

## A MODEL OF CAPITALIST LIBERALIZATION:

<http://home.uchicago.edu/~rmyerson/research/caplib.pdf>

With any capital investment  $k \geq 0$ ,  $Y(k)$  is net output production (flow).

Capitalists' rate of time discounting is  $r$ . Assume capital is durable and mobile.

$\rho$  is rate of time discounting for a ruler who has not liberalized.

$\theta$  is the fraction of capital that ruler could expropriate.

Liberalization  $\lambda$  is probability of ruler losing power if he tried to expropriate capital.

False-alarm scandals occur at rate  $\psi$ .

Ruler's present discounted value is  $V(k, \lambda) = (Y(k) - rk) / (\rho + \psi\lambda)$ .

Trying to expropriate would yield  $W(k, \lambda) = (1 - \lambda)(\theta k + Y(0)) / \rho$ .

Optimal  $(\lambda, k)$  maximizes  $V(k, \lambda)$  s.t.  $V(k, \lambda) \geq W(k, \lambda)$ ,  $k \geq 0$ ,  $0 \leq \lambda \leq 1$ .

Let  $Q(k) = (Y(k) - rk) / (\theta k + Y(0)) / \rho$  and  $q(\lambda) = (\rho + \psi\lambda)(1 - \lambda)$ .

So  $V(k, \lambda) \geq W(k, \lambda)$  iff  $Q(k) \geq q(\lambda)$ . Notice  $q(0) = \rho = q(1 - \rho/\psi)$ ,  $q(1) = 0$ .

Let  $\Lambda(k)$  denote the smallest  $\lambda \geq 0$  such that  $Q(k) \geq q(\lambda)$ .

Optimal  $k$  maximizes  $V(k, \Lambda(k))$ . If  $Q(k) \geq \rho$  then  $\Lambda(k) = 0$ .

If  $Q(k) < \rho$  then  $\Lambda(k) = \{\psi - \rho + [(\psi - \rho)^2 + 4\psi(\rho - Q(k))]^{0.5}\} / (2\psi)$ .

Fact. If  $(\lambda, k)$  is optimal and  $\lambda > 0$  then  $\lambda > 1 - \rho/\psi$ ,

and  $Y'(k) = r + \theta\psi(1 - \Lambda(k))^2$ .

Example 0 Let our parameters be  $r = \rho = 0.05$ ,  $\psi = 0.1$ ,  $\theta = 1$ , and  $Y(k) = 0.2k^{0.8}$ . The ideal without incentive constraints is  $K^* = 335.5$ ,  $Y(K^*) = 20.97$ ,  $Y'(K^*) = 0.05$ . With incentive constraints, optimal regime is  $k = 32$ ,  $\lambda = 0$ ,  $Y(k) = 3.2$ ,  $V(k, \lambda) = 32$ . Reducing  $\psi$  to  $\psi = 0.09$  would change to  $k = 303.9$ ,  $\lambda = 0.895$ ,  $Y = 19.38$ ,  $V = 32.02$ . (Further reduction of the scandal rate  $\psi$  would reduce  $\lambda$  a bit, as the  $V \geq W$  constraint is relaxed.)

Example 1 Let our basic parameters be  $r = \rho = 0.05$ ,  $\psi = 0.1$ ,  $\theta = 1$ .

Suppose that output is produced by three factors, which we may call labor (L), capital (K), and land (D) according to the function  $L^{0.4}K^{0.5}D^{0.1}$ .

Each island is endowed with 1 unit of fixed labor (L=1), 1 unit of fixed land (D=1).

With no fixed capital, ideal would be feasible:  $K^*=100$ ,  $\lambda=0$ ,  $Y=10$ ,  $V=100$ .

Suppose instead that each island is endowed with 25 units of fixed capital (F=25), which can be augmented by capitalist investment k to yield output

$Y(k) = L^{0.4} (F+k)^{0.5} D^{0.1} = (25+k)^{0.5}$  with the given endowments L=1, F=25, D=1.

