ECONOMICS 30300b ASSIGNMENT 3: ADVERSE-SELECTION EXERCISES

1. We are to design an optimal contract for a principal who needs to buy the services of a skilled artisan when the artisan's cost of supplying his services are not known by the principal.

If the artisan's cost type is t, then his net utility gain from providing q hours of service and receiving wage w would be $u(w,q|t) = w-tq$. The principal's net benefit from paying w for q hours of service from either type would be $6\sqrt{q}-w$. The number of hours worked (which the principal can observe) and the wage can depend on the artisan's cost type (which the artisan knows), but the artisan could misrepresent his type. The artisan also will not work unless he gets a nonnegative utility gain.

(a) Suppose that the artisan's cost type is either 3 or 5, and the principal thinks that the probabilities are $P(t=3) = 2/3$ and $P(t=5) = 1/3$. Find the contract that maximizes the principals's expected net benefit from her business with the artisan. Show both the amount of service q(t) and the wage payment w(t) for each possible type t.

(b) If we changed part (a) by assuming that the principal would get $A_H\sqrt{q}-w$ from paying w for q hours of service from a high-type artisan with cost 5, for some parameter A_H (keeping A_L=6), what the bound on A_H defines the range of parameters such that the binding constraints remain the same for the principal's optimal contract? Compute the optimal contract for A_H being this bound plus 1.

(c) Now suppose instead that the artisan's cost type is drawn from a Uniform distribution on the interval from 3 to 5. (The principal's gain from trade is $6\sqrt{q}-w$, as in (a).) Find the contract that maximizes the principals's expected net gain from trade with the artisan. Derive formulas for the amount of service q(t) and the wage payment w(t) for each possible type t. Compute the exact numerical values of these formulas at t=3 and t=5; that is, compute q(3), w(3), q(5), and w(5).

2. Consider a variant of Akerlof's lemons problem. Suppose that workers' reservation wages θ are drawn from a Uniform distribution on the interval 240 to 1000, and a worker with reservation wage θ would yield earnings worth 1.5θ to a firm that hired him. We assume that a worker knows his type θ, but can misrepresent it, and wages cannot be conditioned on the earnings actually produced.

(a) Derive a formula for $\mu(w)$, the expected productivity of an employed worker to a firm when firms are hiring workers at the wage w. Show how to compute $\mu(w)$ for all $w>240$.

(b) Derive a formula for Y(w), the per-capita profits of firms when they hire workers at wage w.

(c) If workers could only be hired by one monopsonistic firm, what wage w would maximize its expected profits? What fraction of workers would accept employment at this monopsonistic wage? What would be the expected productivity of a worker if he accepted employment at this wage?

(d) If all trading were regulated by a Walrasian auctioneer, who sets a wage to make net excess demand equal 0, what could the equilibrium wage be? Assume that firms would expect $\mu(w)=360$ when $w<240$. Then explain: how could your answer change if we dropped this assumption?

(e) Now consider the system where each worker announces a wage at which he can be hired, and then firms decide whom to hire at the announced wages. First let us consider equilibria where all workers who announce a wage W get hired but nobody who bids more than W gets hired. Characterize the set of W for which such an equilibrium is possible.

(f) For the worker-bidding game in (e), construct an equilibrium with the following properties: All workers whose cost-type satisfies $\theta<400$ offer one wage w_0 and are hired for sure, but all workers with $\theta>400$ offer another wage w_1 and are hired with some probability q such that $1>q>0$; firms would refuse to hire workers at any wage higher than w_0 other than w_1; and firms expect zero net profit from the workers whom they hire at either w_0 or w_1.

(g) For the worker-bidding game, find a fully separating equilibrium where the wages that workers

demand are a strictly increasing function of their productivity types, and the probability of being hired is a decreasing function of their wage demands, but only a worker making the lowest possible offer would be sure to get employment.
(h) For the worker-bidding game, find an equilibrium such that all workers with cost-type $\theta<400$ offer the wage w_0 from (f) and are hired for sure, but workers with $\theta>400$ offer a wage that is a strictly increasing function of their type and have positive probabilities of getting hired.
(i) Show that the equilibrium from (h) Pareto-dominates the equilibrium from (g).
Is there any other Pareto ranking among the three equilibria that you found in (f),(g), and (h)?

