Abstract: We consider how to extend Kreps and Wilson's 1982 definition of sequential equilibrium to multi-stage games with infinite sets of types and actions. A concept of open sequential equilibrium is defined by taking limits of strategy profiles that can consistently satisfy approximate sequential rationality for all players at arbitrarily large finite collections of observable open events. Existence of open sequential equilibria is shown for a broad class of regular projective games. Examples are considered to illustrate the properties of this solution and the difficulties of alternative approaches to the problem of extending sequential equilibrium to infinite games.
**Goal:** formulate a definition of *sequential equilibrium* for multi-stage games with infinite type sets and infinite action sets, and prove existence for some broad class of games. Sequential equilibria were defined for finite games by Kreps-Wilson 1982, but rigorously defined extensions to infinite games have been lacking. ("Perfect Bayesian equilibrium" defined for *finite* games in Fudenberg-Tirole 1991, Harris-Stinchcombe-Zame 2000 explored definitions with nonstandard analysis.)

It is natural to try to define sequential equilibria of an infinite game by taking limits of sequential equilibria of finite games that "approximate" it. But no general definition of "good finite approximation" has been found. It is easy to define sequences of finite games that seem to be converging to the infinite game (in some sense) but have limits of equilibria that seem wrong.

Instead we look at limits of strategy profiles that are approximately optimal (among all strategies in the game) on finite sets of events that can be observed by players in the game.

A strategy profile is an $\varepsilon$-approximate sequential equilibrium on a set of observable events $\mathcal{F}$ iff it gives positive probability to each event $C$ in $\mathcal{F}$, and each player $i$ who can observe $C$ has no strategy that could improve $i$'s conditional expected payoff given $C$ by more than $\varepsilon$.

An open sequential equilibrium is defined as a limit of a net of $(\varepsilon,\mathcal{F})$-sequential equilibria as $\varepsilon \to 0$ and $\mathcal{F}$ expands to include all finite subsets of a neighborhood basis for all players' observable events.
Multi-stage games \( \Gamma = (N,K,A,\Theta,T,\mathcal{T},\mathcal{M},\tau,p,u) \)

\( i \in N = \{ \text{players} \} \), a finite set; \( k \in K = \{ 1,\ldots,#K \} = \{ \text{dates} \} \) of the game.

\( L = \{(i,k): i \in N, k \in K\} = \{ \text{dated players} \} \). We write \( i_k \) for \( (i,k) \).

\( A_{ik} = \{ \text{possible actions} \} \) for player \( i \) at date \( k \}; \) history independent.

\( T_{ik} = \{ \text{possible types} \} \) for player \( i \) at date \( k \} \) has open sets \( \mathcal{T}_{ik} \) with a countable basis.

\( \Theta_k = \{ \text{possible date } k \text{ states} \} \).

\( \sigma \)-algebras (closed under countable \( \cap \) and complements) of measurable subsets are specified for \( A_{ik}, T_{ik}, \) and \( \Theta_k \). \( T_{kj} \) has its Borel \( \sigma \)-algebra. Products are given their product \( \sigma \)-algebras.

\( A = \times_{k \leq K} \times_{i \in N} A_{ik} \). \( T = \times_{k \leq K} \times_{i \in N} T_{ik} \). \( \Theta = \times_{k \leq K} \Theta_k \). \( \Theta \times A = \{ \text{possible outcomes} \} \) of the game.

The subscript, \( <k \), denotes the projection onto periods before \( k \), and \( \leq k \) weakly before.

e.g., \( A_{<k} = \times_{h < k} \times_{i \in N} A_{ih} = \{ \text{possible action sequences before period } k \} \) \( (A_{<1} = \{ \emptyset \}) \),

and for \( a \in A \), \( a_{<k} = \times_{h < k} \times_{i \in N} a_{ih} \) is the partial sequence of actions before period \( k \).

For any of the sets \( X \) above, \( \mathcal{M}(X) \) denotes its set of measurable subsets.

Let \( \Delta(X) \) denote the set of countably additive probability measures on \( \mathcal{M}(X) \).

The date \( k \) state is determined by a regular conditional probability (r.c.p.) \( p_k \) from \( \Theta_{<k} \times A_{<k} \) to \( \Delta(\Theta_k) \). i.e., for each \( (\theta_{<k},a_{<k}) \), \( p_k(\cdot | \theta_{<k},a_{<k}) \) is in \( \Delta(\Theta_k) \), and for each \( B \in \mathcal{M}(\Theta_k) \), \( p_k(B | \theta_{<k},a_{<k}) \) is a measurable function of \( (\theta_{<k},a_{<k}) \). Nature's probability function is \( p=(p_1,\ldots,p_K) \).