With this production function, optimal regime for ruler of each island is  $k=0$ ,  $\lambda=0$ .

Although  $Y'(0) = 0.1 > r$ , any positive investment would require a liberalization  $\lambda > 0.5$ , so that rulers prefer not to encourage any investment.

(This is the curse of natural resources.)

Production on each island is then  $Y(0) = 5$ , and the ruler's value is  $V(0,0)=100$ .

We have ignored the cost of wages, assuming that most laborers are bound serfs who are forced to work for negligible wages. But the marginal product of labor in this solution is

$.4Y(0)/L = 2$ , which would be the wage rate for any free labor on each island.

## Example 2 (Tiebout effects)

Now suppose that one island considers the possibility of attracting additional free laborers from other islands, at the given wage rate  $w=2$ . So with investment  $k$  and new free immigrant labor  $n$  on this island, the product net of wage costs would be  $\hat{Y}(k,n) = (1+n)^{0.4} (25+k)^{0.5} (1)^{0.1} - 2n$ .

For any given  $k$ , the maximum of this net product over all  $n \geq 0$  is achieved when  $2 = 0.4(25+k)^{0.5} / (1+n)^{0.6}$ ,  $2(n+1) = .4(1+n)^{0.4} (25+k)^{0.5}$ , and  $(1+n)^{0.4} = (.2)^{2/3} (25+k)^{1/3}$ .

So  $Y(k) = \max_{n \geq 0} \hat{Y}(k,n) = (1-.4)(.2)^{2/3} (25+k)^{5/6} + 2 = 0.2052(25+k)^{5/6} + 2$ .

With this production function, optimal regime on this island is  $k=1431.6$ ,  $\lambda=0.911$ , so that production is  $Y(k)=90.77$ , with new labor  $n=28.6$ , and the ruler's value is  $V(k,\lambda)=136.0$ .

So mobility of labor (or other resources that are complementary to capital) can encourage a ruler to liberalize in a way that increases production and income for others in society.

### Example 3

$Y(k) = (k+F)^{0.5}$  where  $F =$  (fixed natural resources)  $< 100$ .

Let  $r = 0.05$ ,  $\rho = r+0.05 = 0.10$ ,  $\psi = 0.05$ .

*(With this  $Y(\cdot)$  and  $r$ , optimum without moral hazard would have  $k+F=100$ .)*

With  $F = 12$ , no investment is feasible without liberalization ( $\lambda=0 \Rightarrow k=0$ ),

but the optimal regime has  $\lambda=0.504$ ,  $k = 52.4$ , and so  $k+F=64.4$ .

With  $F = 0$ , the optimal regime has  $\lambda = 0$ ,  $k = 44.4$ .

With  $F = 25$ , the optimal regime has  $\lambda = 0$ ,  $k = 0$ .

## A MODEL OF CAPITALIST LIBERALIZATION by Roger Myerson

*Notes:* <http://home.uchicago.edu/~rmyerson/research/caplib.pdf>

*Spreadsheet:* <http://home.uchicago.edu/~rmyerson/research/caplib.xls>

A fundamental problem for encouraging investment is that the government officials who enforce property rights may be tempted to abuse these powers and expropriate assets that are the results of others' investment. The rulers of a tightly controlled authoritarian state would have little or no fear of losing power if they expropriated investors' assets. But investors would be able to trust the government more in a political system where the current rulers would risk losing power if they tried to expropriate invested assets, but such risk presumes that investors have some (implicit) political influence. In more liberal state, where people have more freedom to speak and organize without government control, an expropriation of assets by government leaders would have a greater probability of creating a scandal that could cause a change of leadership.

