SETTLED EQUILIBRIA

by Roger Myerson and Jörgen Weibull

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The paper: http://home.uchicago.edu/~rmyerson/research/settled.pdf
and http://pzp.hhs.se/jorgen-weibull/working-papers
These notes: http://home.uchicago.edu/~rmyerson/research/settlednts.pdf

Example 1 "Battle of Sexes":

	a_2	b_2
a_1	3, 2	0, 0
b_1	0, 0	2, 3

Three Nash equilibria: (a_1, a_2), (b_1, b_2), $(0.6[a_1]+0.4[b_1], 0.4[a_2]+0.6[b_2])$.

If this game is played only once by players who have no cultural or historical context, the symmetric mixed equilibrium might be an appropriate prediction.

But in culturally familiar settings, we may expect players to develop understandings that coordinate their expectations at (a_1,a_2) or (b_1,b_2).

How can we formalize this intuition?

The totally mixed equilibrium is *perfect* (Selten 1975) and *proper* (Myerson 1978) and *stable* (Kohlberg-Mertens 1986).

Kalai-Samet's (1984) *persistent equilibria* exclude the mixed equilibrium for this example, as does Basu-Weibull's (1991) related concept of minimal *curb sets*.

But persistent equilibria and minimal curb sets fail to exclude the totally mixed equilibria for closely related games that seem similar. (*Example 2*)

Here we develop new refinements, along the lines of persistent equilibria and minimal curb sets, that can exclude the totally mixed equilibrium here.

Goal: Exclude "uncoordinated" equilibria for more games than persistence, while maintaining general existence of a refined equilibrium that is also proper (and so corresponds to sequential equilibrium of any extensive game).

General framework:
Given a finite strategic game $G = (N, S, (u_i)_{i\in N})$ with players $N = \{1,2,...n\}$, strategies $S = \times_{i\in N} S_i$, each S_i a nonempty finite set, utility functions $u_i{:}S\to\mathbb{R}$.

$\Delta(S_i)$ is the set of probability distributions over S_i.
Let $M(S) = \times_{i\in N} \Delta(S_i) = \{\text{mixed strategy profiles}\}$.

We extend $u_i{:}M(S)\to\mathbb{R}$ by $u_i(\sigma) = \sum_{s\in S} (\prod_{i\in N} \sigma_i(s_i))\, u_i(s)$.
Let $u_i(\sigma_{-i},[s_i])$ denote the expected payoff that player i would get from choosing pure strategy s_i when everyone else randomizes according to σ.

A *Nash equilibrium* is any σ in $M(S)$ such that $u_i(\sigma) \geq u_i(\sigma_{-i},[s_i])$ $\forall s_i\in S_i$, $\forall i\in N$.

$M^0(S) = \times_{i\in N} \Delta^0(S_i) = \times_{i\in N} \{\sigma_i\in M(S_i)|\ \sigma_i(s_i)>0\ \forall s_i\in S_i\} = \{$*totally mixed strategies*$\}$.
For any $\varepsilon>0$, a mixed strategy profile σ is *ε-proper* in G iff $\sigma \in M^0(S)$ and:
$\forall i\in N$, $\forall s_i\in S_i$, $\forall r_i\in S_i$, if $u_i(\sigma_{-i},[s_i]) < u_i(\sigma_{-i},[r_i])$ then $\sigma_i(s_i) < \varepsilon\sigma_i(r_i)$.
A *proper equilibrium* is any limit of ε-proper strategy profiles as $\varepsilon\to 0$.

Fact (van Damme 1984). A proper equilibrium of G corresponds to a sequential equilibrium in any extensive game having G as its normal form.

Blocks of strategies

Def. Given $G=(N,S,u)$, a *block* is any $T = \times_{i\in N} T_i$ where $\varnothing \neq T_i \subseteq S_i \ \ \forall i\in N$.
(It represents pure strategies that might be "conventional" under some social norm.)
The associated *block game* is $G_T = (N, T, u)$.

We embed its mixed strategies in $M(T) = \{\sigma\in M(S) |\ \sigma_i(s_i)=0\ \forall s_i\notin T_i, \forall i\in N\} \subseteq M(S)$.
Then the *ε-neighborhood* of $M(T)$ in $M(S)$ is $\{\sigma\in M(S) |\ \sigma_i(s_i)<\varepsilon\ \forall s_i\notin T_i, \forall i\in N\}$.
The *support* of σ is contained in a block T iff $\{s_i|\sigma_i(s_i)>0\} \subseteq T_i, \forall i\in N$.