Player i's date k information is given by a measurable onto type function \( \tau_{ik}:\Theta_{\leq k} \times A_{<k} \rightarrow T_{ik} \).

Assume perfect recall: \( \forall i_k \in L, \forall h < k \), there is a measurable \( \varphi_{ikh}:T_{ik} \rightarrow T_{ih} \times A_{ih} \) such that

\( \varphi_{ikh}(\tau_{ik}(\theta_{\leq k},a_{<k})) = (\tau_{ih}(\theta_{\leq h},a_{<h}),a_{ih}), \forall \theta \in \Theta, \forall a \in A. \)

Each player i has a measurable and bounded utility function \( u_i: \Theta \times A \rightarrow \mathbb{R} \).
Strategies and induced distributions

A strategy $s_{ik}$, for any $ik \in L$, is any regular conditional probability from $T_{ik}$ to $\Delta(A_{ik})$. i.e., for each $t_{ik} \in T_{ik}$, $s_{ik}(\cdot | t_{ik})$ is a countably additive probability on the measurable subsets of $A_{ik}$, and for each measurable subset $B$ of $A_{iks}$, $s_{ik}(B | t_{ik})$ is a measurable function of $t_{ik}$. Let $S_{ik}$ denote $ik$'s set of strategies, and let $S = \times_{ik \in L} S_{ik}$ denote the set of strategy profiles. Also we may write $S_i = \times_{k \in K} S_{ik}$, $S_k = \times_{i \in N} S_{ik}$, $S_{\leq k} = \times_{h \leq k} S_h$. Given any $s \in S$, we let $s_{ik}$, $s_i$, $s_k$, or $s_{\leq k}$ respectively denote the components of $s$ in $S_{ik}$, $S_i$, $S_k$, or $S_{\leq k}$.

Each $s_{ik} \in S_{ik}$ determines a regular conditional probability $P_k$ from $\Theta_{<k} \times A_{<k}$ to $\Delta(\Theta_k \times A_k)$ such that, for any measurable product set $Z = Z_0 \times (\times_{i \in N} Z_i) \subseteq \Theta_k \times A_k$, and any $(\theta_{<k}, a_{<k}) \in \Theta_{<k} \times A_{<k}$,

$$P_k(Z | \theta_{<k}, a_{<k}, s_{ik}) = \int_{\theta_k \in Z_0} \prod_{i \in N} s_{ik}(Z_i | \tau_{ik}(\theta_{\leq k}, a_{<k})) \, p_k(d\theta_k | \theta_{<k}, a_{<k}).$$

For any measurable set $B \subseteq \Theta_{\leq k} \times A_{\leq k}$, and any $(\theta_{<k}, a_{<k}) \in \Theta_{<k} \times A_{<k}$,

let $B_k(\theta_{<k}, a_{<k}) = \{(\theta_k, a_k) \in \Theta_k \times (\times_{i \in N} A_{ik}) : ((\theta_{<k}, a_{<k}),(\theta_k, a_k)) \in B\}$. For any strategy profile $s$, we inductively define measures $P_{\leq k}(\cdot | s_{\leq k})$ on $\Theta_{\leq k} \times A_{\leq k}$ so that $P_{\leq 1}(\cdot | \emptyset, \emptyset, s_{\leq 1}) = P_1(\cdot | s_1)$ and, for any $k \in \{2, \ldots, #K\}$, for any measurable set $B \subseteq \Theta_{\leq k} \times A_{\leq k}$,

$$P_{\leq k}(B | s_{\leq k}) = \int_{(\theta_{<k}, a_{<k}) \in \Theta_{<k} \times A_{<k}} P_k(B_k(\theta_{<k}, a_{<k}) | \theta_{<k}, a_{<k}, s_{ik}) P_{\leq k-1}(d(\theta_{<k}, a_{<k}) | s_{\leq k-1}).$$

Let $P(\cdot | s) = P_{\leq K}(\cdot | s)$ be the distribution over outcomes in $\Theta \times A$ induced by strategy profile $s$. We may also write $P(\cdot | s; p)$ to indicate the dependence on the probability function $p$. 
Conditional probabilities and payoffs

Any set of types for a dated player \( i_k \) may be identified with the set of outcomes in \( \Theta \times A \) that would yield such types, identifying any \( C \subseteq T_{ik} \) with \( \{ (\theta, a) : \tau_{ik}(\theta \leq k, a < k) \in C \} = \langle C \rangle \subseteq \Theta \times A \). This is an open subset of \( \Theta \times A \) when \( C \) is open in \( T_{ik} \) and \( \tau_{ik} \) is continuous.