For our purposes here, the probability of political change if the established rulers wrongfully expropriated investments may be considered as a measure of liberalization. A ruler may find some advantages in liberalizing the regime, even though such liberalization creates political risks for him, because such liberalization can encourage greater investment by providing more credible guarantees to investors, and greater investments increase the size of his tax base.

### The model:

With any capital investment  $k \geq 0$ , let  $Y(k)$  denote the net output production flow in the economy. Here  $Y(k)$  is a flow per unit time. To produce the output  $Y(k)$ , the capital  $k$  must be used and controlled by many individuals in the general population, whom we may call capitalists, and their control over the capital would enable them to take it abroad at any time.

The capitalists' rate of time discounting is  $r$ . So to deter capital flight, the capitalists must enjoy an income flow worth  $rk$  from their capital holdings.

We may assume that  $Y(k)$  is net of labor and resource costs, so the authoritarian rulers of the government can take (in taxes) the remaining flow  $Y(k) - rk$ .

Let  $\rho$  denote the rate of time discounting for a ruler who has not liberalized, which may be different from  $r$  because, for example, the ruler might face some exogenous risk of losing power

without liberalization, which would increase  $\rho$  above  $r$ .

When the regime has liberalization  $\lambda$ , the probability of the ruler losing power if he tried to expropriate capital (or if he tried to reduce liberalization) would be  $\lambda$ .

But there are also false-alarm scandals that occur at some Poisson rate  $\psi$ , and people react to such scandals exactly as they would to a genuine attempt to expropriate capital.

So for a regime with liberalization  $\lambda$ , when the government is actually not trying to expropriate capital, still even in any short time interval of length  $\varepsilon$  there is approximately  $\psi\varepsilon$  probability of a scandal and approximately  $\psi\varepsilon\lambda$  of the rulers being replaced because of such a scandal.

So in a regime with liberalization  $\lambda$ , the current ruler discounts future revenue at rate  $\rho+\psi\lambda$ , and so with invested capital  $k$ , the ruler's present discounted value is  $V(k,\lambda) = (Y(k)-rk)/(\rho+\psi\lambda)$ .

But consider what would happen if the ruler tried to expropriate the capital. With probability  $\lambda$  the ruler would lose power and get 0 thereafter. Otherwise, with probability  $1-\lambda$ , the ruler would successfully seize some fraction  $\theta$  of the capital, where  $1-\theta$  denotes the fraction of capital that would be lost or destroyed in the struggle or taken abroad by fleeing capitalists. Thereafter the ruler would have lost any reputation for protecting capital and so may as well deliberalize, and so the value of his continuation in power, without any free investment or liberalization, would be  $Y(0)/\rho$  after a successful expropriation. So the ruler's expected discounted value from trying to expropriate capital would be  $W(k,\lambda) = (1-\lambda)(\theta k + Y(0)/\rho)$ .

Capital  $k$  can be safely invested in a regime with liberalization  $\lambda$  iff  $k$  and  $\lambda$  satisfy the ruler's incentive constraint  $V(k,\lambda) \geq W(k,\lambda)$ . Of course capital must be nonnegative  $k \geq 0$ .

As liberalization  $\lambda$  here is a probability, it must satisfy the probability constraints  $0 \leq \lambda \leq 1$ .

The ruler's optimal regime  $(\lambda,k)$  should maximize  $V(k,\lambda)$  subject to these constraints.

We assume that  $Y(k)$  is a continuously differentiable and strictly concave function of  $k$ , with  $Y(0) \geq 0$  and  $\lim_{k \rightarrow \infty} Y'(k) = 0$ . The other given parameters satisfy  $r > 0$ ,  $\rho > 0$ ,  $\psi > 0$ ,  $0 < \theta \leq 1$ .

### Basic analysis.

Let  $K^*$  denote the ideal unconstrained investment level that satisfies  $Y'(K^*) = r$ .

