COMPARISON OF SCORING RULES IN POISSON VOTING GAMES

http://home.uchicago.edu/~rmyerson/research/

Impossibility theorems of social choice theory show no ideal voting rule can extend majority rule by a unique pure-strategy equilibrium in all social choice situations. Moving beyond such impossibilities, we should now study how the sets of equilibrium outcomes in an election may be systematically affected by changes in the voting rule. With 3 or more candidates, rational voters must ask which close races are more likely.

Condorcet cycle example: three candidates K={1,2,3}, three voter types T={**1**,**2**,**3**}, types' preferences are **1**:$1\succ2\succ3$, **2**:$2\succ3\succ1$, **3**:$3\succ1\succ2$, we expect 1/3 of each type.
With symmetry of candidates and voter types, any anonymous neutral voting rule must yield a symmetric equilibrium where all candidates have equal proby of winning.

Plurality voting (ballots permute (1,0,0)) also yields a <u>discriminatory</u> eqm where candidate 3 is not serious, types **1** and **3** voters vote (1,0,0), type **2** voters vote (0,1,0). Type **3** voters feel a vote for 3 would be wasted if there is a close {1,2} race, which is considered much more likely than a close race involving candidate 3.

Under negative voting (ballots permute (1,1,0)), no such discriminatory eqm exists.
If voters thought only {1,2} race was serious, we'd have types **1** and **3** voting (1,0,1) and type **2** voters voting (0,1,1), and so 3 would win with {1,3} as most serious race!

Another voting rule: ballots permute (1,0.4,0). Do discriminatory eqms exist?
If voters thought only {1,2} race was serious, we'd expect 2/3 voting (1,0,0.4) and 1/3 voting (0,1,0.4), and so expected scores would be (2/3, 1/3, 0.4) per voter.
Myerson-Weber APSR 1993 argue: serious race then is {1,3}, because $2/3>1/3<0.4$.
But with everybody voting (1,0,0.4) or (0,1,0.4), 1's and 2's scores always average 1/2, while 3's score is always 0.4, and so 3 could not be involved in a close race with any positive turnout, thus confirming our assumption of {1,2} as only serious race.
Thus, we should find that this intermediate voting rule yields discriminatory equilibria for Condorcet cycle, like plurality but unlike negative voting.
The simple Myerson-Weber criterion for identifying serious races is not consistent with any reasonable probability model here.

To avoid such inconsistencies when we identify eqms under different voting rules, we should use some reasonable and tractable probability model in which the probabilities of different close races can be estimated and compared, given any voting strategy. Poisson games may be most tractable (Myerson, IJGT 1998, GEB 1998, JET 2000.)

General formulation.
In an <u>(A,B)-scoring rule</u> for three-candidate elections, each voter must choose a vote vector that is a permutation either of (1,B,0) or (1,A,0), where $0 \leq A \leq B \leq 1$.
Vote vectors are summed; winner has most total points; randomize if tie.
We consider extreme cases: A=B=0 is <u>plurality</u> voting; A=B=1 is <u>negative</u> voting; A=0 with B=1 is <u>approval</u> voting. (A=B=1/2 is <u>Borda</u> voting.)

In voting game, let $K = \{\text{candidates}\} = \{1,2,3\}$,
$n = E(\text{number of voters})$, $T = \{\text{types}\}$, $C = \{\text{permissible ballots}\}$
$u_i(t)$ = (utility for type t voter if candidate i wins), $r(t)$ = (prob'y that voter has type t).

Let $Z(C) = \{x \in \mathbb{R}^C \mid x(c) \text{ is a nonnegative integer}, \forall c \in C\}$ = {<u>vote profiles</u>}.
In a game of expected size n, beliefs about the behavior of other players may be described by some <u>expected vote profile</u> $n\tau_n = (n\tau_n(c))_{c \in C}$, where τ_n in $\Delta(C)$ is the probability distribution for the vote of a randomly sampled player in this game.
Candidate i's expected score per voter is then $S_i(\tau) = \sum_{c \in C} c_i \, \tau(c)$.
Prob'y of vote profile $x = (x(c))_{c \in C}$ is $P(x|n\tau_n) = \prod_{c \in C} \exp(-n\tau_n(c))(n\tau_n(c))^{x(c)} / x(c)!$
(Note independent actions in Poisson games.)

Let $W(x) = \operatorname{argmax}_{j \in K} \sum_{c \in C} x(c)c_j$ = (win set at x).
For each candidate i, let $Q_i(x) = 1/\#W(x)$ if $i \in W(x)$, $Q_i(x) = 0$ if $i \notin W(x)$.

A <u>strategy function</u> for players is any $\sigma: T \rightarrow \Delta(C)$. ($\sigma(c|t)$ = prob'y of type t voting c)
A strategy σ_n generates expected vote probabilities $\tau_n(c) = \sum_{t \in T} r(t)\sigma_n(c|t)$, $\forall c \in C$.
Corresponding win probabilities are $q_n = (\sum_{x \in Z(C)} P(x|n\tau_n) \sum_i Q_i(x))_{i \in K}$.
So σ_n is an <u>equilibrium</u> for a game of size n iff σ_n generates a vote distn τ_n such that, for each c and t, $\sigma(c|t) > 0$ implies $c \in \operatorname{argmax}_{d \in C} \sum_{x \in Z(C)} P(x|n\tau_n) \sum_i Q_i(x+[d]) \, u_i(t)$.
(Here $[d](d)=1$, $[d](c)=0$ if $c \neq d$.) (Note environmental equiv'ce of Poisson games.)

Refinement: for each type t, do not use any weakly dominated d such that, for some c, $\sum_i Q_i(x+[d]) \, u_i(t) \leq \sum_i Q_i(x+[c]) \, u_i(t)$, $\forall x \in Z(C)$, with "<" for some x.

A <u>large equilibrium</u> is any limit of (σ_n, τ_n, q_n) for such equilibria $\{\sigma_n\}$ as $n \rightarrow \infty$.

Let $\Omega(i) = \{x \in Z(C) | \ q_i(x) > 0\}$ = (event of i being a winner or tied to win).
Let $\Lambda(c,i,j) = \{x \in Z(C) | \ q_i(x) < q_i(x+[c]), q_j(x) > q_j(x+[c])\}$.
=(the event that one more c ballot could change the winner from i to j).
Let $\Lambda(i,j) = \cup_{c \in C} (\Lambda(c,i,j) \cup \Lambda(c,j,i))$ = (event of a close race between i and j).
Let $D = \{\{i,j\} | \ u_i(t) \neq u_j(t)$ for some $t\}$ = (distinct pairs of candidates).
Let $\Lambda^* = \cup_{(i,j) \in D} \Lambda(i,j)$ denote the event of a close race among distinct candidates.
Rational voters of any type t only care about their votes when there is a close race:
$\text{argmax}_{d \in C} \sum_{x \in Z(C)} P(x|n\tau_n) \sum_i Q(i|x+[d]) u_i(t)$
$= \text{argmax}_{d \in C} \sum_{x \in \Lambda^*} P(x|n\tau_n) \sum_i Q(i|x+[d]) u_i(t)$

We say that the {i,j} race is serious in a large equilibrium iff $\{i,j\} \in D$ and
$\limsup_{n \to \infty} P(\Lambda(i,j)|n\tau_n) / P(\Lambda^*|n\tau_n) > 0$.
A candidate i is serious iff $\exists j$ such that the {i,j} race is serious.
A candidate is out of contention iff he is not serious.
A large equilibrium is discriminatory iff some candidate is not serious.

Given a sequence of expected vote distributions $\{\tau_n | n \to \infty\}$, let
$\mu_i = \lim_{n \to \infty} \log_e(P(\Omega(i)|n\tau_n)) / n$
$\mu_{i,j} = \lim_{n \to \infty} \log_e(P(\Lambda(i,j)|n\tau_n)) / n$

(We might say that a candidate i is strong iff i's winning has substantial probability, in the sense that $\limsup_{n \to \infty} P(\Omega(i)|n\tau_n) > 0$, or $\mu_i = 0$.)

