Conditional equilibria of multi-stage games with infinite sets of signals and actions

by

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Abstract: We develop concepts of conditional equilibria to extend Kreps and Wilson's concept of sequential equilibrium to games where the sets of actions that players can choose and the sets of signals that players may observe are infinite. A strategy profile is a conditional $\varepsilon$-equilibrium if, given any signal event that a player could observe with positive probability, the player's conditionally expected utility would be within $\varepsilon$ of the best that the player could achieve by deviating. Perfect conditional $\varepsilon$-equilibria are defined by testing conditional $\varepsilon$-rationality also under nets of small perturbations that can make any finite collection of signal events have positive probability. With topologies on actions, if a conditional $\varepsilon$-equilibrium has full support, then the perfect-perturbation tests will not be necessary to evaluate $\varepsilon$-rationality of responses to a dense class of deviations. For a large class of projective games, we prove existence of full perfect conditional $\varepsilon$-equilibria.

http://home.uchicago.edu/~rmyerson/research/seqm_notes.pdf
**Goal:** We aimed to extend a definition of *sequential equilibrium* to multi-stage games with infinite signal sets and infinite action sets, prove existence for a large class of games. It seemed natural to try to define sequential equilibria of an infinite game by taking limits of sequential equilibria of finite games that "approximate" it. But no good definition of "approximation" has been found, as sequences that "converge" to an infinite game may have limits of equilibria that seem wrong. So instead we look at limits of strategy profiles that satisfy, for each player at each date, a condition of approximate optimality among all the player's possible continuation strategies.

Nash equilibrium does not test rationality of strategies in events with probability 0, which can be problematic if a deviation could give positive probability to these untested events. For finite games, sequential equilibrium strategies are *weakly perfect* limits of totally mixed strategy profiles that satisfy conditional $\varepsilon$-rationality for each player at every information set.

We define *conditional $\varepsilon$-equilibria* by $\varepsilon$-rationality in all positive-probability events. But with uncountably infinite signal spaces, it may be impossible to give positive probability to all observable events at once, and so we cannot test rationality in all events at once. We can ask that, for any finite set of observable events, small perturbations of nature and players' strategies should be able to verify strategic rationality in these events (*perfectness*).

But even small perturbations of nature can change a game in counter-intuitive ways. With topologies on players' actions, we can ask that such perturbations should not be needed to test rationality under all possible consequences of a dense set of deviations (*fullness*). We also consider restrictions on the set of admissible perturbations of nature.

We emphasize $\varepsilon$-equilibria because $\varepsilon \to 0$ limits may violate strategic independence, as players can be coordinated by infinitesimal details of signals (*strategic entanglement*).
Multi-stage games $\Gamma = (I,T,A,S,\mathcal{M},\sigma,p,u)$.

$i \in I = \{\text{players}\}$, a finite set, $0 \notin I$. Let $I^* = I \cup \{0\}$, where 0 denotes nature (chance).

t $\in \{1,\ldots,T\} = \{\text{dates of play}\}$. $L = I \times \{1,\ldots,T\} = \{\text{dated players}\}$. We write "it" for $(i,t)$.

$S_{it} = \{\text{possible signals for player } i \text{ at } t\} \ \forall it \in L$. $A_{0t} = \{\text{nature's possible states at } t\} = \Theta_t$.

$A_{it} = \{i's \text{ possible actions at } t\} \ \forall it \in L$.

Sigma-algebras of measurable sets (closed under countable $\cap$, $\cup$ complements) are specified for $A_{0t}$, $A_{it}$, $S_{it}$. All one-point sets are measurable. Products have product sigma-algebras.

$A = \times_{t \leq T} \times_{i \in I^*} A_{it}$. $A = \{\text{possible outcomes of the game}\}$.

The subscript, $<t$, denotes the projection onto periods before $t$, and $\geq t$ weakly after.

e.g., $A_{<t} = \times_{r<t} \times_{i \in I^*} A_{ir} = \{\text{possible state-}&\&\text{-action sequences before period } t\}$. $A_{<1} = \{\emptyset\}$.

For any $a \in A$, $a_{<t} = (a_{ir})_{i \in I^*, r < t}$ is the partial sequence of states and actions before period $t$.

For any of the sets $X$ above, $\mathcal{M}(X)$ is its set of measurable subsets.

Let $\Delta(X)$ denote the set of countably additive probability measures on $\mathcal{M}(X)$.

The state at each date-$t$ is determined by nature's probability function $p_t: A_{<t} \rightarrow \Delta(A_{0t})$, a transition probability. [$\forall C \in \mathcal{M}(A_{0t})$, $p_t(C|a_{<t})$ is a measurable function of $a_{<t}$.]  

Player $i$'s date-$t$ information is given by a measurable onto signal function $\sigma_{it}: A_{<t} \rightarrow S_{it}$.

Assume perfect recall: $\forall it \in L$, $\forall r < t$, there is a measurable $\Psi_{itr}: S_{it} \rightarrow S_{ir}$ and a measurable $\psi_{itr}: S_{it} \rightarrow A_{ir}$ such that $\Psi_{itr}(\sigma_{it}(a_{<t})) = \sigma_{ir}(a_{<r})$, $\psi_{itr}(\sigma_{it}(a_{<t})) = a_{ir}, \ \forall a \in A$.