3. In an identifiable population of entrepreneurial agents, there are two types: good and bad.
When a good agent has an idea for a new venture, it has probability 0.4 of success.
When a bad agent has an idea for a new venture, it has probability 0.1 of success.
Any new venture returns earnings 10 if it is a success, 0 otherwise.
All these agents have constant risk tolerance T=5, and so they want to get some insurance from investment bankers. There are many risk-neutral bankers competing to finance such agents.
(a) Find the best safe incentive compatible plan for the agents. That is, specify how much each type of agent will be promised in the event of success or failure, so as to maximize the expected utility of each type of agent, subject to the constraints that that the bankers should expect to break even on each type of agent, and neither type of agent wants to imitate the other. Compute the certainty equivalent for each type of agent.
(b) Suppose that, in the overall population, 70% of all agents are good. Show that a banker could offer a pooling contract that agents would prefer to the plan in part (a), and would yield positive expected profit for the banker if all agents accepted his offer.
(c) Specify an alternative contract, consisting of one wage w_S in case of success and another wage w_F in case of failure, that a banker might offer which would be better than the pooling contract from (b) for good agents, worse than the pooling contract from (b) for bad agents, and would yield positive expected profit for the banker when he does business with good agents only.
(d) Suppose that there is an educational program which does not increase an agent's productivity or chances of success, but the effort-cost to complete the program is different for good and bad types, and the certificate of completion is observable by bankers. Let c_G and c_B denote the effort cost (in monetary units) of for good and bad types respectively to complete this program. What inequalities must c_G and c_B satisfy for there to be an equilibrium where good types complete the program before entering the new-venture-financing market but bad types do not?

4. Consider a bilateral trading problem where agent 1 is the seller of a unique object and 1's type may be either H or L, with probilities p_H and $p_L=1-p_H$ respectively. Agent 2 is the buyer.
The object would be worth \$100 to 1 or \$190 to 2 if 1's type is L,
but the object would be worth \$200 to 1 or \$220 to 2 if 1's type is H.
(a) Find the best safe incentive-compatible trading plan for the seller, where the probability of trade q_t and the expected amount w_t paid by the buyer depend incentive-compatibly on the seller's type t.
(b) Show that, when the parameter p_H is less than some critical value $\bar{p}$, this separating equilibrium is interim incentive-efficient. Prove this incentive-efficiency by explicitly constructing utility weights and Lagrange multipliers (λ,α), as functions of the parameter p_H, that verify the optimality of this plan. What goes wrong with your construction when $p_H > \bar{p}$?
(c) For the $\bar{p}$ that you found in (b), show that the separating plan is in fact interim pareto-dominated by a pooling contract (in which both types always sell) when $p_H > \bar{p}$.

(d) Show that the buyer's optimal trading plan also depends on whether p_H is above or below $\bar{p}$.
(e) Now suppose that there is a costly public signal that the seller could make before approaching the buyer, and the seller's cost of this signal would be $c_L>0$ or $c_H>0$, depending on the his type. After choosing to make the signal or not, the seller then approaches the buyer and makes a first-and-final offer to sell his object for any selected price, which the buyer can only accept or reject.
[e1] What conditions must c_L and c_H satisfy for there to be an equilibrium where the seller would choose to signal when his type is H but not when his type is L?
[e2] What conditions must c_L and c_H satisfy for there to be an equilibrium where both types of the seller would choose to signal? (These conditions may depend on the parameter p_H.)

5. Consider the problem of minimizing the expected cost to the monopsonistic buyer of some service, when one or more possible suppliers exist, but each supplier's cost of providing the service is his own private information. There are n+1 suppliers, numbered 0, 1, 2, ..., n. Exactly one of them must get the contract to supply the service. Supplier 0 has a cost that is drawn from a Uniform distribution on the interval \$100 to \$200, and every other supplier (1,2,...,n) has a cost that is independently drawn from a Uniform distribution on the interval \$200 to \$300.
(a) Characterize an optimal procurement plan that minimizes the buyer's expected cost subject to incentive compatibility and nonnegative-profit constraints for all suppliers.
(b) Derive formulas (that will depend on n and θ_0) to compute the interim expected profit of supplier 0 given any cost-type θ_0 in [100,200] in the optimal procurement plan from (a). What happens to these expected profits as $n\to\infty$? [Note: $P(\min\{\tilde{\theta}_1,...,\tilde{\theta}_n\} > s) = (1-P(\tilde{\theta}_1\leq s))^n$.]
(c) When n=1, what is the probability that the low-cost supplier 0 will get the job in the optimal procurement plan? As $n\to\infty$, what is the limiting probability that supplier 0 will get the job, and what is the limiting expected cost to the buyer?