So for any \( s \in S \), any \( i_k \in L \), and any \( C \in \mathcal{M}(T_{ik}) \), we let \( P(C|s) = P\{ (\theta, a) : \tau_{ik}(\theta \leq k, a < k) \in C \}|s) \).

Let \( Y \) denote the set of measurable subsets \( Y \) of \( \Theta \times A \). So \( Y \) is the set of all outcome events.

If \( P(C|s) > 0 \), then we may define (for any \( Y \in Y \) and any \( i \in N \)):
- **conditional probabilities** \( P(Y|C,s) = \frac{P\{ (\theta, a) \in Y : \tau_{ik}(\theta \leq k, a < k) \in C \}|s)}{P(C|s)} \),
- and **conditional expected payoffs** \( U_i(s|C) = \int_{\Theta \times A} u_i(\theta, a) P(d(\theta, a)|C,s) \).

A set \( C \subseteq T_{ik} \) is observable iff there exists some strategy profile \( s \in S \) such that \( P(C|s) > 0 \).

A type \( t_{ik} \in T_{ik} \) is essential iff every open subset of \( T_{ik} \) that includes \( t_{ik} \) is observable. Types that are not essential are contained in open neighborhoods that have probability 0 whatever strategies players choose, so they can be ignored in positive-probability events.

Let \( \bar{T}_{ik} = \{ \text{essential types in } T_{ik} \} = \text{closure}[\cup_{s \in S} \text{support}(P(\cdot|s) \text{ on } T_{ik})] \).

Let \( \mathcal{T} = \cup_{i_k \in L} T_{ik} \) (a disjoint union) denote the set of all open sets of types for dated players.

Let \( \mathcal{T}^* = \{ C \in \mathcal{T} : \exists s \in S \text{ such that } P(C|s) > 0 \} = \{ \text{open sets of types that are observable} \} \).

A neighborhood basis for the essential types is any set \( \mathcal{B} \subseteq \mathcal{T}^* \) such that \( \forall i_k \in L, T_{ik} \in \mathcal{B} \) and \( \forall t_{ik} \in \bar{T}_{ik}, \forall C \in T_{ik} \), if \( t_{ik} \in C \) then there exists some \( B \in \mathcal{B} \) such that \( t_{ik} \in B \) and \( B \subseteq C \).
(\(\varepsilon, \mathcal{F}\))-sequential equilibria and their limits

Say that \(r_i \in S_i\) is a date-\(k\) continuation of \(s_i \in S_i\) if \(r_{ij} = s_{ij}\) for all dates \(j < k\).

For any \(\varepsilon > 0\), and any \(\mathcal{F} \subseteq \mathcal{T}_*\), say that \(s \in S\) is an \((\varepsilon, \mathcal{F})\)-sequential equilibrium of \(\Gamma\) iff
\[
\forall i k \in L, \ \forall C \in \mathcal{F} \cap \mathcal{T}_{ik}, \ \forall r_i\text{ that is a date-}k\text{ continuation of }s_i:\n\]
\[
P(C|s) > 0 \text{ and } U_i(r_i, s_{-i}|C) \leq U_i(s|C) + \varepsilon.
\]
(\textit{Note:} with \(C \in \mathcal{T}_{ik}\) and \(r_i\) a date-\(k\) continuation of \(s_i\), we have \(P(C|r_i, s_{-i},) = P(C|s)\).)

A function \(\mu: \mathcal{Y} \times \mathcal{B} \rightarrow [0,1]\) is an open sequential equilibrium of \(\Gamma\) (conditioned on \(\mathcal{B}\)) iff \(\mathcal{B}\) is a neighborhood basis for the essential types, and: \(\forall \varepsilon > 0, \ \forall G \) a finite subset of \(\mathcal{Y}\), and \(\forall F\) a finite subset of \(\mathcal{B}\), there exists an \((\varepsilon, \mathcal{F})\)-sequential equilibrium \(s^{\varepsilon, \mathcal{F}, G}\) such that
\[
|\mu(Y|C) - P(Y|C, s^{\varepsilon, \mathcal{F}, G})| < \varepsilon \ \forall Y \in G, \ \forall C \in \mathcal{F}.
\]
(\textit{Note:} We get \(\mu(\bullet) = \mu(\bullet|T_{ik})\ \forall ik,\) but \(\mu\) may be only finitely additive.

We may say that \(\nu\) is the open sequential equilibrium distribution induced by \(\mu\) iff \(\nu\) is a countably additive probability measure on \(\mathcal{Y}\) and there exists a subset of \(\{Y \in \mathcal{Y}: \nu(Y) = \mu(Y)\}\) that is closed under finite intersections and generates the \(\sigma\)-algebra \(\mathcal{Y}\). (Given \(\mu\), there cannot be two such \(\nu\).)