Let $\psi(\theta) = \theta(1-\log(\theta))-1$, $\forall\theta>0$; $\psi(0) = -1$. ψ is concave, $\psi'(\theta) = -\log_e(\theta)$.
For $M\subseteq Z(C)$, let $M/(n\tau_n) = \{\alpha\in\mathbb{R}_+^C \mid (n\tau_n(c)\alpha(c))_{c\in C} \in M\}$ = {offset ratios in M}.
We may write $n\tau_n\alpha = (n\tau_n(c)\alpha(c))_{c\in C}$. So $\alpha \in M/(n\tau_n)$ iff $n\tau_n\alpha \in M$.

Magnitude Thm. Given $M \subseteq Z(C)$, the magnitude of M is
$\mu(M) = \lim_{n\to\infty} \log(P(M|n\tau_n))/n$
$= \lim_{n\to\infty} \max_{y\in M} \log(P(y|n\tau_n))/n = \lim_{n\to\infty} \max_{\alpha\in M/(n\tau_n)} \sum_{c\in C} \tau_n(c)\, \psi(\alpha(c))$.
All probability in M is concentrated in limit in region where max is achieved.

Offset Thm. Suppose offset vector α is the limit of solutions to magnitude problem for M. Then for any integer k, $\lim_{n\to\infty} P(M-k[d]|n\tau_n)/P(M|n\tau_n) = \alpha(d)^k$.

When M is a cone, $M/(n\tau) = M/\tau$.
Dual Magnitude Thm. Let τ be a given vote distribution, and let M be cone defined by

$$M = \{x\in\mathbb{R}_+^C \mid \sum_{c\in C} b_k(c)\, x(c) \geq 0\ \forall k\in L\},$$

where L is a finite set, and the numbers $b_k(c)$ are given $\forall k\ \forall c$.
Suppose that λ in $\mathbb{R}_+^L$ is an optimal solution to the dual problem

$$\text{minimize}_\lambda\ \sum_{c\in C} \tau(c)(\exp(\textstyle\sum_k \lambda_k\, b_k(c))-1) \ \text{ subject to }\ \lambda_k \geq 0\ \forall k\in L.$$

Then letting

$$\alpha(c) = \exp(\textstyle\sum_k \lambda_k\, b_k(c)),\ \forall c\in C,$$

yields the optimal solution to the magnitude problem

$$\text{maximize}_{\alpha\in M/\tau} \sum_{c\in C} \tau(c)\, \psi(\alpha(c)),$$

and the optimal values of the objectives in these two problems are equal.

Fact. Given two disjoint populations of expected size $n\gamma_n$ and $n\nu_n$ (in each game of expected size n), the event M that these two populations are equal has magnitude
$\mu(M) = \lim_{n\to\infty} \log_e(P(M|n\tau_n))/n = \lim_{n\to\infty} -(\gamma_n{}^{1/2} - \nu_n{}^{1/2})^2$.
This magnitude is achieved where the both populations' sizes are near $n(\gamma_n\nu_n)^{1/2}$.
(Pf: $\gamma_n\,(e^\lambda-1) + \nu_n\,(e^{-\lambda}-1)$ is minimized over λ when $e^\lambda = (\nu_n/\gamma_n)^{1/2}$.)

Fact. The event M that a population of expected size $n\nu_n$ completely fails to appear has magnitude $\mu(M) = \lim_{n\to\infty} -\nu_n$. (No cone has magnitude less than -1.)

Fact If x has offset α from $\omega = n\tau_n$ then $\partial\log P(x|\omega)/\partial\omega(c)$
$= \partial(-\omega(c) + x(c)\log(\omega(c)) - \log(x(c)!))/\partial\omega(c) = x(c)/\omega(c) - 1 = \alpha(c) - 1$.

Fact. Near its expected value (offset ratios close to 1), a Poisson random variable is approximately integer-rounded Normal, with stddev = $(\text{mean})^{1/2}$.

Prop 1 For any (A,B)-rule with $B < 0.5$, for any pair of candidates $\{i,j\}$, if all voters have strict preferences on $\{i,j\}$ and neither i nor j is universally preferred over the other, then a discriminatory equilibrium exists in which $\{i,j\}$ is the only serious race.
(Recall that B denotes the upper bound on points for the middle candidate.)

Example 1 ("Above the Fray")
Candidates = $\{1,2,3\}$. T = $\{\mathbf{1},\mathbf{2}\}$.
$u(\mathbf{1}) = (6,0,9)$, $r(\mathbf{1}) = 0.5$, $u(\mathbf{2}) = (0,6,9)$, $r(\mathbf{2}) = 0.5$
In this case, the Pareto-dominant outcome cannot be guaranteed in equilibrium if there exists a discriminatory equilibrium where candidate 3 is not serious.
Such a discriminatory equilibrium exists in our scoring rule if and only if the upper bound on the number of points for the middle candidate satisfies $B < 0.5$,
This discriminatory eqm then has $\tau(1,0,B) = \tau(0,1,B) = 0.5$, $\mu_{1,2} = 0$, $\mu_{1,3} = \mu_{2,3} = -1$.

PLURALITY voting yields a discriminatory equilibrium with 3 not serious:
$\sigma(1,0,0|\mathbf{1}) = 1$, $\sigma(0,1,0|\mathbf{2}) = 1$. Then $\tau(1,0,0) = \tau(0,1,0) = 0.5$, $S = (0.5, 0.5, 0)$,
$\mu_{1,2} = 0$, $\mu_{1,3} = -1 = \mu_{2,3}$, and so only $\{1,2\}$ is serious.
Of course, plurality also yields an equilibrium where 3 is the likely winner:
$\sigma(0,0,1|\mathbf{1}) = 1 = \sigma(0,0,1|\mathbf{2})$. $\tau(0,0,1) = 1$, $S = (1,0,0)$, $\mu_{i,j} = -1$ for all $\{i,j\}$.

Under APPROVAL voting, the unique equilibrium is one where 3 is the likely winner:
$\sigma(0,0,1|\mathbf{1}) = 1 = \sigma(0,0,1|\mathbf{2})$. Then $\tau(0,0,1) = 1$, $S = (0, 0, 1)$, $\mu_{i,j} = -1$ $\forall\{i,j\}$.
(Uniqueness: Candidate 3 must get unanimous approval by dominance.
If anybody also voted for 1 or 2 then the only serious races would be with 3, and so nobody should approve 1 or 2.)
So even though the unique approval-voting equilibrium uses only single votes that are feasible in plurality, the possibility of double votes changes the equilibrium set.

Under NEGATIVE voting, the unique equilibrium is one where 3 is the likely winner:
$\sigma(1,0,1|\mathbf{1}) = 1$, $\sigma(0,1,1|\mathbf{2}) = 1$.   Then $\tau(1,0,1) = \tau(0,1,1) = 0.5$, $S = (0.5, 0.5, 1)$,
$\mu_{1,3} = \mu_{2,3} = -0.5$, $\mu_{1,2} = -1$. $\{1,3\}$ and $\{2,3\}$ are serious races.
(Uniqueness: By dominance, nobody votes against 3. If we expected fewer against 1 than against 2 then only $\{1,3\}$ would be serious, but then all should vote against 1.)

In Above-Fray under negative voting, what if r(**1**) = 0.6, r(**2**)=0.4?

Example 1b ("Nonsymmetric Above the Fray")

Candidates = {1,2,3}. T = {**1**,**2**}.

u(**1**) = (6,0,9), r(**1**) = 0.6, u(**2**) = (0,6,9), r(**2**) = 0.4

Under NEGATIVE voting, **1**'s must randomize between voting against 1 or 2, which they'll do only if 1 is slightly greater threat against 3.

Eqm expected vote distn is of form: $\tau_n(1,0,1) = 0.5 + \varepsilon_n$, $\tau(0,1,1) = 0.5 - \varepsilon_n$, yielded by $\sigma_n(1,0,1|\mathbf{1}) = 5/6 + \varepsilon_n/0.6$, $\sigma(0,1,1|\mathbf{1}) = 1/6 - \varepsilon_n/0.6$, $\sigma(0,1,1|\mathbf{2}) = 1$.

