Incentives to cultivate favored minorities under alternative electoral systems by Roger Myerson, American Political Science Review (1993).

To win, should a politician appeal to all voters, or concentrate on special groups?
A model to show that the answer may depend on the electoral system:

Given K candidates in election to choose M winners (K>M, M=1...),
large number of voters (∞).
Winner gets a budget of \$1 per voter to allocate as promised in the campaign.
Each candidate i chooses a <u>feasible offer distribution</u> F_i on $\mathbb{R}_+$ with $\int_0^\infty x\, dF(x) = 1$.
Then each voter gets a promise from each candidate i independently drawn from F_i.
(If candidates were not independent, last-to-offer could win.)
Consider symmetric equilibria, where all candidates use same distribution F,
and voters believe any pair of candidates has same chance of being in close race.

Results for K=2, M=1, majority voting:
A feasible distribution F is an equilibrium iff
$\forall G$ feasible (s.t: $\int_0^\infty x\, dG(x) = 1$), $\int_0^\infty F(x)\, dG(x) \leq 1/2$.

Unique equilibrium is Uniform [0,2]: $F(x) = x/2$ if $x \in [0,2]$, $F(x)=1$ if $x>2$.
Then $\int_0^\infty F(x)\, dG(x) \leq \int_0^\infty x/2\, dG(x) = [\int_0^\infty x\, dG(x)]/2 = 1/2$.

Rank-scoring rules with K candidates.

A <u>rank-scoring rule</u> is characterized by $1 = s_1 \geq s_2 \geq \ldots \geq s_K = 0$.
Each voter ranks the K candidates, gives s_1 to top-ranked, s_2 to second,
s_j to the candidate ranked above K−j others.
Candidate's score is his average points per voter. M high-scorers win (ties random).
Let S^* denote the average points per voter, $S^* = (s_1+\ldots+s_K)/K$.

<u>Single-positive voting</u> (<u>plurality</u>): $s_1=1$, $0=s_2=\ldots=s_K$. $S^* = 1/K$. [...*best-rewarding*]
<u>Negative voting</u>: $s_1=s_2=\ldots=s_{K-1}=1$, $0=s_K$. $S^* = 1-1/K$. [...*worst-punishing*]
<u>V noncumulative votes</u>: $s_1=\ldots=s_V=1$, $0=s_{V+1}=\ldots=s_K$. $S^* = V/K$.
<u>Borda voting</u>: $sj = (K-j)/(K-1)$. $S^* = 1/2$.

Let $R(p) = \sum_{j=1}^{K} s_j\, p^{K-j} (1-p)^{j-1} (K-1)!/[(j-1)!(K-j)!]$,
so R(p) is the expected value of s_j when K−j has a Binomial (n=K−1, p) dist'n.
(Single-positive voting has $R(p) = p^{K-1}$.)

A feasible distribution F is an equilibrium iff $\forall$G feasible,
$\int_0^\infty R(F(x))\, dG(x) \leq \int_0^\infty R(F(x))\, dF(x) = S^*$.

Theorem 1. The unique symmetric equilibrium F has support $[0, 1/S^*]$, and satisfies
$x = R(F(x))/S^*$ $\forall x \in [0, 1/S^*]$, and so $F^{-1}(p) = R(p)/S^*$ $\forall p \in [0,1]$.