Each player $i$ has a measurable and bounded utility function $u_i: A \rightarrow \mathbb{R}$.
Behavioral strategies and induced distributions

A strategy $b_{it}$, for any $i \in L$, is any transition probability $b_{it}: S_{it} \rightarrow \Delta(A_{it})$, i.e.: $\forall s_{it} \in S_{it}$, $b_{it}(\cdot | s_{it})$ is a countably additive probability on the measurable subsets of $A_{it}$; and, $\forall C \in \mathcal{M}(A_{it})$, $b_{it}(C | s_{it})$ is a measurable function of the signal $s_{it}$.

We let $B_{it}$ be the set of strategies for $i$ at $t$, and let $B = \times_{i \in L} B_{it}$ be the set of strategy profiles. Also we may write $B_i = \times_{t \leq T} B_{it}$, $B_{\leq t} = \times_{i \in I} B_{it}$, $B_{< t} = \times_{r < t} B_{ir}$, $B_{\geq t} = \times_{r \geq t} B_{ir}$.

Given $b \in B$, we let $b_{it}$, $b_i$, $b_{\leq t}$, $b_{< t}$, or $b_{\geq t}$ denote the components of $b$ in $B_{it}$, $B_i$, $B_{\leq t}$, $B_{< t}$, or $B_{\geq t}$. For any $a_{< t}$ in $A_{< t}$ and $b_{\leq t}$ in $B_{\leq t}$, we let $P_t(\cdot | a_{< t}, b_{\leq t})$ be the measure in $\Delta(A_{\leq t})$ such that, for any measurable product set $H = \times_{i \in I} H_{it}$, $P_t(H | a_{< t}, b_{\leq t}) = p_t(H_0 | a_{< t}) \prod_{i \in I} b_{it}(H_{it} | \sigma_{it}(a_{< t}))$.

For any $b$ in $B$, we inductively define measures $P_{t}(\cdot | b)$ in $\Delta(A_{< t})$ so that $P_{1}(\emptyset | b) = 1$ and, $\forall t \in \{1, \ldots, T\}$, $\forall H \in \mathcal{M}(A_{< t+1})$, $P_{t+1}(H | b) = \int P_t(\{a_{t}: (a_{< t}, a_{t}) \in H\} | a_{< t}, b_{\leq t}) P_t(da_{\geq t} | b)$.

We may also write $P(\cdot | b; p) = P(\cdot | b)$ to indicate the dependence on the probability function $p$.

At any time $t$, the conditional expected utility for player $i$ with strategies $b$ given history $a_{< t}$ is $U_i(b | a_{< t}) = \int u_i(a_{< t}, a_{\geq t}) P_{t}(da_{\geq t} | a_{< t}, b)$. (Notice, $U_i(b | a_{< t})$ depends only on $b_{\geq t}$.)

Player $i$'s ex-ante expected utility is $U_i(b) = \int u_i(a) P(da | b) = \int U_i(b | a_{< t}) P_{t}(da_{< t} | b)$. 


Conditional ε-equilibria

∀s_{it} ∈ S_{it}, let \( \sigma_{it}^{-1}(s_{it}) = \{a_{<t} ∈ A_{<t} : \sigma_{it}(a_{<t}) = s_{it}\}. \) ∀Z ⊆ S_{it}, let \( \sigma_{it}^{-1}(Z) = \{a_{<t} ∈ A_{<t} : \sigma_{it}(a_{<t}) ∈ Z\}. \)

For any strategy profile \( b \) in \( B \), any player \( i \), any date \( t \), and any \( Z ∈ \mathcal{M}(S_{it}) \), we let \( P_{it}(Z|b) = P_{<t}(\sigma_{it}^{-1}(Z)|b) = P(\{a ∈ A : \sigma_{it}(a_{<t}) ∈ Z\}|b). \)

When \( P_{it}(Z|b) > 0 \), we may define the following conditional probabilities:

∀H ∈ \( \mathcal{M}(A_{<t}) \), \( P_{<t}(H|Z,b) = P_{<t}(H \cap \sigma_{it}^{-1}(Z)|b)/P_{it}(Z|b) \),

∀H ∈ \( \mathcal{M}(A) \), \( P(H|Z,b) = P(\{a ∈ H : \sigma_{it}(a_{<t}) ∈ Z\}|b)/P_{it}(Z|b) \).

We can also define conditional expected utility, ∀\( b ∈ B \): \( U_{i}(b|Z) = \int_{A} u_{i}(a) \text{P}(da|Z,b). \)

A signal domain is a disjoint union \( Y = \bigcup_{it ∈ L} Y_{it} \) such that \( Y_{it} ∈ \mathcal{M}(S_{it}), \forall it ∈ L. \)

A signal domain \( Y \) is \( ε \)-sure iff \( P_{it}(Y_{it}|b) ≥ 1 – ε^{2} \) ∀\( b ∈ B, ∀ it ∈ L. \)

Given any event with probability \( ε \) or greater under some strategy profile, the signals outside an \( ε \)-sure set would have conditional probability less than \( ε \) even after a deviation.