$\mu_{1,3} = \mu_{2,3} = -0.5$, $\mu_{1,2} = -1$. {1,3} and {2,3} are serious races.

We need $P(\Lambda(1,3)|n\tau_n)/P(\Lambda(2,3)|n\tau_n) = (9-0)/(9-6) = 3$ to make **1**'s randomize.

When $\varepsilon_n = 0$, get $\log P(\Lambda(1,3)|n\tau_n) - \log P(\Lambda(2,3)|n\tau_n) = 0$

Want $\log P(\Lambda(1,3)|n\tau_n) - \log P(\Lambda(2,3)|n\tau_n) = \log(3)$.

Fact If x has offset α from $\omega = n\tau_n$ then

$\partial \log P(x|\omega)/\partial\omega(c) = \partial(-\omega(c) + x(c)\log(\omega(c)) - \log(x(c)!))/\partial\omega(c) = x(c)/\omega(c) - 1$
$= \alpha(c) - 1$.

At $\Lambda(1,3)$ have $\alpha(0,1,1)=0$, $\alpha(1,0,1)=1$.

At $\Lambda(2,3)$ have $\alpha(0,1,1)=1$, $\alpha(1,0,1)=0$.

So need $(0-1)(-n\varepsilon_n) - (0-1)(n\varepsilon_n) = \log(3)$

Implies $\varepsilon_n = \log(3)/(2n) = 0.55/n$ in eqm with this Poisson model.

Prop 2 For any (A,B)-rule, if $A \geq 0.5$ then discriminatory equilibria do not exist for three-candidate elections where all voters have strict preferences.
(Recall that A denotes the lower bound on points for the middle candidate.)

Example 2 ("One Bad Apple")
Candidates = {1,2,3}. T = {**1**,**2**}. u(**1**) = (9,6,0), r(**1**) = 0.5, u(**2**) = (6,9,0), r(**2**) = 0.5.
In this case, the majority-preferred outcome can be guaranteed only if there exists a discriminatory equilibrium where candidate 3 is not serious.
Such a discriminatory equilibrium exists in our scoring rule if and only if the lower bound on the number of points for the middle candidate satisfies $A < 0.5$.
This discriminatory eqm then has $\tau(1,0,A) = \tau(0,1,A) = 0.5$, $\mu_{1,2} = 0$, $\mu_{1,3} = \mu_{2,3} = -1$.

Under any (A,B)-scoring rule with $A<0.5$, such as PLURALITY or APPROVAL, we get an equilibrium: $\sigma(1,0,A|\mathbf{1}) = 1$, $\sigma(0,1,A|\mathbf{2}) = 1$, $\tau(1,0,A) = \tau(0,1,A) = 0.5$, $S = (0.5, 0.5, A)$, $\mu_{1,2} = 0$, $\mu_{1,3} = \mu_{2,3} = -1$, and so 3 is not serious.

But with NEGATIVE VOTING, if candidate 3 was not serious then
1s would vote (1,0,1) and **2**s would vote (0,1,1), and so candidate 3 would win!

The unique equilibrium with negative voting has limit
$\sigma(1,0,1|\mathbf{1}) = 2/3$, $\sigma(1,1,0|\mathbf{1}) = 1/3$, $\sigma(0,1,1|\mathbf{2}) = 2/3$, $\sigma(1,1,0|\mathbf{1}) = 1/3$.
Then $\tau(1,1,0) = \tau(1,0,1) = \tau(0,1,1) = 1/3$, $S = (2/3, 2/3, 2/3)$, all races serious.
(So major offsets are 1 at close races.) For voters to be willing to randomize, relative pivot probabilities $p_{ij} = \lim_{n\to\infty} P(\Lambda(i,j)|n\tau_n)/P(\Lambda^*|n\tau_n)$ must satisfy
$9p_{13} + 6p_{23} = 3p_{12} - 6p_{23}$ and $6p_{13} + 9p_{23} = 3p_{12} - 6p_{13}$,
which implies $p_{13} = p_{23} = p_{12}/7$.
For the game of expected size n, we may look for an equilibrium of the form
$\sigma(1,0,1|\mathbf{1}) = 2/3-\varepsilon_n$, $\sigma(1,1,0|\mathbf{1}) = 1/3+\varepsilon_n$, $\sigma(0,1,1|\mathbf{2}) = 2/3-\varepsilon_n$, $\sigma(1,1,0|\mathbf{1}) = 1/3+\varepsilon_n$.
so $\tau_n(1,1,0) = 1/3 + \varepsilon_n$, $\tau_n(1,0,1) = 1/3 - 0.5\varepsilon_n = \tau_n(0,1,1)$.
$S = (2/3+0.5\varepsilon_n, 2/3+0.5\varepsilon_n, 2/3-\varepsilon_n)$.
Near the expected vote profile, the number of votes against each candidate can be approximated by the integer-rounding of a Normal distribution with standard deviation equal to the square root of the expected number of votes ($\approx (n/3)^{1/2}$).
This approximation yields relative pivot probabilities as above when $\varepsilon_n = 0.628n^{-1/2}$.
Simulation shows that P(3 wins) ≈ 0.044 in limit as $n\to\infty$.
If n = 9,000,000, get $n\tau_n(1,1,0) = 3,001,884$, $n\tau_n(1,0,1) = n\tau_n(0,1,1) = 2,999,058$.

Example 3 (replicated Condorcet cycle)

Candidates = {1,2,3}, T={**1**,**2**,**3**},
u(**1**) = (9,6,0), r(**1**) = 1/3, u(**2**) = (0,9,6), r(**2**) = 1/3, u(**3**) = (6,0,9), r(**3**) = 1/3.

Under any (A,B)-scoring rule, this symmetric example has a symmetric equilibrium with $\sigma(1,B,0|\mathbf{1}) = \sigma(0,1,B|\mathbf{2}) = \sigma(B,0,1|\mathbf{3}) = 1$. $\tau(1,B,0) = \tau(0,1,B) = \tau(B,0,1) = 1/3$, $\mu_{1,2} = \mu_{1,3} = \mu_{2,3} = 0$, all races serious, each candidate wins with probability 1/3.

Under PLURALITY voting, we also have three discriminatory equilibrium, one where each candidate is out of contention. The equilibrium with 3 not serious is
$\sigma(1,0,0|\mathbf{1}) = 1 = \sigma(1,0,0|\mathbf{3})$, $\sigma(0,1,0|\mathbf{2}) = 1$. $\tau(1,0,0) = 2/3$, $\tau(0,1,0) = 1/3$,
$S = (2/3, 1/3, 0)$, $\mu_{1,2} = -((2/3)^{1/2} - (1/3)^{1/2})^2 = -0.0572$,
$\mu_{1,3} = \mu_{2,3} = -1$. Only {1,2} race is serious. 1 is likely winner.

Under NEGATIVE voting, the symmetric equilibrium is unique.
Sketch of uniqueness proof:
If only {1,2} were serious, everybody would vote against 1 or 2, so 3 would win.
If the serious races were {1,2} and {1,3}, then all **2**s would vote (0,1,1),
so $\tau(0,1,1) \geq 1/3$, and so candidate 1 could not be the unique likely winner.

Under APPROVAL voting, this Condorcet-cycle example has no discriminatory equilibria, and all races must be serious in equilibrium.

Ex 3: $u(\mathbf{1}) = (9,6,0)$, $r(\mathbf{1}) = 1/3$, $u(\mathbf{2}) = (0,9,6)$, $r(\mathbf{2}) = 1/3$, $u(\mathbf{3}) = (6,0,9)$, $r(\mathbf{3}) = 1/3$.