In a finite game (with finite \(A_{ik}\) and \(T_{ik}\)), when \(\mathcal{F}\) includes every type (as a discrete open set), any \((\varepsilon, \mathcal{F})\)-sequential equilibrium satisfies \(\varepsilon\)-sequential rationality with positive probability at each type, and so yields Krep-Wilson sequential-equilibrium strategies and beliefs as \(\varepsilon \rightarrow 0\).
Strategic entanglement in limits of approximate equilibria (Harris-Reny-Robson 1995)

**Example 1:** On date 1, player 1 chooses $a_1$ from $[-1,1]$, player 2 chooses from $\{L,R\}$. On date 2, players 3 and 4 observe the date 1 choices and each choose from $\{L,R\}$. For $i \in \{3,4\}$, player $i$'s payoff is $-a_1$ if $i$ chooses $L$ but is $a_1$ if $i$ chooses $R$.

If player 2 chooses $a_2=L$ then player 2 gets $+1$ if $a_3=L$ but gets $-1$ if $a_3=R$; if player 2 chooses $a_2=R$ then player 2 gets $-2$ if $a_3=L$ but gets $+2$ if $a_3=R$.

Player 1’s payoff is the sum of three terms:

(first term) if 3 and 4 match he gets 10, if they mismatch he gets 0;

plus (second term) if 2 and 3 match he gets $-|a_1|$, if they mismatch he gets $|a_1|$;

plus (third term) he gets $-|a_1|^2$.

There is no subgame-perfect equilibrium of this game, but it has an obvious solution which is the limit of strategy profiles where everyone's strategy is arbitrarily close to optimal.

For any $\alpha>0$ and $\varepsilon>0$, when players 3 and 4 $\varepsilon$-optimize on $\{a_1<\alpha\}$ and on $\{a_1>\alpha\}$, they must each, with at least probability $1-\varepsilon/(2\alpha)$, do L on $\{a_1<\alpha\}$ and do R on $\{a_1>\alpha\}$.

To prevent player 2 from matching player 3, player 1 should lead 3 to randomize, which 1 can do optimally by randomizing over small positive and negative $a_1$.

The limiting distribution is only finitely additive, as, for any $\delta>0$, the events that player 1's action is in $\{a_1: -\delta<a_1<0\}$ or in $\{a_1: 0<a_1<\delta\}$ must each have limiting probability 1/2.

The weak* limit of players' strategies is $a_1 = 0$, and $a_i = 0.5[L]+0.5[R]$ $\forall i \in \{2,3,4\}$.

But in this limit, 3's and 4's actions are perfectly correlated independently of 1's and 2's. So players 3 and 4 seem strategically entangled in this induced equilibrium distribution.
Problems of spurious signaling in naïve finite approximations

Example 2:  Nature chooses $\theta \in \{1,2\}$, $p(\theta) = \theta/3$.
Player 1 observes $t_1 = \emptyset$ and chooses $a_1 \in [0,1]$.  
Player 2 observes $t_2 = (a_1)^{\theta}$ and chooses $a_2 \in \{1,2\}$.  Payoffs $(u_1,u_2)$ are as follows.

<table>
<thead>
<tr>
<th></th>
<th>$a_2 = 1$</th>
<th>$a_2 = 2$</th>
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<tbody>
<tr>
<td>$\theta = 1$</td>
<td>(1,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>$\theta = 2$</td>
<td>(1,0)</td>
<td>(0,1)</td>
</tr>
</tbody>
</table>

Consider subgame perfect equilibria of any finite approximate version of the game where player 1 chooses $a_1$ in some $\hat{A}_1$ that is a finite subset of $[0,1]$ including at least one $0 < a_1 < 1$.  
Then player 1 can get expected payoff at least $1/3$ by choosing the largest feasible $\bar{a}_1 < 1$, as 2 should choose $a_2 = 1$ when $t_2 = \bar{a}_1 > (\bar{a}_1)^2$ indicates $\theta = 1$.  
Player 1's expected payoff cannot be more than $1/3$, as 1's choice of the smallest $\underline{a}_1$ in his equilibrium support would lead player 2 to choose $a_2 = 2$ when $t_2 = (\underline{a}_1)^2 < \underline{a}_1$ indicates $\theta = 2$.  
But such a scenario cannot be even an approximate equilibrium of the real game, because player 1 could get an expected payoff at least $2/3$ by deviating to $(\bar{a}_1)^{0.5}$.  
Player 1 must get expected payoff 0 in sequential equilibria of the infinite game (by the logic of the preceding two sentences).
More spurious signaling in finite approximating games