We let \( \mathcal{M}(Y_{it}) \) denote the measurable subsets of \( Y_{it} (\mathcal{M}(Y_{it}) ⊆ \mathcal{M}(S_{it})). \)

For any player \( i \) at any date \( t \), \( c_{i} ∈ B_{i} \) is a date-\( t \) continuation of \( b_{i} ∈ B_{i} \) iff \( c_{ir} = b_{ir} \) ∀\( r < t. \)

Let \( \tilde{B}_{i,\geq t}(b_{i}) \) denote the set of date-\( t \) continuations of \( b_{i}. \)

Let \( (b_{-i},c_{i}) \) denote the strategy profile that differs from \( b \) in that player \( i \) deviates to \( c_{i}. \)

A strategy profile \( b \) is a conditional \( ε \)-equilibrium on the signal domain \( Y \) iff: \( Y \) is \( ε \)-sure, and ∀\( it ∈ L, ∀ c_{i} ∈ \tilde{B}_{i,\geq t}(b_{i}), ∀ Z ∈ \mathcal{M}(Y_{it}), \) if \( P_{it}(Z|b) > 0 \) then \( U_{i}(c_{i},b_{-i}|Z) ≤ U_{i}(b|Z) + ε. \)

(Hellman (2014) constructs a game that has conditional \( ε \)-equilibria only with \( Y_{it} \neq S_{it}. \))
Perfect conditional $\varepsilon$-equilibria
To evaluate rationality of an equilibrium in 0-probability events, we must perturb it. An infinite net of perturbations of both players' strategies and nature's behavior may be needed to test all events that have zero probability, to verify *perfectness* of the equilibrium.

Given any $\delta>0$ and any strategy profile $b\in B$, $\hat{b}$ is a $\delta$-perturbation of $b$ iff $\hat{b}\in B$ and $|\hat{b}_{it}(C|s_{it}) - b_{it}(C|s_{it})| < \delta$, $\forall s_{it}\in S_{it}$, $\forall C\in M(A_{it})$, $\forall it\in L$.

Given any $\delta>0$, $\hat{p}$ is a $\delta$-perturbation of nature's probability function $p$ iff:

$\hat{p} = (\hat{p}_{1},...,\hat{p}_{T})$ where each $\hat{p}_{t}:A_{<t}\rightarrow\Delta(A_{0t})$ is a transition probability,

$|\hat{p}_{t}(C|a_{<t}) - p_{t}(C|a_{<t})| < \delta$, $\forall a_{<t}\in A_{<t}$, $\forall C\in M(A_{0t})$, $\forall t\in \{1,..,T\}$.

Let $\Gamma(\hat{p})$ denote the *perturbed game* where nature's probability function is $\hat{p}$ instead of $p$.

A *perfect conditional $\varepsilon$-equilibrium* is any $b\in B$ such that, there is an $\varepsilon$-sure signal domain $Y$ such that, for every finite subset $W\subset Y$, for every $\delta>0$, there exist $\hat{b}^{\delta,W}$ and $\hat{p}^{\delta,W}$ such that $\hat{b}^{\delta,W}$ is a $\delta$-perturbation of $b$, $\hat{p}^{\delta,W}$ is a $\delta$-perturbation of $p$,

$P_{it}(\{s_{it}\}|\hat{b}^{\delta,W},\hat{p}^{\delta,W}) > 0$, $\forall s_{it}\in W\cap Y_{it}, \forall it\in L$, and

$\hat{b}^{\delta,W}$ is a conditional $\varepsilon$-equilibrium on $Y$ in the perturbed game $\Gamma(\hat{p}^{\delta,W})$. 
Beliefs and subgame-perfectness, for a perfect conditional $\varepsilon$-equilibrium $b$.

Perfectness gives us a net of perturbations indexed on $\delta>0$ and finite sets $\mathcal{W}\subset\mathcal{Y}$, such that, conditional probabilities $P_{<t}(H|Z,\hat{b}^{\delta},\hat{\mathcal{W}};\hat{p}^{\delta},\mathcal{W})$ are defined for all $(\delta,\mathcal{W})$ such that $\mathcal{W}\cap Z\neq\emptyset$. By Tychonoff’s theorem, there is a subnet which yields well-defined belief probabilities,

$$\beta_{it}(H|Z) = \lim_{\delta,\mathcal{W}} P_{<t}(H|Z,\hat{b}^{\delta},\hat{\mathcal{W}};\hat{p}^{\delta},\mathcal{W}), \quad \forall it\in L, \forall Z\in \mathcal{M}(Y_{it}), \forall H\in \mathcal{M}(A_{<t}).$$

If $P_{it}(Z|b;p)>0$ then the beliefs $\beta_{it}(\bullet|Z)$ do not depend on the perturbations $(\hat{b}^{\delta},\hat{\mathcal{W}};\hat{p}^{\delta},\mathcal{W})\rightarrow(b;p)$.

**Facts.** The beliefs $\beta_{it}(\bullet|Z)$ are finitely additive probability distributions on $A_{<t}$.

With these beliefs, the perfect conditional $\varepsilon$-equilibrium $b$ satisfies sequential $\varepsilon$-rationality:

$$\int U_i(c_i,b_{-i}|a_{<t}) \beta_{it}(da_{<t}|Z) \leq \int U_i(b|a_{<t}) \beta_{it}(da_{<t}|Z) + \varepsilon, \quad \forall it\in L, \forall c_i\in \bar{B}_{i,\geq}(b_i), \forall Z\in \mathcal{M}(Y_{it}).$$

In examples, we can get $\beta_{it}(\bullet|Z)$ not countably additive and $\beta_{it}(H|Z) < \min_{sit\in Z} \beta_{it}(H|\{s_{it}\})$.

With perfect recall, a date-t history $a_{<t}$ in $A_{<t}$ is a *subgame* of $\Gamma$ iff $\sigma^{-1}(\sigma_{it}(a_{<t})) = \{a_{<t}\} \quad \forall i\in I$.

Given any $\varepsilon>0$, a strategy profile $b$ is a *subgame-perfect $\varepsilon$-equilibrium* of $\Gamma$ iff there exists an $\varepsilon$-sure signal domain $Y$ such that for every $it\in L$ and every subgame $a_{<t}$, if $\sigma_{it}(a_{<t})\in Y_{it}$, then $U_i(c_i,b_{-i}|a_{<t}) \leq U_i(b|a_{<t}) + \varepsilon \quad \forall c_i\in \bar{B}_{i,\geq}(b_i)$.