...Approval voting yields no discriminatory equilibria for this Condorcet-cycle example.
Proof:
An equilibrium where the only serious race is {1,2} must have, for some ρ in $[0,1]$,
$\sigma(1,0,0|\mathbf{1}) = 1 = \sigma(1,0,1|\mathbf{3})$, $\sigma(0,1,0|\mathbf{2}) = 1-\rho$, $\sigma(0,1,1|\mathbf{2}) = \rho$.
Then $\tau(1,0,0) = 1/3 = \tau(1,0,1)$, $\tau(0,1,0) = (1-\rho)/3$, $\tau(0,1,1) = \rho/3$.
The event of 3 winning has magnitude $\mu_3 = -((1/3)^{1/2} - (\rho/3)^{1/2})^2$, achieved at
$\alpha\tau(1,0,0) = \rho^{1/2}/3 = \alpha\tau(0,1,1)$, $\alpha\tau(1,0,1) = 1/3$, $\alpha\tau(0,1,0) = (1-\rho)/3$.

If $0<\rho<1$, then $\alpha\tau$ yields expected scores such that $S_1(\alpha\tau) = S_3(\alpha\tau) > S_2(\alpha\tau)$,
so $\mu_3 = \mu_{1,3} > \mu_{2,3}$, so **2**s should vote (0,1,1), implying $\rho=1$.

Let $L = \Lambda(1,3) \cap \Lambda(2,3)$ (close threeway race).
If $\rho = 1$, then $\mu_3 = \mu_{1,3} = \mu_{2,3} = \mu(L)$,
all achieved at offset ratios $\alpha(1,0,0) = 0$, $\alpha(1,0,1) = 1 = \alpha(0,1,0)$;
but $\Lambda(1,3) = \cup_{k\geq 0} (L - k[0,1,0])$ and $\Lambda(2,3) = \cup_{k\geq 0} (L - k[1,0,0])$,
and so $\lim_{n\to\infty} P(\Lambda(2,3)|n\tau_n)/P(\Lambda(1,3)|n\tau_n) = (1-\alpha(0,1,0))/(1-\alpha(1,0,0)) = 0$;
and so **2**s should vote (0,1,1), again implying $\rho=1$.

This $\rho=1$ means that, if a discriminatory equilibrium with 3 not serious exists, it must have type **2** voters giving the most middle points (here B=1) to candidate 3, because 3 is more likely to be in a close race with 1 (the likely winner) than with 2.

But with $\rho=1$, $S = (2/3, 1/3, 2/3)$, so $\mu_{1,3} = 0 > \mu_{1,2}$, a contradiction.

Approval voting also yields no equilibria with two serious races for this example.
Proof: If the serious races were {1,2} and {1,3}, then **1**s would vote (1,0,0), **2**s would vote (0,1,1), and **3**s would vote (0,0,1) or (1,0,1). So 2 could not win with any **3** turnout, so $\mu_{1,2} = -1/3 < \mu_{1,3} = -((2/3)^{1/2} - (1/3)^{1/2})^2 = -0.0572$, a contradiction.

Ex 3: $u(\mathbf{1}) = (9,6,0)$, $r(\mathbf{1}) = 1/3$, $u(\mathbf{2}) = (0,9,6)$, $r(\mathbf{2}) = 1/3$, $u(\mathbf{3}) = (6,0,9)$, $r(\mathbf{3}) = 1/3$.

Search for discriminatory equilibria of Condorcet cycle for general (A,B)-scoring rules.
An equilibrium where the only serious race is $\{1,2\}$ must have, for some ρ in $[0,1]$,
$\sigma(1,0,A|\mathbf{1}) = 1 = \sigma(1,0,B|\mathbf{3})$, $\sigma(0,1,A|\mathbf{2}) = 1-\rho$, $\sigma(0,1,B|\mathbf{2}) = \rho$.
$\tau(1,0,A) = 1/3 = \tau(1,0,B)$, $\tau(0,1,A) = (1-\rho)/3$, $\tau(0,1,B) = \rho/3$.
Must have $\rho=1$ because **2**'s prefer $(0,1,B)$ when $P(\Lambda(2,3)|n\tau_n) \leq P(\Lambda(1,3)|n\tau_n)$.

To show this inequality, look at dual of magnitude problem for $\Omega(3)$,
where λ_i is Lagrange multiplier for constraint "(3's score) $\geq$ (i's score)"

$$\begin{aligned} &\min_{\lambda\geq(0,0)} [\exp((A-1)\lambda_1+A\lambda_2) + \exp((B-1)\lambda_1+B\lambda_2) \\ &\qquad + (1-\rho)\exp(A\lambda_1+(A-1)\lambda_2) + \rho\exp(B\lambda_1+(B-1)\lambda_2]/3 - 1 \\ &= \exp(A(\lambda_1+\lambda_2))\,[\exp(-\lambda_1) + (1-\rho)\exp(-\lambda_2)]/3 \\ &\qquad + \exp(B(\lambda_1+\lambda_2))\,[\exp(-\lambda_1) + \rho\exp(-\lambda_2)]/3 - 1 \end{aligned}$$

If $\lambda_1 < \lambda_2$ then switching the values of λ_1 and λ_2 would strictly reduce this objective.
So $\lambda_1 \geq \lambda_2 \geq 0$ at optimal solution, not both 0.
Case 1, $\lambda_1 > \lambda_2 = 0$: Most-likely 3 win is when 1 is in close race with 3, but 2 is not.
Case 2, $\lambda_1 \geq \lambda_2 > 0$: Most-likely 3 win is when both 1 and 2 are in close race.
In this region, adding k (1,0,B) votes and subtracting k (0,1,B) votes to go from $\Lambda(2,3)$ to $\Lambda(1,3)$ would change the probability by a multiplicative factor of

$$\begin{aligned} [\alpha(0,1,B)/\alpha(1,0,B)]^k &= [\exp(\lambda_1 B+\lambda_2(B-1))/\exp(\lambda_1(B-1)+\lambda_2 B)]^k \\ &= \exp(k(\lambda_1-\lambda_2)) \geq 1 \end{aligned}$$

So discriminatory eqm must have $\sigma(1,0,A|\mathbf{1}) = \sigma(1,0,B|\mathbf{3}) = \sigma(0,1,B|\mathbf{2}) =1$,
$\tau(1,0,A) = \tau(1,0,B) = \tau(0,1,B) = 1/3$.
Pivot magnitudes are then $\mu_{1,2} = -((2/3)^{1/2} - (1/3)^{1/2})^2 = -0.05719$, and
$\max\{\mu_{1,3}, \mu_{2,3}\} = \mu_3$ is (by duality) the minimum over $\lambda_1\geq0$ and $\lambda_2\geq0$ of

$$[\exp((A-1)\lambda_1+A\lambda_2) + \exp((B-1)\lambda_1+B\lambda_2) + \exp(B\lambda_1+(B-1)\lambda_2]/3 - 1.$$

Numerical analysis shows this minimum is less than -0.05719, as required for a discriminatory equilibrium, when (A,B) is below a curve that goes through the points:
(0, 0.649), (0.1, 0.621), (0.2, 0.593), (0.3, 0.563), (0.4, 0.532), (0.5, 0.500).
Discriminatory equilibria for this Condorcet cycle exist below this curve.

(My-Weber APSR 1993 would say "$2B+A\leq1$", obviously wrong at A=B=0.4)

Majoritarian outcomes

In a simple bipolar voting game, T = {**1**,**2**}, and the set of candidates K is partitioned into sets K_1 and K_2 such that $u_i(t) = 1$ if $i \in K_t$, $u_i(t)=0$ if $i \notin K_t$.

K_t = {type t candidates}.

An electoral system is majoritarian if we can guarantee that, in all equilibria of simple bipolar voting games, whenever a majority of one type exists, the winning candidate will always be of the majority type.

Propn 3. In the family of (A,B)-scoring rules for three candidate elections, only approval voting is majoritarian in this sense.

Proof. Suppose $K_1 = \{1\}$ and $K_2=\{2,3\}$. There is a symmetric equilibrium in which $\sigma(1,A,0|\mathbf{1}) = 1/2 = \sigma(1,0,A|\mathbf{1})$, $\sigma(0,B,1|\mathbf{2}) = 1/2 = \sigma(0,1,B|\mathbf{2})$.