Example 3 (Bargaining for Akerlof's lemons):
First nature chooses \( \theta \) from a Uniform \([0,1]\) distribution.
Player 1 observes \( t_1=\theta \), chooses \( a_1 \in [0,2] \).
Then player 2 observes \( a_1 \), chooses \( a_2 \in \{0,1\} \).
Payoffs are 
\[
    u_1(a_1,a_2,\theta) = a_2 (a_1 - \theta), \quad u_2(a_1,a_2,\theta) = a_2 (1.5 \theta - a_1).
\]

Consider any finite approximate game where player 1 has a given finite set of pure strategies and player 2 observes a given finite partition of \([0,2]\) before choosing \( a_2 \).
For any \( \delta > 0 \), we can construct a function \( f:[0,1] \to [0,1.5] \) such that:
\[
f(y) = 0 \quad \forall y \in [0,\delta), \quad f(\cdot) \text{ takes finitely many values on } [\delta,1] \text{ and, for every } x \in [\delta,1]:
\]
\[
x < f(x) < 1.5x, \quad \text{and } f(x) \text{ has probability 0 under each strategy in 1's given finite set.}
\]
Then there is a larger game where we add the strategy \( f \) for player 1 and give player 2 the ability to recognize \( a_1 \) in the finite range of \( f \).
This finite game has a perfect equilibrium where player 2 would accept any price \( f(\theta) \).

But when 2 would accept \( f(x) \) for any \( x \), player 1 could do strictly better by the strategy of choosing 
\[
a_1 = \max_{x \in [0,1]} f(x) \quad \text{for all } \theta.
\]

So we can eliminate such false equilibria by requiring approximate optimality among all strategies in the original game.
Thus we define optimality for a player relative to the player's entire set of strategies.
Problems of requiring sequential rationality tests with positive probability in all events

Example 4: Player 1 chooses $a_{11} \in \{L,R\}$.
If $a_{11} = L$, then Nature chooses $\theta \in [0,1]$ uniformly, if $a_{11} = R$, then player 1 chooses $a_{12} \in [0,1]$. Player 2 then observes $t_2 = \theta$ if $a_{11} = L$, observes $t_2 = a_{12}$ if $a_{11} = R$, and chooses $a_2 \in \{L,R\}$.

Payoffs (battle of the sexes) are as follows.

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<tr>
<th></th>
<th>$a_2 = L$</th>
<th>$a_2 = R$</th>
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<tbody>
<tr>
<td>$a_{11} = L$</td>
<td>(1,2)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>$a_{11} = R$</td>
<td>(0,0)</td>
<td>(2,1)</td>
</tr>
</tbody>
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All BoS equilibria are reasonable (the choice, $\theta$ or $a_{12}$, from $[0,1]$ is payoff irrelevant).

However, if all events that can have positive probability under some strategies must eventually receive positive probability along a sequence (or net) for "consistency," then the only possible equilibrium payoff is (2,1).

For any $x \in [0,1]$, the event $\{t_2 = x\}$ can have positive probability, but only if positive probability is given to the strategy $(a_{11} = R, a_{12} = x)$, because $\{\theta = x\}$ has probability 0. So, in any scenario where $P(\{t_2 = x\}) > 0$, player 2 should choose $a_2 = R$ when $t_2 = x$.

But then player 1 can obtain a payoff of 2 with the strategy $(a_{11} = R, a_{12} = x)$ and so the unique sequential equilibrium outcome must be (2,1)!

To allow other equilibria, we avoid sequential rationality tests on individual points. With $a_{11} = L$, all open subsets of $T_2 = [0,1]$ have positive probability and $a_2 = L$ is sequentially rational.
Problems from allowing perturbations of nature

**Example 5:** Nature chooses \( \theta=(\omega_1, \omega_2) \) independent and uniform \([-1,3]\).

Player 1 observes \( \omega_1 \), chooses \( a_1 \in \{-1, 1\} \).

Then player 2 observes \( a_1 \), chooses \( a_2 \in \{-1,1\} \).

Payoffs are \( u_1(\omega_1, \omega_2, a_1, a_2) = a_1a_2, \ u_2(\omega_1, \omega_2, a_1, a_2) = \omega_2a_2. \)

So player 2 wants \( a_2 \) to match the sign of \( \omega_2 \) and player 1 wants \( a_1 \) to match \( a_2 \).

No player has any information about \( \omega_2 \), so player 2 should think \( E(\omega_2)=1 > 0 \), and so 2 should (almost) always choose \( a_2=1 \), and so player 1 should choose \( a_1=1 \).