FIGURE 1

Offer Distribution with One Vote, Four Candidates

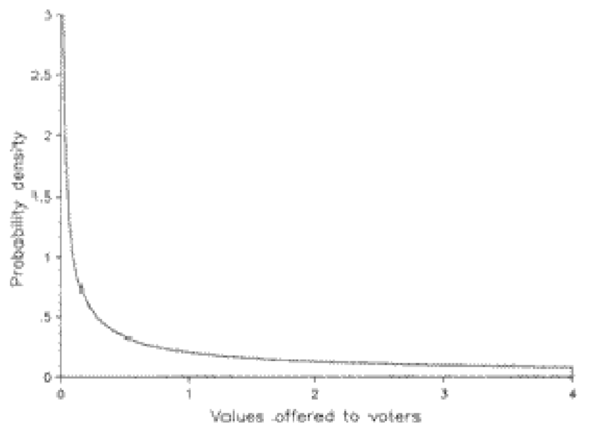


FIGURE 2

Offer Distribution with Two Votes, Four Candidates

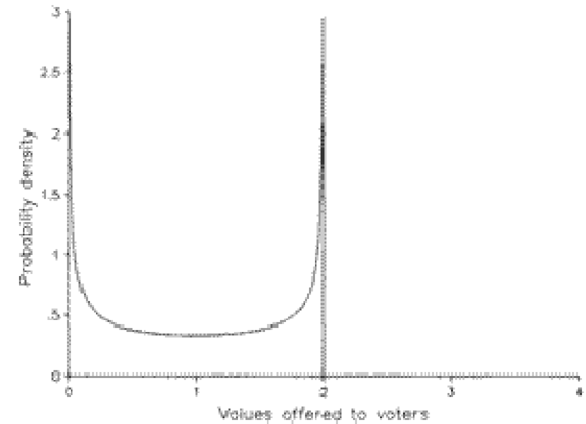


FIGURE 3

Offer Distribution with Three Votes, Four Candidates

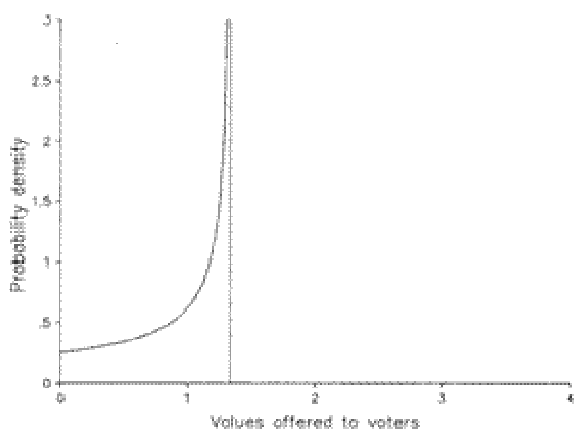
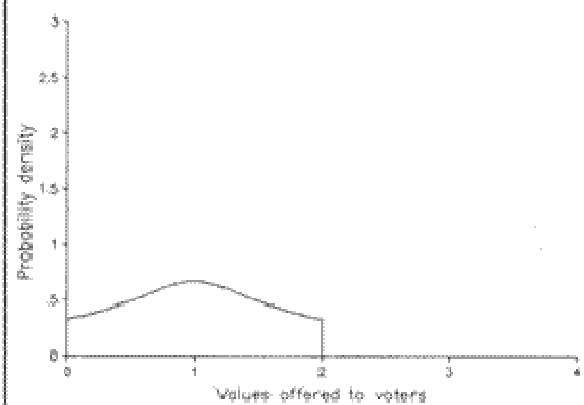


FIGURE 4

Offer Distribution with Ranking Points (1, .5, .5, 0)



Approval voting (a nonrank scoring rule):

In approval voting, each voter can give 0 or 1 point (approval) to each candidate. Winners have the M highest scores (most approvals).

Voter approves candidate i iff (i's offer) > (average of other candidates' offers).

When candidates use distribution F, average of m offers has cumulative A_m, where
$A_1(x) = F(x)$, $A_m(x) = \int_0^{mx} A_{m-1}((mx-z)/(m-1))\, dF(z)$.

When i promises x to a voter, this voter approves i with probability $A_{K-1}(x)$,
and another candidate j is approved by this voter with probability B(x),
where $B(x) = \int_0^{\infty} A_{K-2}([(K-1)y-x]/[K-2])\, dF(y)$.

The feasible distribution F is an equilibrium iff,
$\forall G$ feasible, $\int_0^{\infty} A_{K-1}(x)\, dG(x) \le \int_0^{\infty} B(x)\, dG(x)$.

Approval voting equilibria (discrete numerical approximation)

I found equilibria for discrete approximation, with offers being multiples of 0.05.
The equilibrium offer distribution must have no atoms in continuous case,
but the support seems to have many holes, mass in regions near 0 and in [1,2].
As K increases, fewer get offers near 0, maximal offer decreases toward 1.

FIGURE 5

Offer Distribution with Approval Voting, Four Candidates

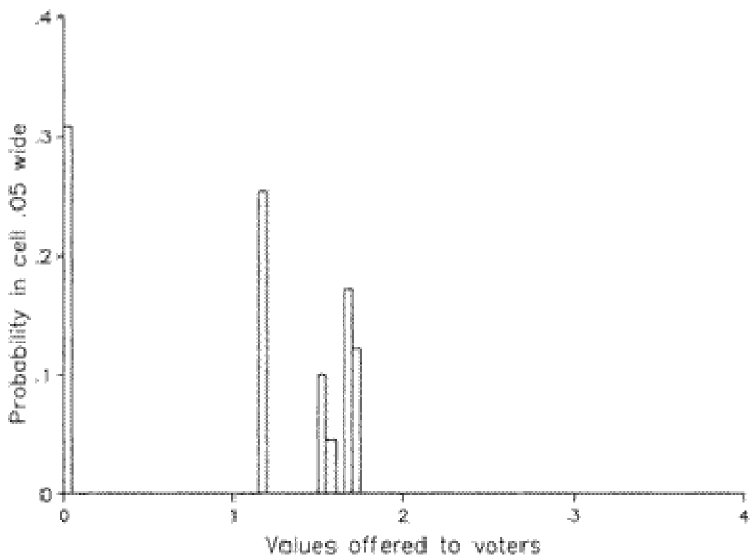


TABLE 2

Candidate Offers in Equilibrium under Approval Voting

NUMBER OF CANDIDATES	S.D.	MAX. OFFER	FRACTION < 1	AVG. SCORE
2	.58	2.00	.50	.50
3	.71	1.90	.34	.55
4	.70	1.70	.31	.60
5	.66	1.60	.29	.63
6	.65	1.50	.29	.66
7	.61	1.45	.27	.69
8	.61	1.40	.27	.71
9	.56	1.35	.24	.74
10	.54	1.35	.23	.74

Note: Statistics were calculated using a .05 discrete approximation.

Single-transferable vote (STV)

In STV, each voter rank-orders the K candidates.

Candidates are eliminated one at a time in recounts, until M winners remain.

At each recount, each voter's ballot gives a point to his highest candidate among those who have not yet been eliminated; and then the lowest scorer is eliminated.

A feasible F is an equilibrium iff there does not exist any feasible G such that $\int_0^\infty (F(x))^{n-1}\, dG(x) \geq 1/n \;\; \forall n \in \{M+1, ..., K-1, K\}$, with > for some n.

Theorem. For any $(\lambda_{M+1}, ... \lambda_{K-1}, \lambda_K)$ such that $\lambda_n > 0 \; \forall n$, $\sum_{n=M+1}^{K} \lambda_n = 1$, there is an equilibrium offer distribution F with STV such that $\forall p \in [0,1]$, $F^{-1}(p) = \sum_{n=M+1}^{K} \lambda_n\, n\, p^{n-1}$.

For any m in {M+1,...,K}, this F can approximate the equilibrium offer distribution for single-positive voting and m candidates, when $\lambda_m \approx 1$ and all other $\lambda_n \approx 0$.