If A>0 then it can happen that **1**'s have a slight majority, but **1**'s all vote (1,A,0) and **2**'s all vote (0,1,B), and so candidate 2 wins.

If A=0 and B<1 then it can happen that **2**'s have a slight majority, but **1**'s all vote (1,0,0) and **2**'s split equally among (0,1,B) and (0,B,1), and so candidate 1 wins.

With approval voting, it is a dominant strategy for all **1**'s to vote 1 for all candidates in K_1 and 0 for all candidates in K_2, and for all **2**'s to vote 1 for all candidates in K_2 and 0 for all candidates in K_1, so that all majority candidates get higher scores than minority candidates.

If A+B > 1 then, in the symmetric equilibrium of this voting game with $K_1 = \{1\}$ and $K_2 = \{2,3\}$, the election of a minority candidate (for **2**) is an almost-sure event when $0.5 < r(\mathbf{1}) < (1+B)/(3+B-A)$ (implies $r(\mathbf{1}) < r(\mathbf{1})(0+A)/2 + (1-r(\mathbf{1}))(1+B)/2$).
So a bloc with duplicate candidates can get a significant advantage if A+B>1.

If A+B < 1 then, in the symmetric equilibrium of this voting game with $K_1 = \{1\}$ and $K_2 = \{2,3\}$, the election of a minority candidate (for **1**) is an almost-sure event when $0.5 > r(\mathbf{1}) > (1+B)/(3+B-A)$ (implies $r(\mathbf{1}) > r(\mathbf{1})(0+A)/2 + (1-r(\mathbf{1}))(1+B)/2$).
So a bloc with duplicate candidates can get a significant disadvantage if A+B<1.

Efficient majoritarian outcomes in two-type multicandidate elections with corruption

Suppose that $T = \{\mathbf{1},\mathbf{2}\}$, $r(\mathbf{1}) > 0.5$, $r(\mathbf{2}) = 1-r(\mathbf{1}) < 0.5$. Write $\sim\mathbf{1} = \mathbf{2}$, $\sim\mathbf{2} = \mathbf{1}$.
The set of candidates K is partitioned into sets K_1 and K_2. $K_t = \{$type t candidates$\}$.
Each candidate k has corruption level $f(k) \geq 0$. k is clean if $f(k)=0$, corrupt if $f(k)>0$.
For a type t voter, the utility from candidate k winning is
$u(k,t) = 1-f(k)$ if $k \in K_t$, $u(k,t) = -f(k)$ if $k \notin K_t$.
Suppose that there exists at least one clean candidate in K_1.

Theorem Under approval voting, in all large equilibria of any election as above, the only winners at the expected vote distribution are clean candidates in K_1.

Proof: Every type **1** voter votes for clean type 1 candidates, by dominance.
If some voters are willing to cross over and vote for a candidate of the other type, then all voters of that candidate's type must strictly prefer to vote for him also.

If propn failed, a close race involving clean guy in K_1 must have magnitude $\geq -r(\mathbf{2})$, because he'd win when type **2** voters disappear.
So we can rule out case of serious candidates expecting no votes (having $\mu_k = -1$).

All serious candidates cannot have same type, else most corrupt of them would have serious races only with others preferred by all voters, so nobody would vote for him.

We now claim that no serious candidate can expect positive approval rate from both types. If not, let g be most corrupt such candidate. Let t be the type of g.

If some serious candidates in K_t are more corrupt than g, let h be most corrupt such.
So h must expect approval only from t voters, and supporters of h are a subset of supporters for g. Also h must have a serious race with some j in $K_{\sim t}$.
In a close {h,j} race, all ~t voters for-g-but-not-h must disappear with offsetRatio=0, while t voters who approve both h and g must have a positive offsetRatio.
Then by adding 2 ~t-votes for-g-but-not-h, and subtracting up to 2 t-votes for-g-and-h, we could reach an infinitely more likely close race involving g but not h. But this contradicts "{h,j} serious". So g must be most corrupt serious candidate in K_t.

Let i be most corrupt serious candidate in $K_{\sim t}$. Type ~t voters must prefer i over g, or else nobody would vote for i. So g is worst serious candidate for ~t voters, contradicting assumed definition of g, and thus proving the above claim.

If a corrupt type 1 candidate expected votes from (almost) all type **1** voters, then the close race of that candidate with a clean type 1 candidate would be a magnitude 0 event, and races involving expected-minority type 2 candidates could not be serious (having magnitude < 0). But we have seen that there must be some serious type 2 candidate h_2.
So only a clean type 1 candidate can expect votes from all type **1** voters, and no other candidates can win at the expected outcome. Q.E.D.

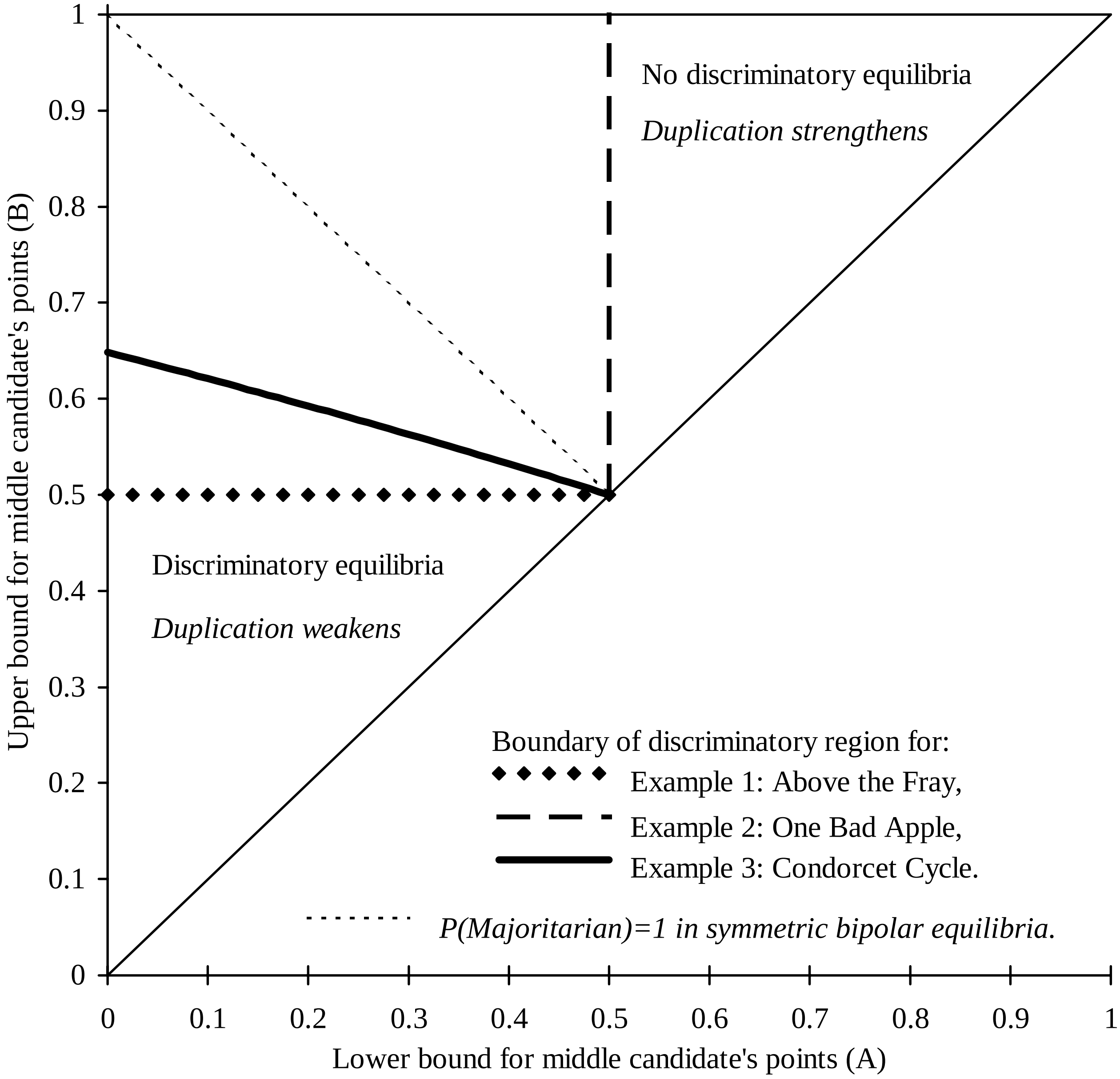

Figure 1. Characterizing Equilibria of (A,B)-Scoring Rules.