**Fact.** If $b$ is a perfect conditional $\varepsilon$-equilibrium then $b$ is a subgame-perfect $\varepsilon$-equilibrium.

Perturbations are used to verify the $\varepsilon$-rationality of the $b_i$ strategies in observable events $Z$ that have $P_{it}(Z|b)=0$ but could be *relevant* (have $P_{it}(Z|c_j,b_{-j})>0$) if some player $j$ deviated. But perturbations of nature can change the game in ways that seem counter-intuitive.

We next define *full* equilibria which do not need such perturbations to verify strategic rationality in all relevant events for a dense set of deviations; but this requires topology.
**Full conditional ε-equilibria**

Suppose that, for each \( i \in L \), the action set \( A_{it} \) has a given separable metric topology, and the measurable sets \( \mathcal{M}(A_{it}) \) are the Borel sets of this topology. With such topologies, we may be able to verify rationality of responses to a dense set of deviations without perturbing nature.

We say that strategy profile \( b \) has **full support** iff: \( \forall i \in L, \forall s_{it} \in S_{it}, b_{it}(C|s_{it}) > 0 \) for every set \( C \) that is a nonempty open subset of \( A_{it} \).

Full-support strategies exist, by separability (\( \exists \) countable dense set).

If \( \hat{b} \in B \) has full support then, \( \forall b \in B, (1-\varepsilon)b+\varepsilon\hat{b} \) also has full support when \( 0<\varepsilon<1 \).

A conditional ε-equilibrium \( b \) is **full** iff it has full support (with the given topologies).

For any player \( i \), given any strategy \( c_i \in B_i \) and transition probabilities \( \varphi_i : A_{it} \times S_{it} \rightarrow \Delta(A_{it}) \), we let \( c_i^*\varphi \) denote the strategy such that, \( \forall t \in \{1,\ldots,T\}, \forall s_{it} \in S_{it}, \forall C \in \mathcal{M}(A_{it}), (c_i^*\varphi)_t(C|s_{it}) = \int_{A_{it}} \varphi_t(C|a_{it},s_{it})c_{it}(da_{it}|s_{it}). \)

This \( c_i^*\varphi \) is a **δ-local perturbation** of \( c_i \) iff \( \varphi_t(\mathcal{B}_\delta(a_{it})|a_{it},s_{it}) = 1 \ \forall a_{it} \in A_{it}, \forall s_{it} \in S_{it}, \) where \( \mathcal{B}_\delta(a_{it}) \) is the ball of radius \( \delta \) around \( a_{it} \).

**Fact.** If \( b \) has full support then, \( \forall i \in I, \forall \delta>0, \) there exists a \( \varphi \) such that, \( \forall c_i \in B_i, c_i^*\varphi \) is a δ-local perturbation of \( c_i \) and \( \{H \in \mathcal{M}(A) : P(H|c_i^*\varphi,b_{-i}) > 0 \} \subseteq \{H \in \mathcal{M}(A) : P(H|b) > 0 \} \).

That is, the measure \( P(\cdot|c_i^*\varphi,b_{-i}) \) on \( \mathcal{M}(A) \) is absolutely continuous with respect to \( P(\cdot|b) \).

(Proof: Let \( \varphi_t(\cdot|a_{it},s_{it}) \) imitate \( b_{it}(\cdot|s_{it}) \) in a small \( \delta \)-neighborhood of \( a_{it} \).)

So for a full conditional ε-equilibrium \( b \), any player's deviation has arbitrarily small local perturbations such that, in all events with positive probability under the perturbed deviations, ε-rationality of responses under \( b \) is verifiable without perturbing nature or other strategies.
Limits of equilibrium distributions as $\varepsilon \to 0$

A **[perfect, full] conditional-equilibrium distribution** is any $\mu$ such that, for any finite $\mathcal{F} \subseteq \mathcal{M}(A)$, for any $\varepsilon > 0$, there exists some $b^{\varepsilon, \mathcal{F}}$ such that $b^{\varepsilon, \mathcal{F}}$ is a [perfect, full] conditional $\varepsilon$-equilibrium and $|\mu(H) - P(H|b^{\varepsilon, \mathcal{F}})| < \varepsilon \ \forall H \in \mathcal{F}$.

A conditional-equilibrium distribution is a finitely additive probability measure on the set of outcomes $A$. It might be only finitely additive (as when $\mu(\{0 < a_{it} < \delta\}) = 1 \ \forall \delta > 0$).

**Fact.** Let $\Gamma^f$ be any finite extensive-form game such that all alternatives have positive probability at each chance node (as KW assumed), with discrete topology on the finite $A_{it}$. Any full conditional $\varepsilon$-equilibrium of $\Gamma^f$ is also a perfect conditional $\varepsilon$-equilibrium. Conversely, for any $b$ that is a perfect conditional $\varepsilon$-equilibrium of $\Gamma^f$, $\forall \delta > 0$ and $\forall \varepsilon' > \varepsilon$, there exists a full conditional $\varepsilon'$-equilibrium $b'$ that is a $\delta$-perturbation of $b$.

Taking $\varepsilon \to 0$ limits, the perfect conditional-equilibrium distributions and the full conditional-equilibrium distributions both coincide with the distributions over outcomes that can result from sequential equilibria.