Cox's threshold of diversity

A two-position model: Given K win-motivated candidates, each must simultaneously choose among two policy positions: Left and Right.

Assume that candidates at the same position are treated symmetrically by voters.

Fraction q of voters prefer Left, $1-q$ prefer Right.

Given K, Cox's threshold of diversity Q^* is the supremum of q such that there is a symmetric equilibrium in which all K candidates choose Right.

Fact. For any rank-scoring rule ($1=s_1 \geq s_2 \geq \ldots \geq s_K=0$), $Q^* = S^* = (s_1+\ldots+s_K)/K$.

Single-positive voting yields $Q^* = 1/K$, and so small minority positions can win when K is large. (Best-rewarding yields low Q*, diversity, favored minorities.)

Single-negative voting yields $Q^* = (K-1)/K$, so a majority can be neglected when K is large! (Ex: K=10 implies $Q^*=.9$; and with $q=.81$, we get $1-q=.19 > q/9=.09$)

(Worst-punishing yields high Q^*, clustering, making few enemies.)

Fact. Approval voting, Borda and STV yield $Q^* = 1/2$ for any K. (Majoritarian.)

A public-goods model: The winner of the election will get a budget of \$1 per voter to distribute as cash or to spend on a public good worth \$B to every voter.

An equilibrium where the public good is guaranteed exists only if $B \geq 1/Q^*$.

When $B < 1/Q^*$, if all other candidates promised the public good, then a candidate could win by promising $\$1/Q^*$ to a Q^* fraction of the voters.

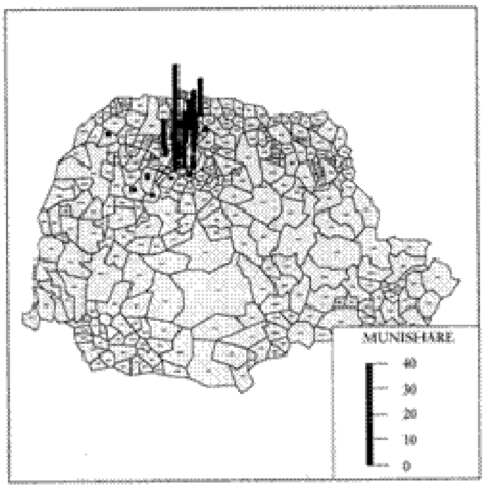
Brazil uses open-list PR voting with single-positive voting for candidates.

Parana: 30 seats.

FIGURE 1

THE CONCENTRATED-DOMINANT VOTE OF A LOCAL POLITICIAN

Municipal Vote Share of Said Ferreira, PMDB—Paraná



Sao Paulo: 60 seats.

FIGURE 2

THE CONCENTRATED-SHARED VOTE OF AN ENVIRONMENTALIST

Personal and Municipal Shares of Fábio Feldmann, PSDB—São Paulo

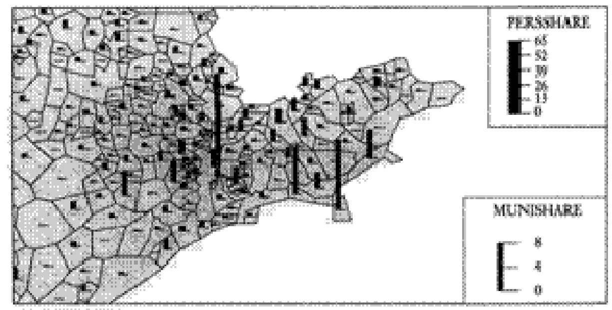
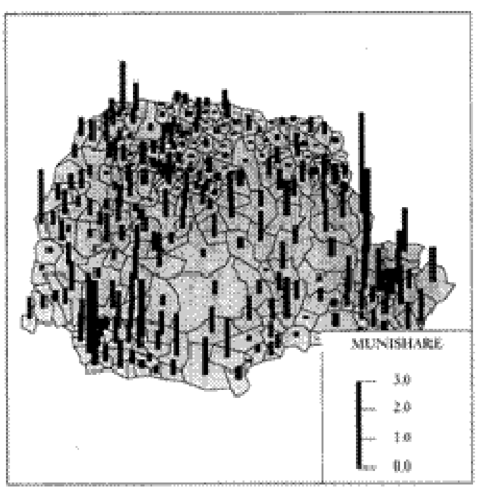


FIGURE 3

THE SCATTERED-SHARED VOTE OF AN EVANGELICAL

Municipal Vote Share of Matheus Iensen, PTB—Paraná

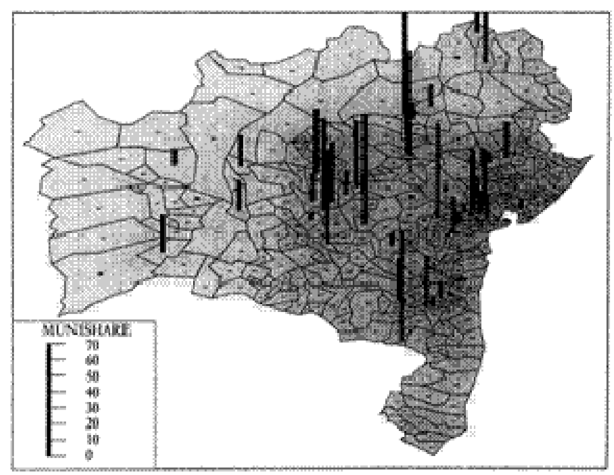


Bahia: 39 seats.

FIGURE 4

A SCATTERED-DOMINANT VOTE DISTRIBUTION

Municipal Vote Share of Jorival Lucas, PDC—Bahia



Figures from Barry Ames, *J. of Politics* 57:324-343 (1995):

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Laurent Bouton, Micael Castanheira 2010. "One person, many votes: divided majority and information aggregation," Econometrica 80: 43-87.