Ex 1: u(**1**) = (6,0,9), r(**1**) = 1/2; u(**2**) = (0,6,9), r(**2**) = 1/2.

Ex 2: u(**1**) = (9,3,0), r(**1**) = 1/2; u(**2**) = (3,9,0), r(**2**) = 1/2.

Ex 3: u(**1**) = (9,6,0), r(**1**) = 1/3; u(**2**) = (0,9,6), r(**2**) = 1/3; u(**3**) = (6,0,9), r(**3**) = 1/3.

R. Myerson, "Comparison of scoring rules in Poisson voting games," Journal of Economic Theory 103:219-251 (2002).

REFERENCES

G. Cox, "Electoral equilibrium under alternative voting institutions." American Journal of Political Science 31 (1987), 82-108.

G. Cox, "Centripetal and centrifugal incentives in electoral systems." American Journal of Political Science 34 (1990), 903-935.

E. Muller and M. Satterthwaite, "The equivalence of strong positive association and strategy-proofness," Journal of Economic Theory 14 (1977), 412-418.

R. Myerson, "Effectiveness of Electoral Systems for Reducing Government Corruption: A Game Theoretic Analysis," Games and Economic Behavior 5 (1993a), 118-132.

R. Myerson, "Incentives to cultivate favored minorities under alternative electoral systems." American Political Science Review 87 (1993b), 856-869.

R. Myerson, "Population uncertainty and Poisson games,"International Journal of Game Theory 27 (1998), 375-392.

R. Myerson, "Large Poisson games," Northwestern University Math Center discussion paper #1189 (1997). To appear in Journal of Economic Theory (2000). [http://www.kellogg.nwu.edu/research/math/Down-papers.htm].

R. Myerson and R. Weber. 1993. "A theory of voting equilibria." American Political Science Review 87 (1993), 102-114.

T. Piketty, "Voting as Communicating," CEPREMAP discussion paper (Paris, 1995).

What if voters care about vote share, not just about who wins?

Example 5 (ratifying a treaty) (T. Piketty) Two alternatives "Yes" and "No".

All voters of same type. With γ = (fraction voting Yes), every voter gets utility = $1-g(\gamma)$ if $\gamma>1/2$, utility = $-g(\gamma)$ if $\gamma \le 1/2$. Suppose $g'(\gamma) > 0, \forall\gamma$.

There exists an equilibrium with $\tau(\text{Yes}) = 0$, but there also exist large equilibria with $\tau(\text{Yes}) = 1/2$, coming from equilibria for finite n of the form

$$\tau_n(\text{Yes}) - 1/2 \approx \pm\frac{0.5}{\sqrt{n}}\sqrt{\log\left(\frac{2n}{(g'(0.5))^2\pi}\right)}.$$

We use the fact that a large Poisson random variable near its expected value (where the offset ratio is close to 1) can be approximated by the integer-rounding of a Normal random variable with variance equal to the expected value.

So the numbers voting for each side are independent and approximately Normal, with a standard deviation that is the square root of the expected value.

So in these equilibria, the difference between the number voting Yes and the number voting No is approximately Normal, with mean $\sqrt{n}\sqrt{\log\left(\frac{2n}{(g'(0.5))^2\pi}\right)}$ and standard deviation $\sqrt{n}$. This yields a pivot probability that is approximately $g'(0.5)/n$.

For equilibrium with "+" in "±", the probability of Yes winning goes to 1 as $n\to\infty$.

With $g(\gamma)=\gamma$ and n= 100million: we get $\tau_n(\text{Yes}) =.500212$,

vote difference has E=42,400, StDev=10,000.

(If we had $\tau_n(\text{Yes}) = \tau_n(\text{No}) = 1/2$ for large n then the probability that one vote could change the outcome would be approximately $\sqrt{2/(\pi n)}$.)

In symmetric equilibria, a tendency of candidates to cluster ideologically and ignore blocs of voters can be measured by Cox's threshold of diversity (R^*).

Suppose that there are two types of voters $T = \{\mathbf{1},\mathbf{2}\}$ and three candidates $K=\{1,2,3\}$.

We consider here symmetric equilibria in which candidates who adopt identical policy positions are treated symmetrically by the voters (neutrality).

In the case where all candidates adopt the ideal position of type **2** voters, they would obviously each have equal 1/3 probabilities of winning in a symmetric equilibrium.

Suppose now that candidate 1 changes to the ideal position of type **1** voters.

Then voters' utility vectors are $u(\mathbf{1}) = (1,0,0)$ and $u(\mathbf{2}) = (0,1,1)$.

In a symmetric equilibrium:

voting strategies are $\sigma(1,0,A|\mathbf{1}) = \sigma(1,A,0|\mathbf{1}) = 0.5$, $\sigma(0,1,B|\mathbf{2}) = \sigma(0,B,1|\mathbf{2}) = 0.5$,

E(vote distribution) is $\tau(1,0,A) = \tau(1,A,0) = r(\mathbf{1})/2$, $\tau(0,1,B) = \tau(0,B,1) = (1-r(\mathbf{1}))/2$,

expected scores per voter are $S_1(\tau) = r(\mathbf{1})$, $S_2(\tau) = S_3(\tau) = (1 + B + r(\mathbf{1})(A-B-1))/2$.

So candidate 1 would win almost surely if $r(\mathbf{1}) > (1 + B + r(\mathbf{1})(A-B-1))/2$,

which holds if $r(\mathbf{1}) > R^* = (1+B)/(3+B-A))$,

where this R^* is Cox's threshold of diversity (AJPS, 1987, 1990).

Candidate 1 would lose almost surely if $r(\mathbf{1}) < R^*$.

If $R^* < r(\mathbf{1}) < 1/2$, we find symmetric equilibria where type **2** voters are a majority and have two candidates, but a single candidate for the type **1** minority wins almost surely.

So when $R^* < 1/2$ ($A+B<1$), having duplicate candidates can weaken a majority bloc.

If $1/2 < r(\mathbf{1}) < R^*$, we get symmetric equilibria where type **1** voters are a majority, but three win-motivated candidates choose to coincide at the type **2** voters' ideal point, because a single deviator to the type **1** ideal would lose almost surely.

So when $R^* > 1/2$ ($A+B>1$), having duplicate candidates can strengthen a minority.

The majoritarian ideal $R^* = 1/2$ holds when $A+B = 1$, on the line from approval ($A=0$, $B=1$) to Borda ($A=B=1/2$) voting. With three candidates, plurality voting ($A=B=0$) has $R^*=1/3$, negative voting ($A=B=1$) has $R^*=2/3$.

Two winners out of three.

Consider now the case where the top two scorers will both win seats (in some multimember council), and each voter gets sum of his utilities for the two winners.

For symmetric equilibria, the analysis of Cox's threshold of diversity remains the same in the two-winner case (except that each candidate has 2/3 probability of winning when they all choose the same position).
The largest fraction of voters who could be ignored by all three candidates in a symmetric common-position equilibrium is still $R^* = (1+B)/(3+B-A)$.
If $1/2 < r(\mathbf{1}) < R^*$, we get symmetric equilibria where type **1** voters are a majority, but three win-motivated candidates coincide at the type **2** voters' ideal point,
because a single deviator to the type **1** ideal would lose almost surely.
So when $R^* > 1/2$ $(A+B>1)$, having duplicate candidates can strengthen a minority.
If $R^* < r(\mathbf{1}) < 1/2$, we find symmetric equilibria where type **2** voters are a majority with two candidates, and a minority type-**1** candidate wins almost surely;
but now, having one minority candidate among two winners may be desirable.