6. Agents 1 and 2 each own 50% of an asset. Each agent i feels that owning 100% would be worth $\tilde{\theta}_i$ to him, and owning 50% is worth $0.5\tilde{\theta}_i$, where $\tilde{\theta}_i$ is i's privately known type. Each agent's belief about the other agent's type is described by a Uniform probability distribution on [0,1].
If we account payoffs so that payoff 0 corresponds to owning 0% and getting no money, then i's payoff from owning q_i and getting payment w_i is $u_i(q_i,w_i,\theta_i) = \theta_i q_i + w_i$. Then feasibility requires $q_1+q_2 = 1$, because their shares must sum to 100%, and $w_1+w_2 = 0$, because payments are only between the two agents. For each possible type θ_i, the minimum payoff for participation is $0.5\theta_i$, because i has the right to keep his 50% share with no payment.
Now suppose we can find an incentive-compatible plan that treats the two agents symmetrically (as a function of their types) and that always allocates a 100% share to whichever agent has the higher (reported) value for the enterprise.
(a) For any type θ_i, what is i's conditionally expected share $Q_i(\theta_i) = E(q_i(\tilde{\theta}_1,\tilde{\theta}_2)|\tilde{\theta}_i=\theta_i)$ in this plan?
(b) Use the expected-payment and informational-rent equations to compute i's interim expected payoff $U_i(\theta_i)$ for each possible type θ_i in the interval [0,1].
(c) Show that this plan satisfies participation constraints for all types θ_i, by finding the type θ_i with the least expected participation gain $U_i(\theta_i)-0.5\theta_i$ and verifying that its gain is positive.
*(d) Consider a game where the agents simultaneously submit bids (β_1,β_2) to buy each other out, and then the low bidder must sell his share to the high bidder for the high bid ($\max_i \beta_i$).
This game has an equilibrium where each bids $\beta_i = \theta_i/3$. Verify that the expected payoff to each type of each agent in this equilibrium of this game is exactly what you predicted in (b).
(Reference: Cramton, Gibbons, Klemperer, Econometrica 1987.)

ECONOMICS 30300b ASSIGNMENT 2: MORAL-HAZARD EXERCISES

1. An agent chooses one of two hidden actions, a_H and a_L. The agent pays a hidden monetary effort cost c_H if he does a_H, or effort cost c_L if he does a_L. The agent's observable output $\tilde{y}$ will be a Normal random variable with a given standard deviation σ, but its mean will be m_H if he chooses a_H, but m_L if he chooses a_L. Suppose $c_H > c_L$ and $m_H > m_L$.

Suppose that the agent has a constant risk tolerance T, but the principal is risk neutral. The agent's reservation wage is w_0, which he can get (with zero effort cost) from other employers.

(a) Suppose that the principal will pay the agent a wage $w(\tilde{y}) = A+B\tilde{y}$ that depends linearly on $\tilde{y}$, for some constants A and B to be chosen by the principal.

For any given B, show (as a formula that depends on B and the given parameters of the problem) what is the smallest value of A such that the agent's certainty equivalent from action a_H is not less than w_0. Then show (as a formula that depends on the given parameters of the problem) what is the smallest share B that makes the agent willing to choose a_H rather than a_L.

(b) Now consider the special case of $c_H=1$, $c_L=0$, $m_H=2$, $m_L=0$, $\sigma=2$, $w_0=0$, $T=2$.

Instead of using a linear formula to determine the wage as a function of the observable output $\tilde{y}$, suppose instead that the principal will use a two-wage rule, paying some wage w_L if $\tilde{y} < 1$ and some other wage w_H if $\tilde{y} > 1$. Find the two-wage rule (w_L,w_H) that minimizes the expected wage cost for the principal subject to the particpation constraint and the incentive constraint that the agent is be willing to work here and do a_H.

(c) Redo part (b), but now suppose that the two-wage rule is to pay some wage w_L if $\tilde{y} < -1$ and some other wage w_H if $\tilde{y} > -1$.

(d) Compare the principal's expected costs from parts (b), (c), and (a) with the same parameters.

2. The observable performance of a shop may be good, fair, or poor, with a probability distribution that depends on the manager's hidden effort, which may be high or low:

P(good|high) = 0.4, P(fair|high) = 0.4, P(poor|high) = 0.2;

P(good|low) = 0.2, P(fair|low) = 0.4, P(poor|low) = 0.4.

The manager's utility depends on his wage $w \geq 0$ and his effort according to the formula

$u(w,\text{high}) = w^{0.5}$, $u(w,\text{low}) = 1.2w^{0.5}$. His outside option is yields utility 10.

(a) Write the optimization problem of minimizing the expected wage subject to the constraints that the manager should be willing to work in this shop and choose high effort.