But if we perturbed nature to put a small positive probability on \( \{(\omega_1, \omega_2)=(-1,-1)\} \), then we could get a sequential equilibrium with \( s_1(\omega_1) = -1 \ \forall \omega_1 > -1, \ s_1(-1) = 1, \ s_2(a_1) = -a_1. \)

When player 2 will always mismatch \( a_1 \), player 1 expects to get \( u_1 = -1 \) whatever he does, and so 1 is willing to apply the strategy of choosing \( a_1=-1 \) except when \( \omega_1=-1 \); but then the perturbation of nature makes \( \{a_1=1\} \) an unlikely signal that 2 should switch to \( a_2 = -1. \)
Open sequential equilibria may not be subgame perfect if payoffs are discontinuous

Example 6: First player 1 chooses \( a_1 \in [0,1] \).
Then player 2 observes \( t_2 = a_1 \) and chooses \( a_2 \in [0,1] \).
Payoffs are \( u_1(a_1, a_2) = u_2(a_1, a_2) = a_2 \) if \( (a_1, a_2) \neq (0.5, 0.5) \), but \( u_1(0.5, 0.5) = u_2(0.5, 0.5) = 2 \).

The unique subgame-perfect equilibrium has \( s_2(a_1) = 1 \) if \( a_1 \neq 0.5 \), \( s_2(0.5) = 0.5 \), and \( a_1 = 0.5 \), with the result that payoffs are \( u_1 = u_2 = 2 \). This is an open sequential equilibrium.

But there is another open sequential equilibrium in which player 1 chooses \( a_1 \) randomly according to a uniform distribution on \([0,1]\), and player 2 always chooses \( a_2 = 1 \), applying the strategy \( s_2(a_1) = 1 \) \( \forall a_1 \in [0,1] \), and so payoffs are \( u_1 = u_2 = 1 \).

When \( a_1 \) has a uniform distribution on \([0,1]\), the observation that \( a_1 \) is in any open neighborhood around 0.5 would still imply a probability 0 of the event \( a_1 = 0.5 \), and so player 2 could not increase her conditionally expected utility by deviating from \( s_2(a_1) = 1 \).
When player 2 would always choose \( a_2 = 1 \), player 1 has no reason not to randomize.
This failure of subgame perfection occurs because sequential rationality is not being applied at the exact event of \( \{a_1 = 0.5\} \), where 2's payoff function is discontinuous.
With open sequential rationality, player 2's behavior at \( \{a_1 = 0.5\} \) is being justified by the possibility that \( a_1 \) was not exactly 0.5 but just very close to it, where she would prefer \( a_2 = 1 \).
(What if changed from usual topology on \( T_2 = [0,1] \) by adding \( \{0.5\} \) as a discrete open set?)
Such payoff discontinuities could also arise from discontinuous responses in later periods.
But subgame-perfect open sequential equilibria exist for a broad class of games.
Discontinuous responses may admit a possibility of other equilibria

Example 7 (Harris-Stinchcombe-Zame 2000):

Nature chooses \((\kappa, \theta)\in \{-1,1\}\times[0,1]\). The coordinates are independent and uniform. Then player 1 observes \(t_1 = \theta\) and chooses \(a_1\in [0,1]\). Then player 2 observes \(t_2 = \kappa|a_1 - \theta|\) and chooses \(a_2\in \{-1,0,1\}\). Payoffs are \(u_1(\kappa, \theta, a_1, a_2) = -|a_2|, \quad u_2(\kappa, \theta, a_1, a_2) = -(a_2 - \kappa)^2\).

Thus, player 2 should choose \(a_2\) to equal her expected value of \(\kappa\), and player 1 wants to hide information about \(\kappa\) from 2.

In any neighborhood of any \(t_2\neq 0\), player 2 knows \(\kappa=1\) if \(t_2>0\), and she knows \(\kappa=-1\) if \(t_2<0\), so sequential rationality implies \(s_2(t_2) = 1\) if \(t_2 > 0\), \(s_2(t_2) = -1\) if \(t_2 < 0\).

There is a sequential equilibrium in which player 1 hides information about \(\kappa\) with the strategy \(s_1(\theta) = \theta\), and player 2 does \(s_2(0) = 0\), but \(s_2(t_2) = -1\) if \(t_2<0\), and \(s_2(t_2) = 1\) if \(t_2>0\).

This equilibrium is reasonable, but 2's behavior is discontinuous at 0.

We admit another equilibrium with \(s_1(\theta) = 1\ \forall \theta\); \(s_2(t_2) = 1\) if \(t_2 > 0\), \(s_2(t_2) = -1\) if \(t_2 \leq 0\).

Player 2's behavior at 0 can be justified by considering neighborhoods \((-\varepsilon, \varepsilon)\) around 0.

Alternative models:

(7') Change from usual topology on \(T_2 = [-1,1]\) by adding \(\{0\}\) as a discrete open set.