Nonsymmetric equilibria are very different from the case of one winner, because the serious race is typically between weakest likely winner and strongest likely loser.
So when the top two candidates are winners, a candidate can be nonserious because he is much stronger (more likely to win) than any other candidate.

In "Above the Fray" [Ex 1: $u(\mathbf{1}) = (6,0,9)$, $r(\mathbf{1}) = 0.5 = r(\mathbf{2})$, $u(\mathbf{2}) = (0,6,9)$],
a discriminatory equilibrium where candidate 3 is not serious would have
$\sigma(1,0,B|\mathbf{1}) = 1 = \sigma(0,1,B|\mathbf{2}) = 1$, $\tau(1,0,B) = \tau(0,1,B) = 0.5$,
and this is a discriminatory equilibrium when $B > 0.5$ (so that 3 is strongest).
With popular 3 among the winners, this discriminatory equilibrium now looks good.

In "One Bad Apple" [Ex 2: $u(\mathbf{1}) = (9,6,0)$, $r(\mathbf{1}) = 0.5 = r(\mathbf{2})$, $u(\mathbf{2}) = (6,9,0)$],
a discriminatory equilibrium where candidate 3 is not serious would have
$\sigma(1,0,A|\mathbf{1}) = 1 = \sigma(0,1,A|\mathbf{2}) = 1$, $\tau(1,0,A) = \tau(0,1,A) = 0.5$,
and this is a discriminatory equilibrium when $A > 0.5$ (so that 3 is strongest).
With unpopular 3 among the winners, this discriminatory equilibrium now looks bad.

notes for lecture in 2002, focusing on last section:

Bipolar multicandidate elections with corruption *(new notes March 2002)*

Suppose the set of candidates K is partitioned into K_1={leftists} and K_2={rightists}.
Each candidate k has corruption level $f(k) \geq 0$. k is clean if f(k)=0, corrupt if f(k)>0.
In game Γ_n, the number of voters is a Poisson random variable with mean n.
Each voter has a type t drawn independently from a probability distribution r that has a continuous positive density on the whole real line $\mathbb{R}$. $r(S) = \text{Prob}(\tilde{t} \in S)\ \forall S \subseteq \mathbb{R}$.
A voter's type t measures his net preference for rightist candidates in K_2,
so t's utility payoff if k wins is $u_k(t) = t - f(k)$ if $k \in K_2$, $u_k(t) = 0 - f(k)$ if $k \in K_1$.
Suppose that, $\forall i \in \{1,2\}$, there exists a clean candidate k in K_i with f(k)=0. (wlog)
To complete the game, we must specify an electoral system (ties broken at random).

An equilibrium in the game Γ_n specifies a (weakly undominated) optimal strategy $\sigma_n(t)$ for each type t, and generates expected fractions $\tau_n(c)$ for each ballot c that is allowed in this electoral system, and win-probabilities $q_n(k)$ for each candidate k.
A large equilibrium (σ,τ,q) is a limit of (σ_n,τ_n,q_n) equilibria of Γ_n as $n \rightarrow \infty$.
A pair of candidates {i,j} is distinct iff $u_i(t) \neq u_j(t)$ for some t in T.
The {i,j}-race is close when adding one vote could change winner from i to j, or j to i.
The {i,j}-race is serious in a large equilibrium iff {i,j} is a distinct pair and there is a strictly positive limit($n \rightarrow \infty$) of the conditional probability of a close {i,j}-race given that some pair of distinct candidates are in a close race.
A candidate is serious iff he is involved in at least one serious race.
A candidate i is strong in a large equilibrium (σ,τ,q) iff $q(k) > 0$ (positive win-proby).

Theorem 1 (effectiveness against corruption). In a large equilibrium under approval voting, no corrupt candidates can be strong or serious.
Theorem 2 (majoritarianism). In a large equilibrium under approval voting, with probability 1, the winner will be a candidate who is considered best by at least half of the voters.

Failures of effective majoritarianism for other electoral systems:

In three-candidate elections, consider rank-scoring rules where ballots are vectors that are permutations of (1,A,0), for some number A such that $0 \leq A \leq 1$.
Suppose $K_1=\{1\}$, $K_2=\{2,3\}$, 1 and 2 are clean, 3 is corrupt.
If A<1/2 then there is an equilibrium where {1,3} is the only serious race. In this equilibrium, everybody votes (1,A,0) or (0,A,1), and so the winner will be either 1 or 3.
If $A \geq 1/2$ then 3 must be serious in all equilibria, because if people thought 3 was not serious then everybody would vote (1,0,A) or (0,1,A), but then 3 would always be in first place when 1 and 2 are tied!

Now consider more general scoring rules where ballots are vectors that are permutations of (1,A,0) and (1,B,0), where $0 \leq A \leq B \leq 1$.
Approval voting is (A,B)=(0,1), plurality voting is (0,0),
Borda voting is (1/2,1/2), negative voting is (1,1).
Suppose now $K_1=\{1\}$, $K_2=\{2,3\}$, all three candidates are clean.
There is a symmetric equilibrium where leftist voters randomize equally among (1,A,0) and (1,0,A), while rightist voters randomize equally among (0,B,1) and (0,1,B).
Notice $r < r(0+A)/2 + (1-r)(1+B)/2$ (1 loses) iff $r < (1+B)/(3+B-A)$.
$(1+B)/(3+B-A)$ is Cox's threshold of diversity here.
Also $1/2 < (1+B)/(3+B-A)$ iff $1 < A+B$.
When $1 < A+B$ and $1/2 < r(\mathbb{R}_-) < (1+B)/(3+B-A)$, then almost-surely leftists are a majority, but a rightist candidate wins (duplication helps righists).
When $1 > A+B$ and $1/2 > r(\mathbb{R}_-) > (1+B)/(3+B-A)$ then almost-surely rightists are a majority, but the leftist candidate wins (duplication hurts rightists).

The magnitude of any event M is $\mu(M) = \lim_{n\to\infty} \log_e(P_n(M))/n$.

A type's offset ratio in an outcome is number of such voters divided by its expectation.

MagThm. The magnitude of an event is determined by the most likely outcome in the event, and is a concave function of the offset ratios there. In the limit, all probability in the event is concentrated where this maximal magnitude is achieved.

$\mu(M) = \lim_{n\to\infty} \max_{x\in M} \sum_{c\in\{\text{ballots}\}} \tau_n(c)\, \psi\big(x(c)/(n\tau_n(c))\big)$, $\psi(\theta) = \theta(1-\log_e(\theta))-1$.

Cor. If $\{S_0, S_1, S_2, S_3\}$ is a partition of all voter-types, then the event "equal numbers in S_1 and S_2 but none in S_0" has magnitude $2\sqrt{r(S_1)r(S_2)}+r(S_3)-1$, which is achieved at offset ratios 0 in S_0, $\sqrt{r(S_2)/r(S_1)}$ in S_1, $\sqrt{r(S_1)/r(S_2)}$ in S_2, 1 in S_3.

Proof of Theorem 1. All leftist voters with types in $\mathbb{R}_-$ approve clean candidates in K_1, as best among all candidates, because approving-best weakly dominates not-approving.

Similarly, all rightist voters in $\mathbb{R}_+ = [0,+\infty]$ approve clean candidates in K_2.

If type t approves candidate i in K_1 and $s < t$ then type s also approves i

(because $u_i(s) - u_k(s) \geq u_i(t) - u_k(t)$ $\forall k\in K$, with "=" if $k\in K_1$ and ">" if $k\in K_2$).

So for i in K_1, exists $\theta_n(i)$ such that t approves i in σ_n if $t < \theta_n(i)$ but not if $t > \theta_n(i)$.

For j in K_2, exists $\theta_n(j)$ such that t approves j in σ_n if $t > \theta_n(j)$ but not if $t < \theta_n(j)$.

Let $\theta(k) = \lim_{n\to\infty} \theta_n(k)$.