2 states $\{\alpha,\beta\}$, $p(\alpha)=p(\beta)=0.5$. 3 candidates $\{A,B,C\}$. 3 voter types $\{a,b,c\}$.
$r(a|\alpha) = r(b|\beta) = .6\times.6 = 0.36$, $r(b|\alpha) = r(a|\beta) = .6\times.4 = 0.24$, $r(c|\alpha) = r(c|\beta) = 0.4$.
For $t\in\{a,b\}$: $u_A(t|\alpha) = u_B(t|\beta) = 1$, $u_B(t|\alpha) = u_A(t|\beta) = 0$, $u_C(t|\alpha) = u_C(t|\beta) = -1$.
For $\omega\in\{\alpha,\beta\}$. $u_C(c|\omega) = 1$, $u_A(c|\omega) = u_B(c|\omega) = 0$.

With plurality voting:
in one eqm, a's vote A, b's vote B, c's vote C, C almost-surely wins (.4>.36>.24);
in another eqm, a's and b's vote A, c's vote C, A almost-surely wins (.6>.4>0);
in another eqm, a's and b's vote B, c's vote C, B almost-surely wins (.6>.4>0).

Suppose the number of voters is a Poisson random variable with mean n (take limit as $n\rightarrow\infty$), and each voter's type is independent according to $r(\bullet|\omega)$ given state ω.
Given state ω, the number of each permissible vote v cast by voters will be an independent Poisson random variable with mean $n\tau(v|\omega) = n \sum_t r(t|\omega)\sigma(v|t)$, where σ denotes the voters' type-conditional voting strategy.

Fact. For any candidates i and j with $\tau_i > \tau_j$, the magnitude of the event "$1+X_j \geq X_i$" is $\lim_{n\rightarrow\infty} \mathrm{LN}(P(1+X_j \geq X_i \mid \tau,n))/n = 2(\tau_i\tau_j)^{0.5} - \tau_i - \tau_j = -(\tau_i^{0.5}-\tau_j^{0.5})^2$.

With approval voting, consider symmetric scenarios where c's vote C,
a's vote either AB with prob'y δ or A with proby $1-\delta$,
b's vote either AB with prob'y δ or B with proby $1-\delta$.

Expected vote fractions:
$\tau(C|\alpha) = 0.4$, $\tau(A|\alpha) = 0.36(1-\delta)$, $\tau(AB|\alpha) = 0.6\delta$, $\tau(B|\alpha) = 0.24(1-\delta)$,
$\tau(C|\beta) = 0.4$, $\tau(B|\beta) = 0.36(1-\delta)$, $\tau(AB|\beta) = 0.6\delta$, $\tau(A|\beta) = 0.24(1-\delta)$.

In state α, magnitude of close AC race ($\lim_{n\to\infty} P(\mathrm{piv}_{AC})/n$) is
$\mu(AC|\alpha) = -([.36(1-\delta)+.6\delta]^{0.5} - .4^{0.5})^2$,
the magnitude of close BC race is smaller (more negative),
the magnitude of close AB race is $\mu(AB|\alpha) = -(1-\delta)(.36^{0.5} - .24^{0.5})^2$.
In state β, magnitudes are similar, just reversing the roles of A and B.

With δ=0.569, get $\mu(AC|\alpha) = \mu(BC|\beta) = -0.0052 = \mu(AB|\alpha) = \mu(AB|\beta)$
$> \mu(AC|\beta) = \mu(BC|\alpha)$,
and so we get an eqm with better-in-{A,B} likely to win (0.497 > 0.445 > 0.4).
($\delta > 0.569 \Rightarrow \mu(AC|\alpha) = \mu(BC|\beta) < \mu(AB|\alpha) = \mu(AB|\beta)$, so δ too large!)
($\delta < 0.569 \Rightarrow \mu(AC|\alpha) = \mu(BC|\beta) > \mu(AB|\alpha) = \mu(AB|\beta)$, so δ too small!)

Bipolar multicandidate elections with corruption

Myerson, Scandinavian Journal of Economics (2006) 108:727-742.

Set of candidates K is partitioned into K_1={leftists} and K_2={rightists}.

Each candidate k has corruption level $f(k) \geq 0$.

k is <u>clean</u> if f(k)=0, <u>corrupt</u> if f(k)>0.

In game Γ_n, the number of voters is a Poisson random variable with mean n.

Each voter has a type t drawn independently from a probability distribution r that has a continuous positive density on the real line $\mathbb{R}$. $r(S) = \text{Prob}(\in S)\ \ \forall S \subseteq \mathbb{R}$.

A voter's type t measures his net preference for rightist candidates in K_2,

so t's utility payoff if k wins is $u_k(t) = t - f(k)$ if $k \in K_2$, $u_k(t) = 0 - f(k)$ if $k \in K_1$.

Suppose $\forall i \in \{1,2\}$, there exists a clean candidate k in K_i with f(k)=0. (wlog)

To complete the game, we must specify an electoral system (ties broken at random).

An equilibrium in game Γ_n specifies a (weakly undominated) optimal strategy $\sigma_n(t)$ for each type t, and generates expected fractions $\tau_n(c)$ for each ballot c that is allowed in this electoral system, and win-probabilities $q_n(k)$ for each candidate k.

A large equilibrium (σ,τ,q) is a limit of (σ_n,τ_n,q_n) equilibria of Γ_n as $n\rightarrow\infty$.

A pair of candidates {i,j} is distinct iff $u_i(t)u_j(t)$ for some t in T.

