from $\alpha_i$ here would be $EU_i(\alpha_i|t_i) = E(X|t_i) = t_i$.

By definition of $A$, player $i$ knows that, in equilibrium, the other player would not choose $\alpha_i$ unless $t_i \leq A$. Thus, given the type $t_i$, player $i$'s expected payoff from $\beta_i$ would satisfy $EU_i(\beta_i|t_i) \leq 4 P(t_i \leq A|t_i)$, where $P(t_i \leq A|t_i)$ is $i$'s probability of the other player's type being less than or equal to $A$, given $i$'s type $t_i$.

The probability $P(t_i \leq A|t_i)$ is a decreasing continuous function of $t_i = 1$ when $t_i < A-2\epsilon$, $=0$ when $t_i > A+2\epsilon$.

When $t_i = A$, $i$'s beliefs about $X$ and $t_i$ are symmetric around $A$, and so we get $P(t_i \leq A|t_i = A) = 1/2$, and $EU_i(\alpha_i|t_i = A) = EU_i(\beta_i|t_i = A) = E(X|t_i = A) - 4 P(t_i \leq A|t_i = A) = A - 4(1/2) = A-2$.

So if we had $A-2 > 0$ then by continuity we could find some small $\delta > 0$ such that, for all $t_i > A-\delta$: $EU_i(\alpha_i|t_i) > EU_i(\beta_i|t_i) \geq E(X|t_i) - 4 P(t_i \leq A|t_i = A) = t_i - 4 P(t_i \leq A|t_i) > t_i$.

But then player $i$ would play $\alpha_i$ in equilibrium with all types such that $t_i > A-\delta$, which would contradict the assumption that $A$ was the lowest number with this property. Thus, $A-2 \leq 0$, and so $A \leq 2$.

Similarly, as player $i$ knows that the other player $-i$ plays $\beta_i$ whenever $t_i < B$,

$EU_i(\beta_i|t_i) - EU_i(\alpha_i|t_i) \geq 4 P(t_i < B|t_i) - E(X|t_i) = 4 P(t_i < B|t_i) - t_i$.

At $t_i = B$ this becomes $4 P(t_i < B|t_i = B) - E(X|t_i = B) = 4(1/2) = 2 - B$.

$P(t_i < B|t_i)$ is a continuous and decreasing function of $t_i$.

So if we had $2-B > 0$ then we could find some some small $\delta > 0$ such that, for all $t_i < B+\delta$:

$EU_i(\beta_i|t_i) - EU_i(\alpha_i|t_i) \geq 4 P(t_i < B|t_i) - E(X|t_i) = 4 P(t_i < B|t_i) - t_i > 0$.

But then player $i$ would play $\beta_i$ in equilibrium with all types such that $t_i < B+\delta$, which would contradict the assumption that $B$ was the greatest number with this property. Thus, $2-B \leq 0$, and so $B \geq 2$.

Obviously $B$ cannot be greater than $A$. Thus $A = B = 2$. That is, the global game has a unique Bayesian equilibrium in which each player $i$ chooses $\alpha_i$ whenever $t_i > 2$, and $i$ chooses $\beta_i$ whenever $t_i < 2$.

As $\epsilon \to 0$, the players' information about $X$ becomes almost perfect, but their Bayesian-equilibrium choices become the risk-dominant equilibria of the games where $X$ is common knowledge, as the choices can switch only at the $X$ where each player is indifferent between $\alpha_i$ and $\beta_i$ when the other is equally likely to do either.

This result can be extended to more general symmetric $2 \times 2$ games. The key is to have types with overlapping ranges of local uncertainty that cover a continuous interval of possible payoff-relevant states which includes some extreme states where each action becomes a dominant strategy for both players.