Let $h_1\in H_1 = \operatorname{argmax}_{i\in K_1}\theta(i)$, $h_2\in H_2 = \operatorname{argmin}_{j\in K_2}\theta(j)$ (highest E scores on each side).

A clean candidate in K_1 has $\theta\geq 0$; a clean candidate in K_2 has $\theta\leq 0$. So $\theta(h_2)\leq 0\leq\theta(h_1)$.

Let $r_1 = r([-\infty, \theta(h_2)])$, $r_2 = r([\theta(h_1), +\infty])$, $r_3 = r([\theta(h_2), \theta(h_1)])$.

The event of a close $\{h_1,h_2\}$-race has magnitude $2\sqrt{r_1 r_2}+r_3-1$ (> -1).

Let i and j be any other candidates in K_1 and K_2 respectively.

Let $s_0 = r([\theta(h_2),\theta(j)] \cup [\theta(i),\theta(h_1)])$ (E fractn for-h_2-but-not-j or for-h_1-but-not-i),

$s_1 = r([-\infty, \min\{\theta(i),\theta(h_2)\}])$ (E fraction for-i-but-not-h_2),

$s_2 = r([\max\{\theta(j),\theta(h_1)\}, +\infty])$ (E fraction for-j-but-not-h_1),

$s_3 = r([\theta(j), \theta(i)]$ (E fraction for-i-and-j). Here $s_3 = 0$ if $\theta(j)\geq\theta(i)$.

The event of a close $\{i,j\}$-race has magnitude $2\sqrt{s_1 s_2}+s_3-1$.

If $\theta(i) < \theta(h_1)$ or $\theta(h_2) < \theta(j)$ then $s_1 \leq r_1$, $s_2 \leq r_2$, $s_3 < r_3$, and so a close $\{i,j\}$-race has strictly lower magnitude than a close $\{h_1,h_2\}$-race.

So a serious race between a leftist and rightist candidate can only involve candidates in H_1 and H_2 (those with highest expected scores on each side as $n \to \infty$).

Now suppose, contrary to the theorem, that some corrupt candidate is serious.

Let i denote the most corrupt serious candidate. Suppose w.l.o.g. that $i \in K_1$.

There must exist some j in H_2 such that the {i,j} race is serious, because nobody would vote for i if i's serious races were all with other less-corrupt candidates in K_1.

Candidate i is the worst serious candidate for all voters in $\mathbb{R}_+$, so $\theta_n(i) < 0 \ \forall n$.

Let g be a clean candidate in K_1, who is approved by all voters in $\mathbb{R}_-$, so $\theta_n(g) \geq 0 \ \forall n$.

So the set of voters approving i is a subset of those approving g.

Candidate i can win only when all voters for-g-but-not-for-i vanish, leaving g in a tie with i. So whenever an additional vote for i could make i win, there is a positive limiting conditional probability that the winner would be g otherwise.

But for type-0 voters, g is strictly better than i, and no serious candidate is worse than i.

So in the limit, there are strictly negative conditional expected gains for type-0 voters from approving i, given the event that some serious race is close.

So $\theta(i) < 0 \leq \theta(g)$. Thus, i is not in H_1.

But then a close {i,j}-race must have lower magnitude than some other close race involving a higher-expected-scoring candidate in H_1.

So the {i,j} race cannot be serious.

This contradiction shows that no corrupt candidate i can be serious.

Thus, all serious candidates must be clean.

A pair of clean candidates who are both in K_1 (or both in K_2) would not be distinct, so every serious race involves a clean candidate in K_1 and a clean candidate in K_2.

In a single-winner election, a strong candidate must be serious, and so all strong candidates must be clean.

Proof of Theorem 2.

From Theorem 1 all serious races are between clean candidates in K_1 and K_2.
So leftist voters in $\mathbb{R}_-$ will all approve the clean candidates in K_1 but not in K_2,
while rightist voters in $\mathbb{R}_+$ will all approve the clean candidates in K_2 but not in K_1.
Corrupt candidates may get some votes, but only from an expected-strict subset of the voters on their same side of the political spectrum (so not serious contenders).
So with probability 1, the winner will be a clean candidate from the side of the political spectrum that has a majority (or at least half) of the electorate,
and so the winner will be an optimal candidate for at least half of the voters.

References:

"Comparison of scoring rules in Poisson voting games," Journal of Economic Theory (2002) 103:219-251.
"Effectiveness of electoral systems for reducing government corruption: a game-theoretic analysis," Games and Economic Behavior (1993) 5:118-132.
"Theoretical Comparisons of Electoral Systems," European Economic Review 43 (1999), 671-697.

`http://home.uchicago.edu/~rmyerson/research/corrup.pdf`

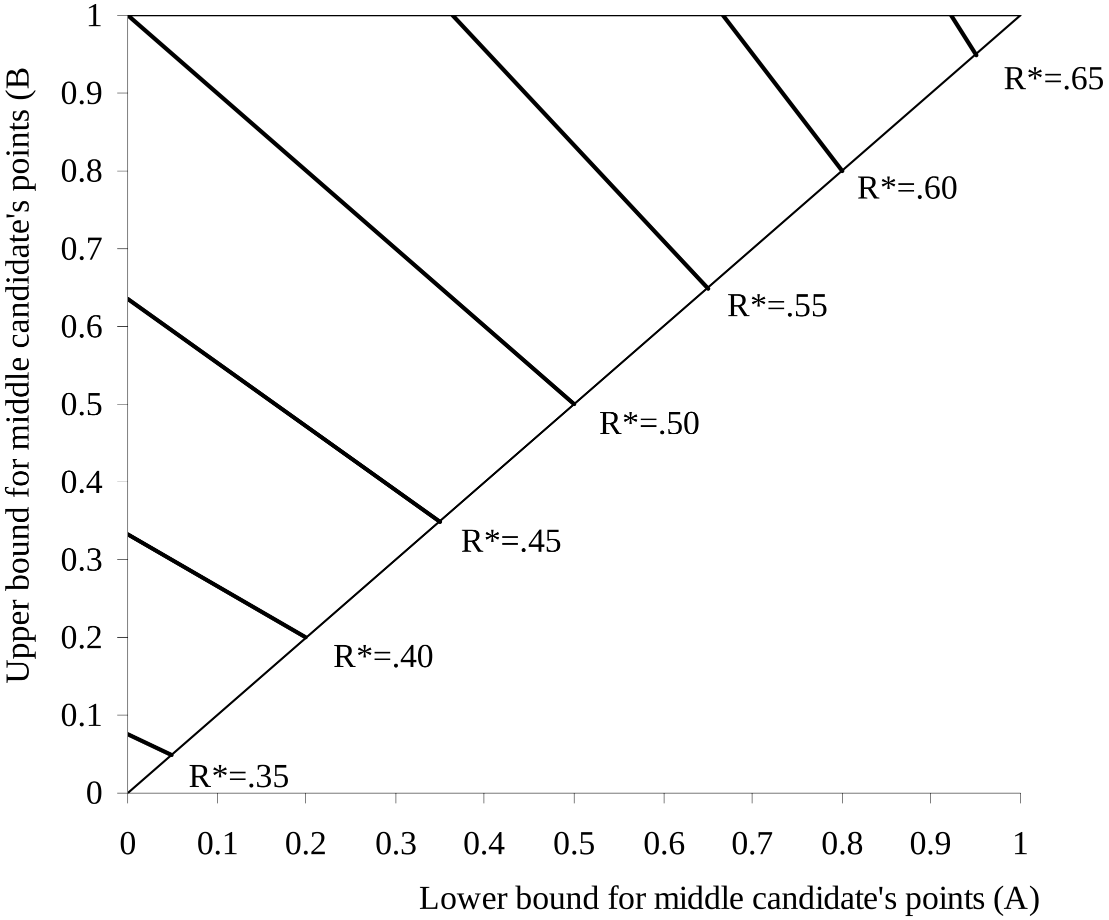

Figure 2. Cox's threshold of diversity (R*) for (A,B)-scoring rules.

$R^* = (1+B)/(3+B-A)$

= [largest fraction that can lose with one candidate against two in a symmetric eqm]