If  $K^*$  were incentive-feasible with  $\lambda=0$  then  $(K^*,0)$  would be trivially optimal. To rule out trivial solutions, suppose that  $(Y(K^*)-rK^*)/\rho < \theta K^* + Y(0)/\rho$ , so that  $K^*$  is infeasible,

and also suppose that  $Y(k) - rk > 0$  for at least some small  $k$  (as if  $Y(0) > 0$  or  $Y'(0) > r$ )

The basic incentive constraint  $V(k, \lambda) = (Y(k) - rk) / (\rho + \psi\lambda) \geq W(k, \lambda) = (1 - \lambda)(\theta k + Y(0) / \rho)$  is equivalent to the inequality  $(Y(k) - rk) / (\theta k + Y(0) / \rho) \geq (\rho + \psi\lambda)(1 - \lambda)$ .

So let  $Q(k) = (Y(k) - rk) / (\theta k + Y(0) / \rho)$ , with  $Q(0) = \rho$ , and let  $q(\lambda) = (\rho + \psi\lambda)(1 - \lambda)$ .

The quotient  $Q(k) = (Y(k) - rk) / (\theta k + Y(0) / \rho)$  is the ruler's rate of revenue per unit of expropriable wealth when capitalist investment is  $k$ . The quadratic  $q(\lambda) = (\rho + \psi\lambda)(1 - \lambda)$  is the ruler's required rate of return on expropriable assets when liberalization is  $\lambda$ .

Then we can rewrite the incentive constraint as  $Q(k) \geq q(\lambda)$ .

Given any  $k$ ,  $V(k, \lambda)$  is decreasing in  $\lambda$ , so the rulers would prefer the smallest feasible  $\lambda$ .

So for any  $k$  such that  $Y(k) - rk \geq 0$ , let  $\Lambda(k)$  denote the smallest  $\lambda \geq 0$  such that  $Q(k) \geq q(\lambda)$ .

Notice  $q(0) = \rho$ . So if  $Q(k) \geq \rho$  then  $\Lambda(k) = 0$ .

That is, investment  $k$  is compatible with the ruler's ideal of nonliberalization ( $\lambda = 0$ ) iff  $Q(k) \geq \rho$  or equivalently  $Y(k) - (r + \rho\theta)k \geq Y(0)$ . With  $Y()$  concave, the set of  $k$  that satisfy this inequality is an interval  $[0, K_0]$  for some  $K_0 \geq 0$  which satisfies the binding constraint equation  $Y(K_0) - (r + \rho\theta)K_0 = Y(0)$ , and so  $Q(K_0) = \rho$ . With the nontriviality assumption, we have  $K_0 < K^*$ , and so  $V(k, 0)$  is increasing over  $k$  in  $[0, K_0]$ , and so the best investment without liberalization is  $K_0$ .

Now consider what can be achieved with positive liberalization  $\lambda > 0$ .

An optimal investment  $k$  that requires positive liberalization must have  $0 < Q(k) < \rho$ .

( $Q(k) \leq 0$  would imply  $F(k) \leq rk$  and so  $V(k, \lambda) \leq 0$  for all  $\lambda$ , and so such  $k$  could not be optimal.)

For any feasible investment  $k$ , if the incentive constraint were not binding then the ruler could increase his objective  $V(k, \lambda)$  by decreasing  $\lambda$  slightly.

So any optimal regime  $(k, \lambda)$  with  $\lambda > 0$  must have  $V(k, \lambda) = W(k, \lambda)$  and so  $0 < Q(k) = q(\lambda) < \rho$ .

With  $0 < Q(k) < \rho$ , the unique  $\lambda > 0$  that satisfies  $Q(k) = q(\lambda) = \rho + (\psi - \rho)\lambda - \psi\lambda^2$  is  $\lambda = \Lambda(k)$  where  $\Lambda(k) = \{ \psi - \rho + [(\psi - \rho)^2 + 4\psi(\rho - Q(k))]^{0.5} \} / (2\psi) = \{ \psi - \rho + [(\psi + \rho)^2 - 4\psi Q(k)]^{0.5} \} / (2\psi)$ .