{i,j}-race is close when adding 1 vote could change winner from i to j, or j to i.

The {i,j}-race is serious in a large equilibrium iff {i,j} is a distinct pair and there is a strictly positive limit($n\rightarrow\infty$) of the conditional probability of a close {i,j}-race given that some pair of distinct candidates are in a close race.

A candidate is serious iff he is involved in at least one serious race.

A candidate i is strong in a large eqm (σ,τ,q) iff $q(k) > 0$ (positive win-proby).

Theorem 1 (effectiveness against corruption). In a large equilibrium under approval voting, no corrupt candidates can be strong or serious.

Theorem 2 (majoritarianism). In a large equilibrium under approval voting, with probability 1, the winner will be a candidate who is considered best by at least half of the voters.

Failures of effective majoritarianism for other electoral systems (A):

In 3-candidate elections, consider rank-scoring rules where ballots are permutations of (1,A,0), for some A such that $0 \leq A \leq 1$.

Suppose $K_1=\{1\}$, $K_2=\{2,3\}$, 1 and 2 are clean, 3 is corrupt.

If A<1/2 then there is an equilibrium where {1,3} is the only serious race.

In this eqm, everybody votes (1,A,0) or (0,A,1), so winner will be either 1 or 3.

If $A \geq 1/2$ then 3 must be serious in all equilibria.

Otherwise, if 3 were not serious, then everybody would vote (1,0,A) or (0,1,A), but then 3 would always be in first place when 1 and 2 tie!

Failures of effective majoritarianism for other electoral systems (A,B):

Now consider scoring rules where ballots are permutations of (1,A,0) and (1,B,0), where $0 \leq A \leq B \leq 1$.

Approval voting is (A,B)=(0,1), plurality voting is (0,0),

Borda voting is (1/2,1/2), negative voting is (1,1).

Suppose now $K_1=\{1\}$, $K_2=\{2,3\}$, all three candidates are clean.

In a symmetric eqm, leftists randomize equally among (1,A,0) and (1,0,A), while rightists randomize equally among (0,B,1) and (0,1,B).

Notice $r < r(0+A)/2 + (1-r)(1+B)/2$ (1 loses) iff $r < (1+B)/(3+B-A)$.

$(1+B)/(3+B-A)$ is Cox's threshold of diversity here.

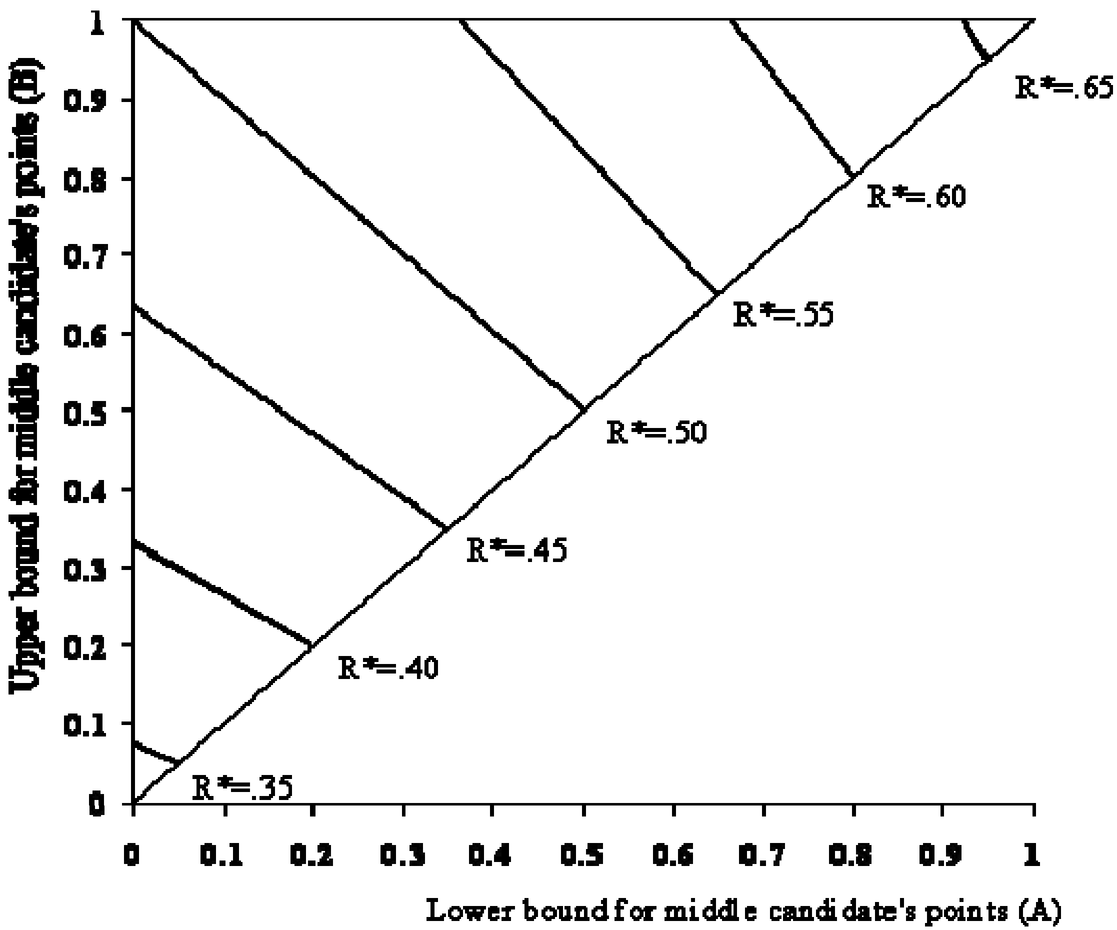
Also $1/2 < (1+B)/(3+B-A)$ iff $1 < A+B$.