(b) Write the first-order conditions for the optimal wages in terms of the Lagrange multipliers for the constraints in this problem, assuming that the optimal wages are strictly positive.

(c) Show that the equations from (b) imply a formula that can be used to compute the Lagrange multiplier of the participation constraint from the Lagrange multiplier of the incentive constraint.

(d) Numerically solve for the optimal wage plan, as a function of the shop's performance, and show the Lagrange multipliers for each constraint.

3. Consider the moral hazard model with risk neutrality but limited liability (from Tirole). Suppose the required investment is I=20, the project's return is R=50 if successful but is 0 if unsuccessful, the probability of success is $p_H=0.6$ if the agent behaves well but is $p_L=0.2$ if the agent misbehaves, the agent's outside option is $\bar{w}=0$, and the agent's private benefit from misbehavior is B=8.

(a) Suppose first that the agent has no collateral (C=0) and punishment is not allowed. What is the highest expected net profit that can be achieved with the agent willing to participate and behave well?

(b) Find the optimal contract and optimized net profit with C=0 but with punishment allowed.

(c) How much collateral C would the agent need for our optimal contract without punishment to yield the same expected profit as in part (b) (with punishment but C=0).

4. Consider a Shapiro-Stiglitz moral-hazard problem where agents control a Poisson process in which accidents occur at a rate of $\alpha=0.2$ per unit time when the agent's hidden action is diligent a_H, but accidents occur at a rate of $\beta=0.5$ per unit time when the agent hidden action is shirking a_L.

Shirking give the agent a hidden income worth D=6 per unit time. Agents are risk-neutral, and use the discount rate per unit time r=0.1. Agents can get the competitive market wage $\bar{w}=1$, and so the present discounted value of an agent's alternative in the labor market is $\bar{V}=\bar{w}/r=10$.

(a) Consider a incentive plan in which the agent will be paid a constant wage w until an accident occurs, and then he will be dismissed and return to the competitive labor market forever.

What is the lowest wage w that will motivate the agent to always be diligent?

(b) Now suppose that the principal will randomize if an accident occurs, dismissing the agent with some fixed probability q, and retaining him with probability $1-q$ (regardless of any history of past accidents). The agent will be paid a constant wage w until he is dismissed, and then he will return to the competitive labor market forever. As a function of q, what is the lowest wage w that will motivate the agent to always be diligent? What q between 0 and 1 minimizes this wage?

5. An agent will choose efforts a_1 and a_2 to put into tasks 1 and 2. He can choose any nonnegative numbers $a_1 \geq 0$ and $a_2 \geq 0$, and then his cost of effort will be $C(a_1,a_2) = 2(a_1+a_2)^2 + (a_1)^2 + (a_2)^2$ per unit time. The principal can observe two outcomes $\tilde{Y}_1$ and $\tilde{Y}_2$ which are independent Normal random variables, each with the variance $\sigma^2 = 1$, and with means $E(\tilde{Y}_1) = a_1$, $E(\tilde{Y}_2) = a_2$.

The principal will get income $R(\tilde{Y}_1,\tilde{Y}_2) = 24\tilde{Y}_1 + 24\tilde{Y}_2$. Following Holmstrom and Milgrom's advice, the principal will pay the agent a wage of the form $w(\tilde{Y}_1,\tilde{Y}_2) = A + B_1\tilde{Y}_1 + B_2\tilde{Y}_2$.

Then the principal's profit will be $R(\tilde{Y}_1,\tilde{Y}_2)-w(\tilde{Y}_1,\tilde{Y}_2)$. The agent's utility function $u(w-C)$ has constant risk tolerance T=2, and the agent's outside option pays $\bar{w}=0$.

(a) Express the agent's certainty equivalent payoff as a function of his actions (a_1,a_2), and the wage-plan's parameters (A,B_1,B_2).

(b) Show how the agent's optimal actions (a_1,a_2) would depend on the principal's (A,B_1,B_2).

(Remember that these efforts $a_1 \geq 0$ and $a_2 \geq 0$ cannot be negative numbers.)

(c) The principal wants to choose an incentive plan (A,B_1,B_2) that maximizes his expected profit $R-w$, subject to a moral-hazard constraint that the agent's actions (a_1,a_2) be optimal for the agent, and an ex-ante participation constraint that the agents certainty equivalent is not less than $\bar{w}=0$.

Noting the obvious symmetry in this problem, let us consider just plans where $B_1=B_2$.

What plan (A,B_1,B_1) would maximize the principal's expected profit subject to the moral-hazard and participation constraints? What is the principal's expected profit?