Notice that this formula yields  $(\psi - \rho) / \psi < \Lambda(k) < 1$  when  $0 < Q(k) < \rho$ .

Then the optimal investment is the  $k$  that maximizes  $V(k, \Lambda(k))$  over all  $k$  between 0 and  $K^*$ .

Further analysis The objective function  $V(k, \Lambda(k))$  may not be concave in  $k$ .

Notice that the quadratic  $q(\lambda)$  is maximal at  $\lambda = 0.5(1 - \rho / \psi)$ , and  $q(1 - \rho / \psi) = \rho = q(0)$ .

So if  $\rho < \psi$  then  $\Lambda(k)$  has a discontinuity at  $k = K_0$ .

By definition of  $K_0$  as the largest feasible  $k$  with  $\lambda=0$ , we have  $Q(K_0)=\rho$  and  $\Lambda(K_0) = 0$ .

But with any  $k>K_0$ , we get  $Q(k)<\rho$ , and then the formula for  $\Lambda(k)$  yields  $\lambda > (\psi-\rho)/\psi = 1-\rho/\psi$ .

At  $k>K_0$ , however  $\Lambda(k)$  is continuously differentiable, and we have derivatives

$$Q'(k) = (Y'(k) - r - \theta Q(k))/(\theta k + Y(0)/\rho), \quad q'(\lambda) = \psi - \rho - 2\psi\lambda,$$

$$\Lambda'(k) = Q'(k)/q'(\Lambda(k)) = -[Y'(k) - r - \theta q(\Lambda(k))]/[(\theta k + Y(0)/\rho)(2\psi\Lambda(k) + \rho - \psi)].$$

Here  $(2\psi\Lambda(k)+\rho-\psi)$  is positive, as  $\lambda>\max\{0,1-\rho/\psi\} \Rightarrow 2\psi\lambda+\rho-\psi>\rho-\psi$  and  $2\psi\lambda+\rho-\psi>\psi-\rho$ .

The ruler's marginal value of additional investment (with the necessary liberalization) is then

$$\begin{aligned} d/dk V(k,\Lambda(k)) &= d/dk W(k,\Lambda(k)) = (1-\Lambda(k))\theta - (\theta k + Y(0)/\rho)\Lambda'(k) \\ &= (1-\Lambda(k))\theta + [Y'(k) - r - \theta(1-\Lambda(k))(\rho+\psi\Lambda(k))]/(2\psi\Lambda(k) + \rho - \psi) \\ &= [Y'(k) - r - \theta\psi(1-\Lambda(k))^2]/(2\psi\Lambda(k) + \rho - \psi). \end{aligned}$$

So a locally optimal capital with  $d/d V(k,\Lambda(k)) = 0$  must have  $Y'(k) = r+\theta\psi(1-\Lambda(k))^2$ .

When  $k$  satisfies the local-optimality condition, we have

$$0 = d/dk W(k,\Lambda(k)) = (1-\Lambda(k))\theta - (\theta k + Y(0)/\rho)\Lambda'(k) = 0,$$

and so  $\Lambda'(k) = (1-\Lambda(k))\theta/(\theta k + Y(0)/\rho) > 0$ .

Summarizing, we get the following characterization of an optimal liberalized regime.

Fact If the optimal regime  $(k,\lambda)$  has  $\lambda=0$  then  $k=K_0$  where  $Y(K_0) - (r + \rho\theta)K_0 = Y(0)$ .

On the other hand, if the optimal regime  $(k,\lambda)$  has  $\lambda>0$  then it must satisfy the equations

$$(Y(k)-rk)/(\theta k + Y(0)/\rho) = (\rho+\psi\lambda)(1-\lambda) \quad \text{and} \quad Y'(k) = r+\theta\psi(1-\Lambda(k))^2,$$

and it must satisfy the inequalities  $\max\{0,1-\rho/\psi\} < \lambda < 1$  and  $k > K_0$ .