The fact that this coincidence of perfectness and fullness does not extend to infinite games is a basic reason why it has been so difficult to define sequential equilibria for infinite games. An uncountable infinity of outcomes cannot all get positive probability from one strategy profile, and so one must either let the strategy profile satisfy a weaker topological condition of full support, or one must consider a net of perturbed strategies that can test rationality in all events but may yield only finite additivity in the limit. Simon-Stinchcombe 1995 and Bajoori-Flesch-Vermuelen 2013&2016 use the topological full-support condition in defining solutions that they call "perfect." But "perfect" in English comes from a Latin word meaning "complete", and so it seems more appropriate for the condition of testing rationality everywhere by a net of perturbations.
Strategic entanglement in limits of approximate equilibria (Harris-Reny-Robson 1995)

**Example 1:** Date 1: Player 1 chooses $a_{11}$ from $[-1,1]$, player 2 chooses $a_{21}$ from \{L,R\}. 
Date 2: Players 3 and 4 observe the date 1 choices and each choose $a_{i2}$ from \{L,R\}.

For $i=3,4$, player $i$’s payoff is $u_i = -a_{11}$ if $i$ chooses L and $u_i = a_{11}$ if $i$ chooses R.

Player 2’s payoff depends on whether she matches 3’s choice.
If 2 chooses L then she gets 1 if player 3 chooses L but $-1$ if 3 chooses R; and
If 2 chooses R then she gets 2 if player 3 chooses R but $-2$ if 3 chooses L.

Player 1’s payoff is the sum of three terms:
(First term) If 2 and 3 match he gets $-|a_{11}|$, if they mismatch he gets $|a_{11}|$;
plus (second term) if 3 and 4 match he gets 0, if they mismatch he gets $-10$;
plus (third term) he gets $-|a_{11}|^2$.

This game has no subgame perfect equilibrium; but it has an obvious "solution" which is the limit of strategy profiles where everyone's strategy is arbitrarily close to optimal.

Approximations in which 3 and 4 can distinguish between $a_{11} = +,0,-$ and in which 1’s action set is \{-1,\ldots,-2/m,-1/m,1/m,2/m,\ldots,1\} have a unique subgame perfect equilibrium in which player 1 chooses $\pm 1/m$ with probability $\frac{1}{2}$ each, player 2 chooses L and R each with probability $\frac{1}{2}$, and players $i=3,4$ both choose L if $a_{11}=-1/m$ and both choose R if $a_{11}=1/m$.

The limiting outcome distribution is $a_{11} = 0$, $a_{21} = L$ or R each with probability $\frac{1}{2}$, and $(a_{32},a_{42}) = (L,L)$ or (R,R) each with probability $\frac{1}{2}$.

Player 3’s and player 4’s strategies are entangled in the limit, that is, independent strategies cannot yield their joint behavior in the limit distribution.
Problems of spurious signaling in naïve finite approximations

Example 2: First, nature chooses $\tilde{a}_{01}=\theta \in \{1,2\}$, with $p(1)=1/4$ and $p(2)=3/4$, while player 1 observes $s_{11}=\emptyset$ and chooses $a_{11} \in [0,1]$. Then player 2 observes $s_{22} = (a_{11})^\theta$, and then 2 chooses $a_{22} \in \{1,2\}$. Payoffs $(u_1,u_2)$ are as follows: ($a_{11}$ is payoff irrelevant)

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<th>$a_{22}$</th>
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<td>$\theta = 1$</td>
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<td>$\theta = 2$</td>
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Consider subgame perfect equilibria of any finite approximate version of the game where player 1 chooses $a_{11}$ in some $\tilde{A}_{11}$ that is a finite subset of $[0,1]$ with at least one $0 < a_{11} < 1$. Then player 1 can get expected payoff at least $1/4$ by choosing the largest feasible $\tilde{a}_{11} < 1$, as player 2 should choose $a_{22} = 1$ when $s_{22} = \tilde{a}_{11} > (\tilde{a}_{11})^2$ indicates $\theta=1$.

Player 1's expected payoff cannot be more than $1/4$, as the smallest $\tilde{a}_{11} > 0$ in his equilibrium support would lead player 2 to choose $a_{22} = 2$ when $s_{22} = (\tilde{a}_{11})^2 < \tilde{a}_{11}$ indicates $\theta=2$.

But such a scenario cannot be even an approximate equilibrium of the real game, because player 1 could get an expected payoff at least $3/4$ by deviating to $(\tilde{a}_{11})^{0.5}$.

Player 1 should get expected payoff 0 in "sequential" equilibria of the infinite game.

A full perfect conditional $\varepsilon$-equilibrium has $b_{22}(\{2\}|x) = 1-\varepsilon \ \forall x \in [0,1]$, $b_{11}(\cdot)=\text{Unif}[0,1]$; perturbations give positive probability to countably many $x$ with $\hat{b}_{11}(\{x^{0.5}\})/\hat{b}_{11}(\{x\}) \geq 1/3$.

This example shows the importance of evaluating the optimality of each player's strategy relative to his entire set of strategies in the actual game (not just some approximating game).
Why perfect conditional-rationality tests must include perturbations of nature

**Example 3:** First, nature chooses $\tilde{\alpha}_0=(\theta_1,\theta_2) \in \{1,2\} \times [0,1]$, with $\theta_1$ and $\theta_2$ independent, $P(\{\theta_1=1\}) = 1/4$, and $\theta_2$ uniform on $[0,1]$. Simultaneously, player 1 chooses $a_{11} \in [0,1]$. On date 2, player 2 observes $s_{22} = a_{11}$ if $\theta_1 = 1$, or $s_{22} = \theta_2$ if $\theta_1 = 2$, and then player 2 chooses $a_{22} \in \{1,2\}$.