(Basu Weibull '91) A block T is *curb* (closed under rational behavior) iff
$\forall\sigma\in M(T),\ \operatorname{argmax}_{s_i\in S_i} u_i(\sigma_{-i},[s_i]) \subseteq T_i,\ \forall i\in N$.

(Kalai Samet '84) A block T is *absorbing* iff $\exists\varepsilon>0$ such that, $\forall\sigma$ in the ε-neighborhood of M(T), $\forall i\in N$, $\max_{t_i\in T_i} u_i(\sigma_{-i},[t_i]) = \max_{s_i\in S_i} u_i(\sigma_{-i},[s_i])$.

Fact. Any curb set is absorbing (curb => absorbing).

An equilibrium is *persistent* iff its support is contained in a minimal absorbing block.

Fact. Any absorbing block contains a proper equilibrium of the original game, and so there exists at least one equilibrium that is both proper and persistent.

Persistence seems too weak in an elaborated version of Example 1

But persistent equilibria and minimal curb sets fail to exclude the totally mixed equilibria for closely related games that seem similar.

Example 2. An elaborated version of the battle of sexes game, where a 0,0 payoff has been replaced by a two-person zero-sum game that has value 0:

	a_2	bx_2	by_2
ax_1	3, 2	4,-4	-4, 4
ay_1	3, 2	-4, 4	4,-4
b_1	0, 0	2, 3	2, 3

Equilibria, for $1/4 \leq p \leq 3/4$:

$(p[ax_1]+(1-p)[ay_1], [a_2])$,

$([b_1], p[bx_2]+(1-p)[by_2])$,

$(0.3[ax_1]+0.3[ay_1]+0.4[b_1], 0.4[a_2]+0.3[bx_2]+0.3[by_2])$.

The proper equilibria are $(0.5[ax_1]+0.5[ay_1], [a_2])$, $([b_1], 0.5[bx_2]+0.5[by_2])$, and the totally mixed equilibrium. (Any subgame perfect equilibrium would have .50-50 x-y randomization in the (a_1,b_2) subgame.)

The only curb or absorbing block is the whole strategy space S.
All equilibria are persistent.

Extending the key idea underlying persistent equilibria and curb sets:

In familiar games, people may view some strategies as "conventional" or "normal," and so most players consider only these, disregarding other strategic alternatives.
Conformity with such social norms helps simplify people's decision-making.
Social institutions are sustained in a larger natural game by viewing such unconventional actions as "illegal" (Hurwicz 2007).

We assume that social conventions develop in accord with the following principles:

Rationality restriction: When people are generally expected to act rationally within the conventional norms, unconventional alternatives should not be advantageous.

Simplification imperative: Conventional norms will tend to simplify the game by excluding as many strategies as possible, subject to the rationality restriction.

Idea: We can relax the definitions of curb and absorbing sets by applying their best-response conditions only at Nash equilibria of the block game.
This corresponds to requiring that unconventional alternatives should not be advantageous when others are expected to behave *rationally* within the norms.

Relaxing the rationality restriction can admit more blocks as conventional norms, but then the simplification imperative to consider a *minimal* admissible block yields a stronger equilibrium refinement that can exclude more Nash equilibria.

Random consideration sets

We want to assume that players are very likely to consider only the strategies in some conventional block, but some might see more or less of the game.

Let $C(S_i)$ denote the set of all nonempty subsets of player i's pure strategy set, any one of which might be the choices that player i considers in the game.

A *random consideration-set game* on $G=(N,S,u)$ is defined by any $\mu = (\mu_1,\ldots, \mu_n)$ such that each μ_i is a probability distribution over $C(S_i)$.

Interpretation: each player i will only consider strategies in some independent random consideration set $\tilde{X}_i \subseteq S_i$ where $P(\tilde{X}_i=R_i) = \mu_i(R_i)$, $\forall R_i \subseteq S_i$.

A mixed strategy profile for a random consideration-set game μ is any τ such that $\tau_i(\bullet|R_i)$ is a probability distribution in $\Delta(R_i)$, $\forall i\in N$, $\forall R_i\in C(S_i)$.

(So $\tau_i(s_i|R_i) = 0$ if $s_i\notin R_i$.)

Each player j's expected strategy is $\tau^\mu_j(s_i) = \sum_{R_j\in C(S_j)} \mu_j(R_j)\tau_j(s_j|R_j)$.

Then τ is an *equilibrium* of the random consideration-set game μ iff $u_i(\tau^\mu_{-i}, \tau_i(\bullet|R_i)) = \max_{r_i\in R_i} u_i(\tau^\mu_{-i},[r_i])$, $\forall i\in N$, $\forall R_i\in C(S_i)$ with $\mu_i(R_i)>0$.