(7'') \(\theta \in \Theta = [-2,-1] \cup [1,2], \ t_1 = |\theta|, \ t_2 = (\text{sgn}(\theta))(a_1 - |\theta|), \ u_2 = -(a_2 - \text{sgn}(\theta))^2\).
A Bayesian game where sequential rationality for all types is not possible

**Example 8 (Hellman 2014):** N={1,2}, K=1. \( \theta = (\theta_0, \theta_1, \theta_2) \in \Theta = \{1,2\} \times [0,1] \times [0,1] \). \( \theta_0 \) is equally likely to be 1 or 2; it names the player who is "on". Types are \( t_1=\theta_1, t_2=\theta_2 \). When \( \theta_0=i \), \( t_i \) is Uniform [0,1], other \( -i \) has type \( t_i = 2t_i \) if \( t_i<0.5 \), \( t_i=2t_i-1 \) if \( t_i\geq0.5 \). (This implies \( \theta_{-i} \) is also Uniform [0,1] when \( \theta_0=i \).) Action sets are \( A_1 = A_2 = \{L,R\} \). When \( \theta_0=i \), the other player \( -i \) just gets \( u_{-i} = 0 \), and \( u_i \) is determined by:

- when \( t_i\geq0.5 \) then \( i \) gets \( u_i = 0.7 \) if \( a_i=L=a_{-i} \), \( u_i = 0.3 \) if \( a_i=R=a_{-i} \), and \( u_i=0 \) if \( a_i\neq a_{-i} \);
- when \( t_i<0.5 \) then \( i \) gets \( u_i = 0.7 \) if \( a_i=L\neq a_{-i} \), \( u_i = 0.3 \) if \( a_i=R\neq a_{-i} \), and \( u_i=0 \) if \( a_i=a_{-i} \).

So \( t_i\geq0.5 \) wants to match \( -i \) when \( i \) is "on" and prefers L if \( -i \)'s probability of R is <0.7; \( t_i<0.5 \) wants to mismatch \( -i \) when \( i \) is "on" and prefers L if \( -i \)'s probability of R is >0.3.

This game has no Bayesian equilibrium in which the strategic functions \( s_i(R|t_i) \) are measurable functions of \( t_i \in [0,1] \), by arguments of Simon (2003) and Hellman (2014). But we can construct \((\varepsilon,F)\)-sequential equilibria for any \( \varepsilon>0 \) and any finite collection \( F \) of open sets of types for 1 and 2. Pick an integer \( m\geq1 \) such that \( P(t_1<2^{-m}|C) < \varepsilon \ \forall C\in F \cap T_1 \). In the binary expansion of any \( t_1 \), find the first string of \( m \) consecutive 0's starting at an odd position, and if the number of prior 0's is odd then let \( s_1(R|t_1)=1 \), else \( s_1(R|t_1)=0 \). In the binary expansion of any \( t_2 \), find the first string of \( m \) consecutive 0's starting at an even position, and if the number of prior 0's is odd then let \( s_2(R|t_2)=1 \), else \( s_2(R|t_2)=0 \). In this specification, only types \( t_1<2^{-m} \) are not best responding (to \( t_{-i}=2t_i \) or \( t_{-i}=2t_i-1 \)). The types not covered here have probability 0, for them we can let \( s_i(R|t_i)=0 \).
Regular projective games

Let $\Gamma = (N,K,A,\Theta,T,M,\tau,p,u)$ be a multi-stage game (with perfect recall and full support). $\Gamma$ is a regular projective game iff there is a finite index set $J$ and sets $\Theta_{kj}$ and $A_{ikj}$ such that, for every $ik \in L$:

(R.1) $\Theta_k = \times_{j \in J} \Theta_{kj}$ and $A_{ik} = \times_{j \in J} A_{ikj},$

(R.2) there exist sets $M_{0ik} \subseteq \{1,\ldots,k\} \times J$ and $M_{1ik} \subseteq N \times \{1,\ldots,k-1\} \times J$ such that $T_{ik} = (\times_{hj \in M_{0ik}} \Theta_{hj}) \times (\times_{nhj \in M_{1ik}} A_{nhj})$ and $\tau_{ik}(\theta_{\leq k},a_{<k}) = (((\theta_{hj})_{hj \in M_{0ik}}, (a_{nhj})_{nhj \in M_{1ik}}) \forall (\theta_{\leq k},a_{<k}),$

that is, i's type at $k$ is a list of state coordinates and action coordinates from periods up to $k$,

(R.3) $\Theta_{kj}$ and $A_{ikj}$ are nonempty compact metric spaces $\forall j \in J$, and all products of these spaces, including the type sets $T_{ik}$, have their product topologies and Borel sigma-algebras,