Payoffs $(u_1,u_2)$ are as follows: (a$_{11}$ is payoff irrelevant)

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This game has many perfect conditional $\varepsilon$-equilibria.

In one eqm, player 1 chooses $a_{11}$ uniformly over $[0,1]$ and player 2 always chooses $a_{22}=2$. In another, player 1 chooses $a_{11}=0.5$, and player 2 chooses $a_{11}=1$ if $s_{22}=0.5$, else $a_{22}=2$.

But there cannot exist any perfect conditional $\varepsilon$-equilibrium that evaluates conditional rationality with a net of perturbations on strategies alone, without perturbing nature.

If we only perturb players' strategies then, for any $x \in [0,1]$, the event $\{s_{22}=x\}$ could have positive probability only when $\theta_1=1$ and 1's strategy assigns positive probability to $a_{11}=x$, and so then conditional $\varepsilon$-rationality for player 2 would require $\hat{b}_{22}(1|x) > 1-\varepsilon$, $\forall x \in [0,1]$.

But such a strategy would not be ex-ante rational for player 2!

Such perturbations would also verify beliefs $\beta_{22}(\{\theta_1=1\}|\{s_{22}=x\}) = 1$, $\forall x \in [0,1]$, even though the event $\{\theta_1=1\}$ has prior probability $1/4 = \beta_{22}(\{\theta_1=1\}|\{s_{22} \in [0,1]\})$. 
Problems of applying sequential rationality with finitely additive beliefs

**Example 4:** First, nature chooses $\hat{\alpha}_{01}=\theta$ uniformly from $A_{01}=(0,1)=$\{\theta: $0<\theta<1\}$. Then player 1 observes $s_{12}=\theta$ and chooses $a_{12} \in A_{12}=$\{0,1\}. Then player 2 observes $s_{23}=a_{12}$ and chooses $a_{23} \in A_{23}=[0,1)=$\{a_{23}: $0 \leq a_{23}<1\}$

If $a_{12}=0$ ("1 quits") then payoffs are $u_1 = u_2 = 0$.
If $a_{12}=1$ but $a_{23}=0$ ("2 quits") then payoffs are $u_1 = -1$ and $u_2 = 0$.
If $a_{12}=1$ and $a_{23}>0$ then $u_1 = 1$, and $u_2 = 1$ if $a_{23} \geq \theta$, but $u_2 = -1$ if $a_{23} < \theta$.

A perfect conditional $\varepsilon$-equilibrium has $a_{12}=1$, $a_{23}=1-\varepsilon/2$. (No optimum for 2 in $A_{23}=[0,1)$.)

Given $a_{12}=1$, for any countably additive conditional belief on $A_{01}$, there would exist some feasible $a_{23}<1$ worth choosing (having probability greater than $1-\varepsilon$ of satisfying $a_{23} > \theta$).

However, consider a strategy profile $b$ such that player 1 chooses $a_{12}=0$ with probability 1, but if player 1 chose $a_{12}=1$ then player 2 would choose $a_{23}=0$ with probability 1.

Consider beliefs for 2 when $a_{12}=1$ generated by a net of perturbations of $b$ indexed on $\delta>0$ in which player 1 would choose $a_{12}=1$ with conditional probability $\varepsilon$ only when $\theta > 1-\delta$.

The limiting beliefs are finitely additive, with $\beta_{23}(\{a_{23}<\theta<1\}|\{a_{12}=1\}) = 1 \forall a_{23}<1$.

So with these beliefs, player 2 would strictly prefer the quit option $a_{23}=0$ over any $a_{23}>0$.

But these perturbations are not conditional $\varepsilon$-equilibria because, for any $\delta>0$ perturbation in the net, player 2 would strictly prefer to choose $a_{23}=1-\delta\varepsilon/2$. 

Problems from allowing perturbations of nature

**Example 5:** Nature chooses $\tilde{\alpha}_{01} = (\theta_1, \theta_2) \in [-1,3] \times [-1,3]$, independent, uniform. Then player 1 observes $s_{12} = \theta_1$ and chooses $a_{12} \in \{-1, 1\}$. Then player 2 observes $s_{23} = a_{12}$ and chooses $a_{23} \in \{-1,1\}$. Payoffs are $u_1 = a_{12}a_{23}$, $u_2 = a_{23}\theta_2$.

So player 2 wants $a_{23}$ to match the sign of $\theta_2$, and player 1 wants to match $a_{23}$. Player 1 has no information about $\theta_2$, and player 2 thinks $E\theta_2 > 0$.

In one reasonable equilibrium, player 2 chooses $a_{23} = 1$ for any $s_{23}$, and player 1 chooses $a_{12} = 1$ regardless of $s_{12}$. This can be verified as a perfect conditional $\varepsilon$-equilibrium.

Add small ($<\varepsilon/2$) probabilities of each player independently trembling to the other action yields a conditional $\varepsilon$-equilibrium that is perfect and full.

Now consider a perturbation of nature that puts a small positive probability $\delta$ on the event $\theta_1 = \theta_2 = -1$, otherwise $(\theta_1, \theta_2)$ are drawn independently from the given uniform distribution. These perturbations verify a perfect conditional $\varepsilon$-equilibrium in which player 1 chooses $a_{12} = -1$ if $\theta_1 < -1$, but 1 chooses $a_{12} = 1$ if $\theta_1 = -1$, and player 2 always chooses $a_{23} = -a_{12}$.