When $1 < A+B$ and $1/2 < r(\mathbb{R}_-) < (1+B)/(3+B-A)$, then almost-surely leftists are a majority, but a rightist candidate wins (duplication helps righists).

When $1 > A+B$ and $1/2 > r(\mathbb{R}_-) > (1+B)/(3+B-A)$ then almost-surely rightists are a majority, but the leftist candidate wins (duplication hurts rightists).

R* = (1+B)/(3+B−A) = [biggest fraction that can lose with one candidate versus two in a symmetric eqm]

R. Myerson, "Comparison of scoring rules in Poisson voting games," JET 103:219-251 (2002).



The **M+1 law** for multi-seat legislative elections under the single nontransferable vote (**SNTV**) system, was observed in Japanese elections by S Reed in *Brit. J. Poli. Sci. 20* (1990), formulated by G. Cox, *Amer. Poli. Sci. Rev. 88* (1994).

Suppose that there are K candidates, numbered 1,2,...,K, in an election with single nontransferable vote where the top M candidates win. Here $M < K$.
Each voter has a type which is drawn independently from some finite set according to some fixed probability distribution r.
Each type t has a strict utility ranking $u_i(t)$ of candidates i.
A voter's payoff from the election is his sum of utility from each of the M winners.
In n'th voting game, number of voters is a Poisson random variable with mean n.
Consider a large equilibrium, a convergent sequence of equilibria as $n\rightarrow\infty$.
The expected fraction who vote for each candidate i is converging to some limit τ_i.
Without loss of generality, we may number the candidates so that $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_K$.

<u>Thm (weak M+1 law for SNTV).</u> Consider a large eqm in SNTV with $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_K$.
For each $i \leq M$, τ_i must equal τ_M. For each $j \geq M+1$, τ_j must be either τ_{M+1} or 0.

Lemma. The magnitude of a close race between candidates M and M+1 is
$-((\tau_M)^{0.5}-(\tau_{M+1})^{0.5})^2$

Consider now some candidate $j > M+1$.

If candidates 1,..,M all got strictly more votes than $1+\tilde{x}_j$, then candidate j would not be in a close race. Thus, when candidate j is in a close race, there at least one candidate i in $\{1,\ldots,M\}$ such that $1+\tilde{x}_j \geq \tilde{x}_i$.

The magnitude of this event is $-((\tau_i)^{0.5}-(\tau_j)^{0.5})^2 \leq -((\tau_M)^{0.5}-(\tau_j)^{0.5})^2$.

So the magnitude of the event that j is in a close race is not more than $-((\tau_M)^{0.5}-(\tau_j)^{0.5})^2$.

But if $\tau_j < \tau_{M+1}$ then this magnitude is strictly less than the magnitude of a close race between candidates M and M+1. Thus we get

Lemma. For any j in $\{M+2,\ldots,K\}$, if $\tau_j < \tau_{M+1}$ then candidate j is not serious.

Consider now some candidate $i < M$.

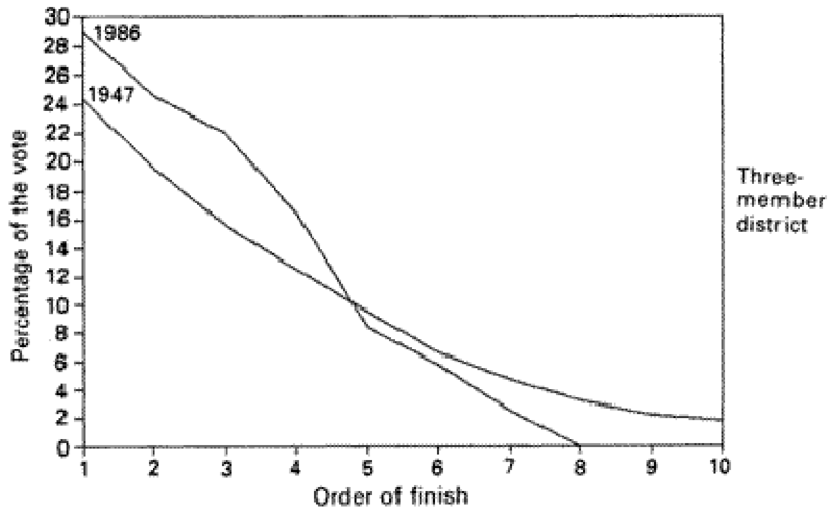
If candidates M+1,...,K all got strictly less votes than $_i-1$, then candidate i would not be in a close race, because he would be a guaranteed winner even with one more vote.

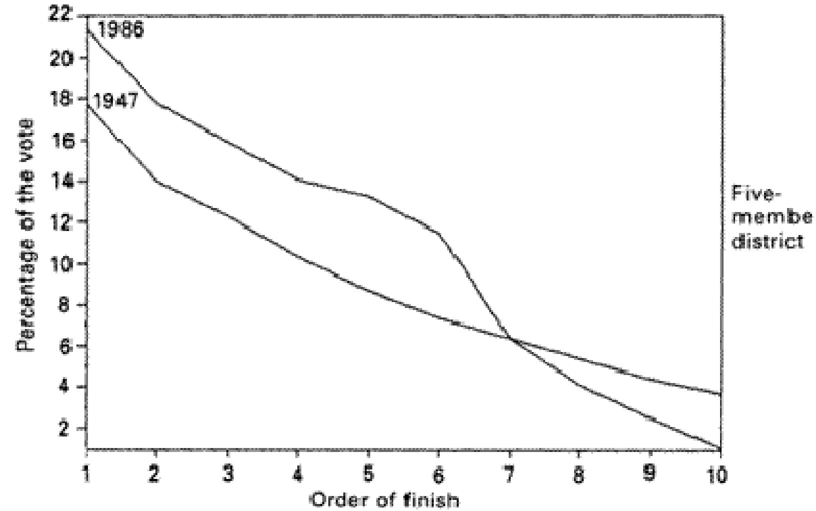
Thus, when candidate i is in a close race, there at least one candidate j in $\{M+1,\ldots,K\}$ such that $1+\tilde{x}_j \geq \tilde{x}_i$. The magnitude of this event is $-((\tau_i)^{0.5}-(\tau_j)^{0.5})^2 \leq -((\tau_i)^{0.5}-(\tau_{M+1})^{0.5})^2$.