Example 0 Let our basic parameters be  $r = \rho = 0.05$ ,  $\psi = 0.1$ ,  $\theta = 1$ , and  $Y(k) = 0.2k^{0.8}$ .

The ideal without incentive constraints is  $K^* = 335.5$ ,  $Y(K^*) = 20.97$ ,  $Y'(K^*) = 0.05$ .

With incentive constraints, the optimal regime is  $k = 32$ ,  $\lambda = 0$ ,  $Y(k) = 3.2$ ,  $V(k, \lambda) = 32$ .

Reducing  $\psi$  to  $\psi = 0.09$  would change the optimum to  $k = 303.9$ ,  $\lambda = 0.895$ ,  $Y = 19.38$ ,  $V = 32.02$ .

(Further reduction of the scandal rate  $\psi$  would reduce  $\lambda$  a bit, as the  $V \geq W$  constraint is relaxed.)

Example 1 Let our basic parameters be  $r = \rho = 0.05$ ,  $\psi = 0.1$ ,  $\theta = 1$ .

Suppose that output is produced by three factors, which we may call labor (L), capital (K), and land (D) according to the function  $L^{0.4}K^{0.5}D^{0.1}$ .

Each island is endowed with 1 unit of fixed labor ( $L=1$ ), one unit of fixed land ( $D=1$ ).

If there were no capital on an island, the ideal would be feasible:  $K^* = 100$ ,  $\lambda = 0$ ,  $Y = 10$ ,  $V = 100$ .

But suppose instead that each island is endowed with 25 units of fixed capital ( $F=25$ ),

which can be augmented by capitalist investment  $k$  to yield output

$Y(k) = L^{0.4} (F+k)^{0.5} D^{0.1} = (25+k)^{0.5}$  with the given endowments  $L=1$ ,  $F=25$ ,  $D=1$ .

With this production function, the optimal regime for the ruler of each island is  $k=0$ ,  $\lambda=0$ .

Although  $Y'(0) = 0.1 > r$ , any positive investment would require a liberalization  $\lambda > 0.5$ , so that rulers prefer not to encourage any investment. (This is the curse of natural resources.)

Production on each island is then  $Y(0) = 5$ , and the ruler's value is  $V(0,0) = 100$ .

We have ignored the cost of wages, assuming that most laborers are bound serfs who are forced to work for negligible wages. But the marginal product of labor in this solution is

$.4Y(0)/L = 2$ , which would be the wage rate for any free labor on each island.

Example 2 (Tiebout effects)

Now suppose that one island considers the possibility of attracting additional free laborers from other islands, at the given wage rate  $w=2$ . So with investment  $k$  and new free immigrant labor  $n$

on this island, the product net of wage costs would be  $\hat{Y}(k,n) = (1+n)^{0.4} (25+k)^{0.5} (1)^{0.1} - 2n$ .

For any given  $k$ , the maximum of this net product over all  $n \geq 0$  is achieved when

$2 = 0.4(25+k)^{0.5}/(1+n)^{0.6}$ ,  $2(n+1) = .4(1+n)^{0.4} (25+k)^{0.5}$ , and  $(1+n)^{0.4} = (.2)^{2/3} (25+k)^{1/3}$ .

So  $Y(k) = \max_{n \geq 0} \hat{Y}(k,n) = (1-.4)(.2)^{2/3} (25+k)^{5/6} + 2 = 0.2052(25+k)^{5/6} + 2$ .

With this production function, the optimal regime on this island is  $k=1431.6$ ,  $\lambda=0.911$ ,

so that production is  $Y(k)=90.77$ , with new labor  $n=28.6$ , and the ruler's value is  $V(k,\lambda)=136.0$ .