Fact. For any sequence of $\{(\mu,\tau)\}$ such that $\mu(T)\to 1$ and each τ is an equilibrium of the random consideration-set game μ, any limit of the expected strategies τ^μ is an equilibrium of the block game $G_T=(N,T,u)$.

Any equilibrium of the block game G_T can be expressed as such a limit (with a constant sequence).

Coarsely tenable blocks and coarsely settled equilibria

A block T is *coarsely tenable* iff there exists some $\varepsilon>0$ such that, for every equilibrium τ of every random consideration-set game μ such that $\mu_i(T_i)>1-\varepsilon$, $\forall i\in N$, we must have $u_i(\tau^\mu_{-i}, \tau_i(\bullet|T_i)) = \max_{s_i\in S_i} u_i(\tau^\mu_{-i},[s_i])$, $\forall i\in N$.

That is, when players are likely to act rationally within the T block, nobody can do better by choosing another strategy outside T.

Fact. The block of pure strategies that are not weakly dominated for each player is coarsely tenable.

Fact. If T is coarsely tenable then the Nash equilibria of its block game G_T are all Nash equilibria of the original game $G = (N,S,u)$ that have support in T.

Fact. If T is coarsely tenable then there exists a proper equilibrium of G that has support in T. (*to be shown...*)

Def. A *coarsely settled* equilibrium is any Nash equilibrium of G that has support in some minimal coarsely tenable block T (with no coarsely tenable subsets).

Example 2.

	a_2	bx_2	by_2
ax_1	3, 2	4,-4	-4, 4
ay_1	3, 2	-4, 4	4,-4
b_1	0, 0	2, 3	2, 3

In $S = \{ax_1,ay_1,b_1\}\times\{a_2,bx_2,by_2\}$, the minimal coarsely tenable blocks are:

$\{ax_1,ay_1\}\times\{a_2,bx_2,by_2\}$, with the coarsely settled equilibria

$(p[ax_1]+(1-p)[ay_1], [a_2])$ for $1/4 \le p \le 3/4$:

$\{ax_1,ay_1,b_1\}\times\{bx_2,by_2\}$ with the coarsely settled equilibria

$([b_1], p[bx_2]+(1-p)[by_2])$ for $1/4 \le p \le 3/4$.

In the tenable block $\{ax_1,ay_1\}\times\{a_2,bx_2,by_2\}$, bx_2 and by_2 are not in the support of any block-game equilibria, but they reduce the set of block-game equilibria.

$\{ax_1,ay_1\}\times\{a_2\}$ would not be coarsely tenable, because its block game has equilibria (such as (ax_1,a_2)) that are not equilibria of the whole game on S.

Other elaborated versions of the battle of sexes may have no coarsely tenable blocks other than the whole strategy space S, however. Something finer is needed!

Example 3.

	ax_2	ay_2	bx_2	by_2
ax_1	3, 2	3, 2	4,-4	-4, 4
ay_1	3, 2	3, 2	-4, 4	4,-4
bx_1	4,-4	-4, 4	2, 3	2, 3
by_1	-4, 4	4,-4	2, 3	2, 3

The Nash equilibria are, for $1/4 \le p \le 3/4$ and $1/8 \le q \le 7/8$:

$(p[ax_1]+(1-p)[ay_1],\ q[ax_2]+(1-q)[ay_2])$,

$(q[bx_1]+(1-q)[by_1],\ p[bx_2]+(1-p)[by_2])$,

$(0.3[ax_1]+0.3[ay_1]+0.2[bx_1]+0.2[by_1],\ 0.2[ax_2]+0.2[ay_2]+0.3[bx_2]+0.3[by_2])$.

The proper equilibria are $(0.5[ax_1]+0.5[ay_1],\ 0.5[ax_2]+0.5[ay_2])$, $(0.5[bx_1]+0.5[by_1],\ 0.5[bx_2]+0.5[by_2])$, and the totally mixed equilibrium.

Subgame perfection requires .5-.5 randomization in the off-diagonal subgames.

We want to identify a narrower block including only the equilibria that generalize the (a_1,a_2) equilibrium of Example 1.

Coarse tenability leaves us no way to exclude the extreme pure equilibria in the $\{ax_1,ay_1\}\times\{ax_2,ay_2\}$ block without adding in all the other strategies.

But imposing some rationality on the out-of-block deviations could narrow the scope to the proper equilibria without enlarging the block.