Here 1 usually avoids $a_{12} = 1$ because 2 would take this surprise as evidence of $\theta_1 = \theta_2 = -1$. The perturbation of nature here admitted a possibility that player 1's signal $\theta_1$ might convey information about $\theta_2$, which affects beliefs even though its probability vanishes as $\delta \to 0$.

However, if we added small-probability trembles to these equilibrium strategies, to yield a full-support strategy profile, then nature's $\delta$-perturbations would not affect 2's limiting beliefs after any $s_{23} \in \{-1,1\}$, and so a **full** conditional $\varepsilon$-equilibrium cannot be so perverse.
Regular projective games

Let $\Gamma = (I,T,A,S,M,\sigma,p,u)$ be a multi-stage game (with perfect recall).

$\Gamma$ is a regular projective game iff there is a finite index set $J$ and sets $A_{ikj}$ such that, $\forall it \in L$:

(R.1) $A_{it} = \times_{j \in J} A_{itj} \ \forall i \in I^*, \ \forall t \in \{1,...,T\}$;
(R.2) there exist sets $M_{it} \subseteq I^* \times \{1,...,t-1\} \times J$ such that $S_{it} = \times_{hrj \in M_{it}} A_{hrj}$ and $\sigma_{it}(a_{<t}) = (a_{hrj})_{hrj \in M_{it}} \ \forall a_{<t} \in A_{<t}$, that is, i's signal at t is a list of state and action coordinates from periods before t;
(R.3) $A_{itj}$ are nonempty compact metric spaces $\forall j \in J$, and all products of these spaces, including the signal sets $S_{it}$, have their product topologies and Borel sigma-algebras,
(R.4) $u_i : A \rightarrow \mathbb{R}$ is continuous,
(R.5) there is a continuous nonnegative density function $f_i : A_{0t} \times A_{<t} \rightarrow [0,\infty)$ and for each $j \in J$ there is a probability measure $\rho_{ij}$ on $\mathcal{M}(A_{0tj})$, with full support on $A_{0tj}$, such that $p_t(C|a_{<t}) = \int_C f_i(a_{0t}|a_{<t}) \prod_{j \in J} \rho_{ij}(da_{0tj}) \ \forall C \in \mathcal{M}(A_{0t}), \ \forall a_{<t} A_{<t}$.

Remarks. One can always reduce the cardinality of $J$ to $(T+1)^\#I$ or less by grouping ($\forall i \in I^*, \ \forall t \in \{1,...,T\}$) the variables $\{a_{itj}\}_{j \in J}$ by the dates when each player observes them, if ever. Regular projective games can include all finite multi-stage games simply by letting each player's signal be a coordinate of the state.
Consequences of large perturbations of nature even with small probability

**Example 6:** First nature chooses $\tilde{\alpha}_{01} = (\theta_1, \theta_2) \in [0,1]^2$; with probability $1/2$, $\theta_1$ and $\theta_2$ are independent uniform $[0,1]$; with probability $1/2$, $\theta_1$ and $\theta_2$ are equal and uniform $[0,1]$. Then player 1 observes $s_{12} = \theta_1$ and chooses $a_{12} \in \{-1,1\}$. Then player 2 observes $s_{23} = a_{12}$ and chooses $a_{23} \in \{-1,1\}$. Payoffs are $u_1 = a_{12}a_{23}$, $u_2 = a_{23}(1/3 + \theta_2 - \theta_1)$.

Given any $\theta_1$, $E(\theta_2 - \theta_1 | \theta_1) = 0.5(0) + 0.5(1/2 - \theta_1) = 1/4 - \theta_1/2 \geq -1/4$.

So player 2 should prefer $a_{23} = 1$, which player 1 would prefer to match with $a_{12} = 1$.

But consider the strategy profile where $b_{12}(\theta_1) = [-1]$ if $\theta_1 \neq 1$, $b_{12}(1) = [1]$, $b_{23}(a_{12}) = [-a_{12}]$.

(Here "[x]" denotes a probability distribution with probability 1 on the point x.)

This is a perfect conditional $\epsilon$-equilibrium supported by a perturbation of nature that leaves $\theta_2$ unchanged but switches $\theta_1$ to $\theta_1 = 1$ with small probability $\delta$.

**Restricting the possible perturbations of nature may help to avoid such problems.**

Consider a regular projective game as defined above.

Given transition probabilities $\varphi_{ij}: A_{0tj} \rightarrow \Delta(A_{0tj})$, let $p*\varphi$ be a perturbation of nature such that:

$$(p*\varphi)_t(C|s_{it}) = \int_{A_{0t}} \prod_{j \in J} \varphi_{ij}(C_j|a_{0tj}) \ p_t(da_{0t}|a_{<t}), \ \forall t, \ \forall a_{<t} \in A_{<t}, \ \forall C = \times_{j \in J} C_j \in \times_{j \in J} M(A_{0tj}).$$

This $p*\varphi$ is a $\delta$-close perturbation of $p$ iff:

$$\varphi_{ij}(B_{\delta}(a_{0tj})|a_{0tj}) = 1 \ \text{and} \ \varphi_{ij}(\{a_{0tj}\}|a_{0tj}) \geq 1 - \delta, \ \forall t \in \{1,...,T\}, \ \forall j \in J, \ \forall a_{0tj} \in A_{0tj}.$$

That is, coordinates of nature should be perturbed independently, with only small probability $\delta$ of actual change in any coordinate, and with no change ever going further than distance $\delta$.