So the magnitude of the event that j is in a close race is not more than $-((\tau_i)^{0.5}-(\tau_{M+1})^{0.5})^2$. But if $\tau_i > \tau_M$ then this magnitude is strictly less than the magnitude of a close race between candidates M and M+1. Thus we get

Lemma. For any i in $\{1,\ldots,M-1\}$, if $\tau_i > \tau_M$ then candidate i is not serious.

From: Steven R. Reed, "Structure and behavior: extending Duverger's law to the Japanese case," British J of Political Science 20(3), 340-341.





[19] The first postwar election in 1946 was held under slightly different rules and in large electora districts.

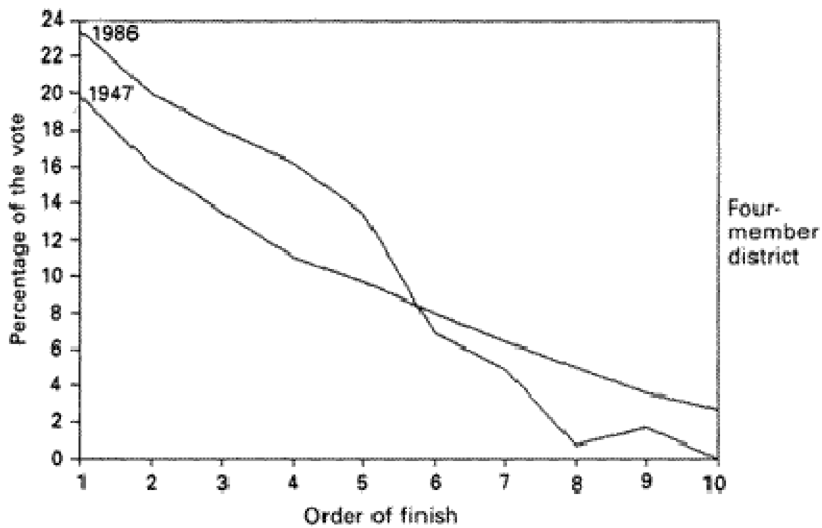


Fig. 1. Percentage of the vote by order of finish

Notes: Entries are the average votes of candidates who finished first, second, third and so on. The denominator is the number of candidates finishing in that order. When there are only five candidates in a district, the sixth place is not set to zero but ignored. Thus, it is possible for ninth-place finishers to have a higher average vote than eighth-place finishers, as in four-member districts, because most districts had only eight candidates.

The M+1 law for single nontransferable vote (SNTV).

A pair of candidates are in a close race when a small number of votes could change the winner from one to the other.

A rational voter knows that his vote matters only in the event of a close race, and so should vote in equilibrium to maximize expected utility in this event.

A race between a pair of candidates is <u>serious</u> iff, conditional on a close race existing, there is a substantial positive probability of this race being close.

A candidate is a serious iff she is in some serious race.

A <u>stronger</u> candidate has greater probability of winning; weaker, less.

A likely winner has probability of winning close to 1; likely loser, close to 0.

Consider an election with M winners under single-positive voting (SNTV).

Voters should not waste votes on nonserious candidates.

Becoming weaker than the strongest likely loser makes a candidate less serious, which weakens her further until she gets almost no votes.

Among likely winners however, being perceived stronger tends to weaken.

Such a tendency towards M+1 serious candidates, with relatively level scores for top M candidates was found by Steven Reed in Japan (*Brit.J.P.S.* 1991).

Other (unstable, non-Duvergerian) equilibria have ties for strongest likely loser.

(Gary Cox, *American Political Science Review* 88:608-621 (1994).)

Short proof of the weak M+1 law for SNTV with a large Poisson electorate

Suppose that there are K candidates, numbered 1,2,...,K, in an election with single nontransferable vote where the top M candidates win. Here $M < K$.

In case of a tie for M'th and M+1'th place, a random ordering of the candidates is generated, and the set of M winners is completed by selecting from the borderline-winning candidate in this order.

Each voter has a type which is drawn independently from some finite set according to some fixed probability distribution.

Each type of voter has a strict utility ranking of the candidates, with $u_i(t)$ denoting the utility of candidate i winning for a voter of type t.

A voter's payoff from the election is the sum of the winners for him.

In the voting game n, the number of voters is a Poisson random variable with mean n. If W denotes the actual number of voters: $P(W=k) = e^{-n}n^k/k!$

A large equilibrium is a convergent sequence of equilibria of these games as $n\to\infty$. By convergent, we mean that the expected fraction of the electorate who vote for each candidate i is converging to some limit τ_i. [$\tau_i = \sum_t r(t)\sigma(i \mid t)$.]

Choosing a subsequence if necessary, we may assume that other probabilities are also convergent to well-defined limits as $n\to\infty$.

Now consider a large equilibrium. Without loss of generality, we may assume that the candidates are numbered so that $\tau_1 \geq \tau_2 \geq ... \geq \tau_K$.