(R.4) $u_i: \Theta \times A \to \mathbb{R}$ is continuous,

(R.5) there is a continuous nonnegative density function $f_k: \Theta_{\leq k} \times A_{<k} \to [0,\infty)$ and for each $j \in J$ there is a probability measure $\rho_{kj}$ on $\mathcal{M}(\Theta_{kj})$ such that $p_k(B|\theta_{<k},a_{<k}) = \int_B f_k(\theta_k|\theta_{<k},a_{<k}) \prod_{j \in J} \rho_{kj}(d\theta_{kj}) \forall B \in \mathcal{M}(\Theta_k), \forall (\theta_{<k},a_{<k}) \in \Theta_{<k} \times A_{<k}.$

Remarks. One can always reduce the cardinality of $J$ to $(K+1)^\#N$ or less by grouping ($\forall ik \in L$) the variables $\{a_{ikj}\}_{j \in J}$ and $\{\theta_{kj}\}_{j \in J}$ by the dates when each player observes them, if ever. Regular projective games can include all finite multi-stage games simply by letting each player's type be a coordinate of the state.
Existence Theorems

In a regular projective game, define $\mathcal{B}^* \subseteq \mathcal{T}^*$ so that $B \in \mathcal{B}^* \cap \mathcal{T}_{ik}$ iff:

$\exists a \in A$ such that $P(B|a) > 0$, and $B = (\times_{(h,j) \in M0ik} B_{0hj}) \times (\times_{(n,h,j) \in M1ik} B_{nhj})$,

where each $B_{0hj}$ is an open subset of $\Theta_{hj}$ and each $B_{nhj}$ is an open subset of $A_{nhj}$.

Fact. $\mathcal{B}^*$ is a neighborhood basis for the essential types in the game.

We may call $\mathcal{B}^*$ the product basis.

A product partition of $\Theta \times A$ is a partition in which every element is a product of Borel subsets of the $\Theta_{kj}$ and $A_{ikj}$ sets.

Fact. For any $F$ that is a finite subset of $\mathcal{B}^*$, there exists a finite product partition $Q$ of $\Theta \times A$ such that every element of $F$ is a union of elements of $Q$.

Theorem. Let $\Gamma$ be a regular projective game and let $Q$ be any finite product partition of $\Theta \times A$. Let $F$ be a finite subset of $\mathcal{T}$ whose elements are unions of elements of $Q$. Then for any $\varepsilon > 0$, $\Gamma$ has an $(\varepsilon, F)$-sequential equilibrium.

Theorem. Any regular projective game has an open sequential equilibrium conditioned on $\mathcal{B}^*$, and any open sequential equilibrium induces an open sequential equilibrium distribution.

The proof applies Tychonoff’s theorem to the compact product topology space $[0,1]^Y$. 
Subgame perfection

A date-k history \((\theta \leq k, a < k)\) is a subgame of \(\Gamma\) iff \(\tau_{ik}^{-1}(\tau_{ik}(\theta \leq k, a < k)) = \{(\theta \leq k, a < k)\}\ \forall i \in N\).

We can define \(U_i(s|\theta \leq k, a < k) = \int_{\Theta \times A} u_i(\tilde{\theta}, \tilde{a}) P(d(\tilde{\theta}, \tilde{a})|(a < k, s \geq k); (\theta \leq k, p > k))\).

For any \(\varepsilon > 0\), a strategy profile \(s\) is an \(\varepsilon\)-subgame-perfect equilibrium of \(\Gamma\) iff for every \(i \in L\), for every subgame \((\theta \leq k, a < k)\), and for every strategy \(r_i\) that is a date-k continuation of \(s_i\), \(U_i(s|\theta \leq k, a < k) + \varepsilon \geq U_i(r_i, s_{-i}|\theta \leq k, a < k)\).

A function \(\mu: \mathcal{Y} \times \mathcal{B} \rightarrow [0,1]\) is a subgame-perfect open sequential equilibrium of \(\Gamma\) iff \(\mathcal{B}\) is a neighborhood basis for the essential types, and for \(\varepsilon > 0\), every finite subset \(G\) of \(\mathcal{Y}\), and every finite subset \(F\) of \(\mathcal{B}\), there exists an \((\varepsilon, F)\)-sequential equilibrium \(s^{\varepsilon, F, G}\) that is \(\varepsilon\)-subgame perfect and satisfies \(|\mu(Y|C) - P(Y|C, s^{\varepsilon, F, G})| < \varepsilon\ \forall Y \in G, \forall C \in F\).

In a regular projective game, if any \((\theta \leq k, a < k)\) is a subgame then all \((\tilde{\theta} \leq k, \tilde{a} < k)\) are subgames.

**Theorem.** Any regular projective game has a subgame-perfect open sequential equilibrium conditioned on \(\mathcal{B}^*\).
References