Finely tenable blocks and finely settled equilibria

Def. For any block T and any $\varepsilon>0$, a random consideration-set game μ is *ε-proper on T* iff, for every player i, $\mu_i(T_i) > 1-\varepsilon$, and, for every Q_i and R_i such that $\varnothing \neq Q_i \subset R_i \subseteq S_i$ with $Q_i \neq T_i$ and $Q_i \neq R_i$, we have $0 < \mu_i(Q_i) < \varepsilon\mu_i(R_i)$.

This says that any player i's consideration set is very likely to be T_i, but i may consider any nonempty subset of S_i with positive probability, and if i considers some set other than T_i then it is much more likely to be a larger set.

A block T is *finely tenable* iff there exists some $\varepsilon>0$ such that, for every equilibrium τ of every random consideration-set game μ that is ε-proper on T, we must have $u_i(\tau^\mu_{-i}, \tau_i(\bullet|T_i)) = \max_{s_i\in S_i} u_i(\tau^\mu_{-i},[s_i])$, $\forall i\in N$.

Fact. Any coarsely tenable block is also finely tenable.

Fact. Any limit as $\varepsilon\to 0$ of equilibria of random consideration-set games that are ε-proper on a finely tenable block is a proper equilibrium of the original game G.

Thus, any finely tenable block contains the support of a proper equilibrium.

Def. A *finely settled* equilibrium is any proper equilibrium of G that has support in some minimal finely tenable block.

Def. A *fully settled* equilibrium is any proper equilibrium of G that has support in a minimal finely tenable block contained in a minimal coarsely tenable block.

Fact. A fully settled equilibrium exists for any finite game G = (N,S,u).

Consider again *Example 3*.

	ax_2	ay_2	bx_2	by_2
ax_1	3, 2	3, 2	4,-4	-4, 4
ay_1	3, 2	3, 2	-4, 4	4,-4
bx_1	4,-4	-4, 4	2, 3	2, 3
by_1	-4, 4	4,-4	2, 3	2, 3

This game has no coarsely tenable blocks other than the whole strategy space, but $\{ax_1,ay_1\}\times\{ax_2,ay_2\}$ and $\{bx_1,by_1\}\times\{bx_2,by_2\}$ are minimal finely tenable blocks.

Then we get two finely and fully settled equilibria:

$(0.5[ax_1]+0.5[ay_1],\ 0.5[ax_2]+0.5[ay_2])$,

$(0.5[bx_1]+0.5[by_1],\ 0.5[bx_2]+0.5[by_2])$.

Example 4.

	a_2	b_2
a_1	3, 1	0, 0
b_1	0, 0	1, 3
c_1	2, 0	2, 0

This is the reduced normal form of an extensive game where player 1 can choose (2,0) or play another version of "battle of sexes" with player 2.

Its Nash equilibria are (a_1,a_2) and $(c_1, p[a_2]+(1-p)[b_2])$ for $0 \le p \le 2/3$.

But b_1 is strictly dominated by c_1 for player 1, and iterative elimination of weakly dominated strategies yields the *forward-induction solution* (a_1,a_2).

Here $\{a_1\}\times\{a_2\}$ is the unique minimal coarsely tenable block, and so it is also finely tenable, so the equilibrium (a_1,a_2) is fully settled.

$\{c_1\}\times\{b_2\}$ is finely tenable, so the equilibrium (c_1,b_2) is settled finely but not fully.

(Notice (c_1,b_2) corresponds to the sequential equilibrium of the extensive game in which (b_1,b_2) would be expected in the battle-of-sexes subgame.)

Example 5. Yet another elaborated version of the battle of sexes game:

	ax_2	ay_2	bx_2	by_2
ax_1	3, 2	3, 2	1,-4	-1, 4
ay_1	3, 2	3, 2	-1, 4	1,-4
bx_1	4,-1	-4, 1	2, 3	2, 3
by_1	-4, 1	4,-1	2, 3	2, 3

The equilibria are:

$(p[ax_1]+(1-p)[ay_1],\ q[ax_2]+(1-q)[ay_2])$ for $1/4 \le p \le 3/4$ and $1/8 \le q \le 7/8$,
$(q[bx_1]+(1-q)[by_1],\ p[bx_2]+(1-p)[by_2])$ for $0 \le p \le 1$ and $0 \le q \le 1$,
$(0.3[ax_1]+0.3[ay_1]+0.2[bx_1]+0.2[by_1],\ 0.2[ax_2]+0.2[ay_2]+0.3[bx_2]+0.3[by_2])$.