Let X_i denote the number of votes for candidate i.

Then X_i is a Poisson random variable with mean $n\tau_i$, independent of other X_j.

The {i,j} race is close when adding one vote for i or j could make one of them replace the other in the set of winners.

If there is no close race, then adding one more vote in cannot matter to anybody.

In equilibrium, each voter must vote cast the ballot that would maximize his conditional expected utility gain, relative to not voting, given that there is at least one close race.

A race between two candidates is serious iff its conditional probability of being close, given that there is some close race, is strictly positive in the limit as $n \to \infty$.

A candidate is serious iff he is involved in at least one close race.

A voter's conditional expected gain, given that there is a close race, from voting for his favorite serious candidate would be strictly positive in the limit.

So each voter's ballot in equilibrium must give him a strictly positive conditional expected gain, given that there is a close race.

Thus, in the large equilibrium, nobody votes for candidates who are not serious;
that is, if h is not a serious candidate then $\tau_h = 0$.

From any standard paper on Poisson voting games, we get:

Fact. For any candidates i and j with $\tau_i > \tau_j$, the magnitude of the event "$1+X_j \geq X_i$" is $\lim_{n\to\infty} LN(P(1+X_j \geq X_i \mid \tau,n))/n = 2(\tau_i\tau_j)^{0.5} - \tau_i - \tau_j = -(\tau_i^{0.5}-\tau_j^{0.5})^2$.

Here is a sketch of the argument:

If Y is a Poisson random variable with large mean m, and $k=\alpha m$ is an integer, then

$$P(Y=k) = e^{-m}m^k/k! \approx e^{-m}\, m^k/[(k/e)^k\,(2\pi k)^{0.5}]$$
$$= e^{-m}\, m^{\alpha m}\, e^{\alpha m}/[(\alpha m)^{\alpha m}\,(2\pi\alpha m)^{0.5}] = e^{m(\alpha-1-\alpha LN(\alpha))}/(2\pi\alpha m)^{0.5}.$$

(Stirling's approximation for k! is used here.)

Thus, $LN(P(Y=\alpha m))/m \approx \alpha-1-\alpha LN(\alpha)$.

$$LN(P(X_1=\beta n=X_2 \mid n))/n = \sum_{i\in\{1,2\}} \tau_i LN(P(X_i=(\beta/\tau_i)\tau_i n))/(\tau_i n)$$
$$\approx \tau_1[\beta/\tau_1-1-(\beta/\tau_1)LN(\beta/\tau_1)] + \tau_2[\beta/\tau_2-1-(\beta/\tau_2)LN(\beta/\tau_2)]$$
$$= 2\beta - \tau_1 - \tau_2 - 2\beta LN(\beta) + \beta LN(\tau_1) + \beta LN(\tau_2).$$

This is maximized when $LN(\beta) = LN(\tau_1) + LN(\tau_2)$ and so $\beta = (\tau_1\tau_2)^{0.5}$.

Thus, the magnitude of the event that X_1 and X_2 are equal (or close) is

$$LN(P(X_1=X_2 \mid n)/n \approx \max_{\beta\geq 0} LN(P(X_1=\beta n=X_2 \mid n))/n$$
$$\approx 2(\tau_1\tau_2)^{0.5} - \tau_1 - \tau_2.$$

The magnitude of a reversal of X_1 and X_2 relative to expectations is the same as the magnitude of a tie.

A close race between candidates M and M+1 can occur when their votes are within one of each other and all other candidates' votes are near their expected values.

Thus, the magnitude of a close race involving M and M+1 is $-(\tau_M^{0.5}-\tau_{M+1}^{0.5})^2$.

Consider now some candidate $j > M+1$.

If candidates 1,..,M all got strictly more votes than $1+X_j$, then candidate j would not be in a close race.

Thus, when candidate j is in a close race, there at least one candidate i in {1,...,M} such that $1+X_j \geq X_i$. The magnitude of this event is $-(\tau_i^{0.5}-\tau_j^{0.5})^2 \leq -(\tau_M^{0.5}-\tau_j^{0.5})^2$.

So the magnitude of the event "j is in a close race" is not more than $-(\tau_M^{0.5}-\tau_j^{0.5})^2$.

But if $\tau_j < \tau_{M+1}$ then this magnitude is strictly less than the magnitude of a close race between candidates M and M+1.

Thus, for any j in {M+2,...,K}, if $\tau_j < \tau_{M+1}$ then candidate j is not serious.

Consider now some candidate $i < M$.

If candidates $M+1,...,K$ all got strictly less votes than X_i-1, then candidate i would not be in a close race, because he would be winner by at least 1 vote.

So when candidate i is in a close race, there at least one candidate j in $\{M+1,...,K\}$ such that $1+X_j \geq X_i$.

The magnitude of this event is $-(\tau_i^{0.5}-\tau_j^{0.5})^2 \leq -(\tau_i^{0.5}-\tau_{M+1}^{0.5})^2$.

So the magnitude of the event that j is in a close race is not more than $-(\tau_i^{0.5}-\tau_{M+1}^{0.5})^2$.

But if $\tau_i > \tau_M$ then this magnitude is strictly less than the magnitude of a close race between candidates M and $M+1$.

Thus, for any i in $\{1,...,M-1\}$, if $\tau_i > \tau_M$ then candidate i is not serious.

We obviously cannot have $\tau_i > \tau_M$, because then i would not be serious and so τ_i would be 0, contradicting $\tau_i > \tau_M \geq 0$. Thus we get the main result:

Theorem. For each i in $\{1,2,...,M\}$, τ_i must be equal to τ_M.

For each j in $\{M+1,...,K\}$, τ_j must be equal to either τ_{M+1} or 0.