When the perfectness conditions can be satisfied with such $\delta$-close $p*\varphi$, for every $\delta > 0$, then we have a perfect conditional $\epsilon$-equilibrium with close perturbations.
Existence Theorems

**Theorem.** Given any regular projective game, for any $\varepsilon > 0$, there exists a conditional $\varepsilon$-equilibrium that is both full and perfect with close perturbations. The proof uses a finite approximating game where each $A_{ij}$ is partitioned into small sets on which the continuous functions $u_i(\theta, a) \prod_{t \in \{1, \ldots, T\}} f(a_{0i|a_{<t}})$ have small variation.

**Theorem.** Any regular projective game has a perfect conditional-equilibrium distribution $\mu$. The proof applies Tychonoff’s theorem to the compact product topology on $[0,1]^{M(A)}$. 
An example with asymptotic strategic entanglement everywhere

Example 7 (Hellman 2014): I=\{1,2\}, T=2. \ a_{01} = (\theta_0, \theta_1, \theta_2) \in A_{01} = \{1,2\} \times [0,1] \times [0,1].

\theta_0 is equally likely to be 1 or 2; it names the player who is "on". Signals are s_1=s_{12}=\theta_1, s_2=s_{22}=\theta_2.

When \theta_0=i, s_i is Uniform [0,1], other -i has signal s_{-i} = 2s_i if s_i<0.5, s_{-i}=2s_i-1 if s_i\geq0.5.

(This implies \theta_{-i} is also Uniform [0,1] when \theta_0=i.) Action sets are \ A_{12} = A_1 = \{L,R\} = A_{22} = A_2.

When \theta_0=i, the other player -i just gets u_{-i} = 0, and u_i is determined by:

when s_i\geq0.5 then i gets u_i = 0.7 if a_i=L=a_{-i}, u_i = 0.3 if a_i=R=a_{-i}, and u_i=0 if a_i\neq a_{-i};
when s_i<0.5 then i gets u_i = 0.7 if a_i=L\neq a_{-i}, u_i = 0.3 if a_i=R\neq a_{-i}, and u_i=0 if a_{i2}=a_{-i2}.

So s_i\geq0.5 wants to match -i when i is "on" and prefers L if -i's probability of R is <0.7;

s_i<0.5 wants to mismatch -i when i is "on" and prefers L if -i's probability of R is >0.3.

This game has no Bayesian equilibrium in which the strategic functions b_i(R|s_i) are measurable functions of s_i \in [0,1], by arguments of Simon (2003) and Hellman (2014).

But for any \varepsilon>0, we can construct strategy profiles that are conditionally \varepsilon-rational on an \varepsilon-sure set.

Pick an integer m\geq1 such that P(s_1<2^{-m}) < \varepsilon^2.

In the binary expansion of any s_1, find the first string of m consecutive 0's starting at an odd position, and if the number of prior 0's is odd then let b_1(R|s_1)=1, else b_1(R|s_1)=0.

In the binary expansion of any s_2, find the first string of m consecutive 0's starting at an even position, and if the number of prior 0's is odd then let b_2(R|s_2)=1, else b_2(R|s_2)=0.

In this specification, only signals s_1<2^{-m} are not best responding (to s_{i2}=2s_i or s_{i2}=2s_i-1).
The signals not covered here have probability 0, for them we can let b_i(R|s_i)=0.

Beliefs and strategies are entangled in limit: on any small open interval, 2 believes that, when \theta_0=2, the unobserved a_1 is equally likely to be L or R, but a_2 = a_1 when s_2>0.5, a_2 \neq a_1 when s_2<0.5.
The players' actions are correlated by infinitesimal details of their signal information.
Example 7 (Hellman 2014) picture

\[ u_{-i}=0 \text{ in state } \theta_0=0. \]

\[ u_i \text{ in state } \theta_0=0 \text{ if } s_i \geq 0.5: \]
\[ a_{-i}=L \quad a_{-i}=R \]
\[ a_i=L \quad 0.7 \quad 0 \]
\[ a_i=R \quad 0 \quad 0.3 \]
\[ (s_i = (1+s_{-i})/2) \]

\[ u_i \text{ in state } \theta_0=0 \text{ if } s_i < 0.5: \]
\[ a_{-i}=L \quad a_{-i}=R \]
\[ a_i=L \quad 0 \quad 0.7 \]
\[ a_i=R \quad 0.3 \quad 0 \]
\[ (s_i = s_{-i}/2) \]

Suppose 1 does L when \( s_1 < 2^{-3} \), that is \( s_1 = 0.000... \) in binary expansion (where "..." can be any string of 0s & 1s). Then rational responses for all other signal-types would imply:

- 2 does R when \( s_2 = 0.0000... \), 2 does L when \( s_2 = 0.1000... \);
- 1 does R when \( s_1 = 0.10000... \), 1 does L when \( s_1 = 0.11000... \);
- 2 does L when \( s_2 = 0.010000... \), 2 does R when \( s_2 = 0.110000... \);
- 2 does L when \( s_2 = 0.0010000... \), 2 does R when \( s_2 = 0.1010000... \);
- 2 does R when \( s_2 = 0.0110000... \), 2 does L when \( s_2 = 0.1110000... \);

In the binary expansion of \( s_1 \), find the first string of 3 consecutive 0's starting at an odd position, and if the number of prior 0's is odd then 1 does R at \( s_1 \), else 1 does L at \( s_1 \).

In the binary expansion of \( s_2 \), find the first string of 3 consecutive 0's starting at an even position, and if the number of prior 0's is odd then 2 does R at \( s_2 \), else 2 does L at \( s_2 \).
References