There is one minimal coarsely tenable block: $\{bx_1,by_1\}\times\{bx_2,by_2\}$.
So the $(q[bx_1]+(1-q)[by_1],\ p[bx_2]+(1-p)[by_2])$ equilibria are coarsely settled.

But $\{ax_1,ay_1\}\times\{ax_2,ay_2\}$ and $\{bx_1,by_1\}\times\{bx_2,by_2\}$ are finely tenable blocks, and are minimal for this property.
In each, a finely settled equilibrium can be found:

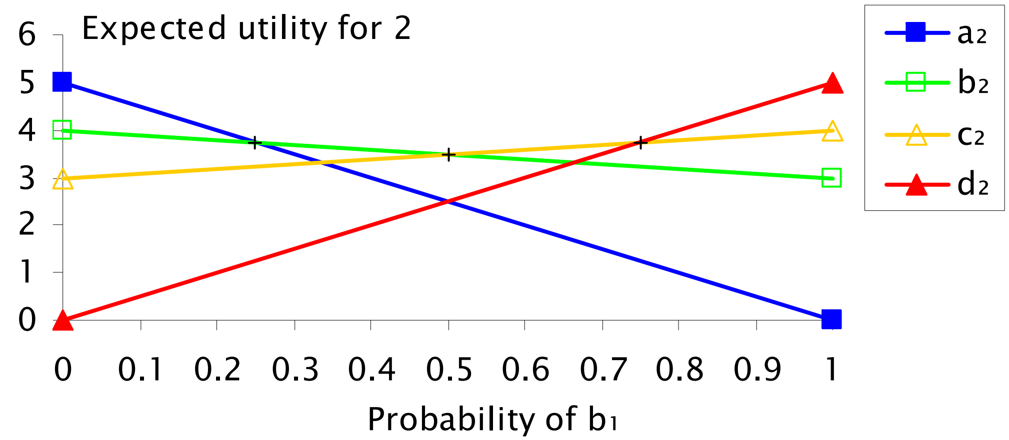
$(0.5[ax_1]+0.5[ay_1],\ 0.5[ax_2]+0.5[ay_2])$ and
$(0.5[bx_1]+0.5[by_1],\ 0.5[bx_2]+0.5[by_2])$.

Only $(0.5[bx_1]+0.5[by_1],\ 0.5[bx_2]+0.5[by_2])$ is fully settled.

A generic example where settledness excludes more than persistence

Example 6.

	a_2	b_2	c_2	d_2
a_1	0, 5	1, 4	0, 3	1, 0
b_1	1, 0	0, 3	1, 4	0, 5



This game has 3 Nash equilibria:

$(0.75[a_1]+0.25[b_1],\ 0.5[a_2]+0.5[b_2])$,
$(0.50[a_1]+0.50[b_1],\ 0.5[b_2]+0.5[c_2])$,
$(0.25[a_1]+0.75[b_1],\ 0.5[c_2]+0.5[d_2])$.

The only curb or absorbing block is the whole game, so all equilibria are persistent.

The block $\{a_1,b_1\}\times\{b_2,c_2\}$ is not coarsely tenable, as its block game has two pure equilibria to which player 2 has strictly better responses outside the block.

Both coarsely and finely: the minimal tenable blocks are $\{a_1,b_1\}\times\{a_2,b_2\}$ and $\{a_1,b_1\}\times\{c_2,d_2\}$, and so only the outer two equilibria are settled:
$(0.75[a_1]+0.25[b_1],\ 0.5[a_2]+0.5[b_2])$,
and $(0.25[a_1]+0.75[b_1],\ 0.5[c_2]+0.5[d_2])$.

Generic properties

Def. A game G = (N,S,u) is *hyper-regular* iff, for every block game T, all Nash equilibria of the block game on T are regular in the sense of van Damme (1987).

Fact. Given a finite set of players N and finite sets of strategies S_i for all players i in N, the set of utility functions u in $\mathbb{R}^{N\times S}$ such that (N,S,u) is not hyper-regular is contained in a closed set of Lesbegue measure 0 in $\mathbb{R}^{N\times S}$.

So, in this sense, almost all finite strategic games are hyper-regular.

Fact. If the game G = (N,S,u) is a hyper-regular, then any block is coarsely tenable if and only if it is finely tenable, and so any equilibrium of G is coarsely settled if and only if it is also finely and fully settled.

So all our concepts of settledness are equivalent for a generic class of finite games in strategic form.

Settledness (like persistence, but unlike perfectness and properness) is not generically equivalent to Nash equilibrium in the class of finite strategic games

An open set of utility functions around Example 6 yield games with three equilibria, all of which are persistent, but one of which is not settled in any sense here.