So mobility of labor (or other resources that are complementary to capital) can encourage a ruler to liberalize in a way that increases production and income for others in society.

### Lagrangean analysis

Consider  $L(k, \lambda; \mu) = V(k, \lambda) + \mu[V(k, \lambda) - W(k, \lambda)]$

$$= (Y(k) - rk) / (\rho + \psi\lambda) + \mu[(Y(k) - rk) / (\rho + \psi\lambda) - (1 - \lambda)(\theta k + Y(0) / \rho)].$$

So  $\partial L / \partial k = (Y'(k) - r) / (\rho + \psi\lambda) + \mu[(Y'(k) - r) / (\rho + \psi\lambda) - (1 - \lambda)\theta]$ , which becomes 0 when  $0 = [Y'(k) - r] + \mu[(Y'(k) - r) - (\rho + \psi\lambda)(1 - \lambda)\theta] = [Y'(k) - r] + \mu[(Y'(k) - r) - q(\lambda)\theta]$ .

So at an optimal solution, the Lagrange multiplier is  $\mu = [Y'(k) - r] / [\theta q(\lambda) + r - Y'(k)]$ .

Then  $1 + \mu = \theta q(\lambda) / [\theta q(\lambda) + r - Y'(k)]$ .

So when the constraint  $V=W$  is satisfied,

$$\begin{aligned} \partial L / \partial \lambda &= -\psi(1 + \mu)V(\lambda, k) / (\rho + \psi\lambda) + \mu(\theta k + Y(0) / \rho) \\ &= (\theta k + Y(0) / \rho) \{ Y'(k) - r - (1 - \lambda)\psi\theta q(\lambda) / (\rho + \psi\lambda) \} / [\theta q(\lambda) + r - Y'(k)] \\ &= (\theta k + Y(0) / \rho) [Y'(k) - r - \psi\theta(1 - \lambda)^2] / [\theta q(\lambda) + r - Y'(k)]. \end{aligned}$$

### Cobb-Douglas production

Consider the case of  $Y(k) = Ak^\alpha$  with  $0 < \alpha < 1$ .

Then the equations for a solution with positive  $\lambda$  become

$$(Ak^{\alpha-1} - r) / \theta = (\rho + \psi\lambda)(1 - \lambda) \text{ and } \alpha Ak^{\alpha-1} = r + \theta\psi(1 - \lambda)^2,$$

and so  $\lambda$  satisfies the quadratic equation  $r + \theta\psi(1 - \lambda)^2 = \alpha r + \alpha\theta(\rho + \psi\lambda)(1 - \lambda)$ .

This quadratic can be rewritten  $0 = (1 - \alpha)\psi\theta\lambda^2 + (2\psi + \alpha\psi - \alpha\rho)\theta\lambda + (\alpha\rho\theta + \alpha r - r - \psi\theta)$ ,

and its positive  $\lambda$  solution (if any) can be computed from the quadratic formula

$$\lambda = \{(\alpha\rho - \alpha\psi - 2\psi) \pm [(\alpha\rho - \alpha\psi - 2\psi)^2 + 4(1 - \alpha)\psi\theta(r + \psi\theta - \alpha\rho\theta - \alpha)]^{0.5}\} / [(1 - \alpha)\psi\theta].$$

Then with such  $\lambda > 0$ , the corresponding investment is  $k = \{(\alpha A) / [r + \theta\psi(1 - \lambda)^2]\}^{1/(1 - \alpha)}$ .

To find the (globally) optimal solution, we only need to compare the  $V(k, \lambda)$  that we can get from such positive- $\lambda$  regimes with the best that can be achieved with  $\lambda = 0$ .

Here the equation for the best investment  $K_0$  with  $\lambda = 0$  becomes

$$AK_0^\alpha - (r + \rho\theta)K_0 = 0, \text{ and so } K_0 = [A / (r + \rho\theta)]^{1/(1 - \alpha)}.$$