(d) Now let us consider plans where $B_2=0$, so the agent is encouraged to specialize in task 1.

What plan $(A,B_1,0)$ would maximize the principal's expected profit subject to the moral-hazard and participation constraints? What is the principal's expected profit?

6. Let us consider a Poisson approximation of the preceding problem, with $\varepsilon=0.1$. As above, the agent chooses $a_1 \geq 0$ and $a_2 \geq 0$ with cost $C(a_1,a_2) = 2(a_1+a_2)^2 + (a_1)^2 + (a_2)^2$ per unit time. The principal observes two types of arrivals, numbered i = 1,2, and the arrivals are independent Poisson processes with rates that depend on the agent's (unobservable) actions.

The expected rate of type-i arrivals per unit time is $\Lambda(a_i) = (\sigma^2+\varepsilon a_i)/\varepsilon^2 = 100 + 10a_i$.

When $\tilde{x}_i$ denotes the number of arrivals of type i in a period of length 1, the corresponding approximate Brownian motion is $\tilde{y}_i = \varepsilon\tilde{x}_i - \sigma^2/\varepsilon = 0.1\tilde{x}_i - 10$. So to match the revenue $24\tilde{Y}_1+24\tilde{Y}_2$ in the previous problem, suppose now that the principal's income over any interval of length 1 is

$\tilde{R} = 2.4\tilde{x}_1 + 2.4\tilde{x}_2 - 480$. That is, the principal gets 2.4 from arrival, but has continuous costs 480 per unit time. Now consider a wage plan of paying the agent a base wage α per unit time plus β_1 per arrival of type 1 plus β_2 per arrival of type 2. As before, the agent has constant risk tolerance T=2,and the agent's outside option pays $\bar{w}=0$.

(a)-(d) Redo the four parts of problem 4 for this Poisson approximation to it.

(d) Compare the mean and standard deviation of the principal's revenue $\tilde{R}$ in a period of length 1 across the Brownian model of 4 and the Poisson model of 5 when the agent chooses $a_1=1$, $a_2=0$.

ECONOMICS 30300b ASSIGNMENT 1: RISK-SHARING EXERCISES

1. (a) For any given utility weights $\lambda_1>0$ and $\lambda_2>0$ and risk tolerances $T_1>0$ and $T_2>0$, derive the sharing rules (x_1,x_2) that maximize $\lambda_1 E(u_1(x_1(\tilde{Y}))+\lambda_2 E(u_2(x_2(\tilde{Y}))$ subject to the constraint $x_1(\tilde{Y})+x_2(\tilde{Y}) = \tilde{Y}$, when $u_1(x) = -EXP(-x/T_1)$ and $u_2(x) = -EXP(-x/T_2)$.
(You will need to find the formula for how $x_1(0)$ depends on λ_1, λ_2, T_1, and T_2.)
(b) For any given utility weights $\lambda_1>0$ and $\lambda_2>0$ and risk tolerance $T_2>0$, derive equations to characterize the sharing rules (x_1,x_2) that maximize $\lambda_1 E(u_1(x_1(\tilde{Y}))+\lambda_2 E(u_2(x_2(\tilde{Y}))$ subject to the constraint $x_1(\tilde{Y})+x_2(\tilde{Y}) = \tilde{Y}$, when $u_1(x) = LN(x)$ and $u_2(x) = -EXP(-x/T_2)$.
(You may characterize $x_2(y)$ in terms of $x_1(y)$. Then, to implement your characterization, you can plot $x_1(y)$ and $x_2(y)$ for the case of $\lambda_1=\lambda_2=1=T_2$, using Excel or any other computational program.)

2. Suppose that the n members of a syndicate have constant risk tolerance, and let T_i denote the constant risk tolerance of individual i.
So i's preferences can be described by a utility function $u_i(x) = -EXP(-x/T_i)$.
Let $T^* = \sum_i T_i$ be the sum of risk tolerances of all members of the syndicate.
Suppose that the total income of the syndicate's investments will be $\tilde{Y}$, a discrete random variable with a finite set of possible values and a given probability distribution p(y) over these values.
The syndicate's income $\tilde{Y}$ will be divided among the syndicate's members according to an efficient rule such that the income of each individual i will be $x_i(\tilde{Y}) = x_i(0) + \tilde{Y}T_i/T^*$.
Let $CE_i(x_i(\tilde{Y}))$ denote i's certainty equivalent of i's share of the income.
For comparison, imagine an alternative situation where the total income $\tilde{Y}$ would be consumed by a single fictitious individual who has constant risk tolerance T^*, and let $CE^*(\tilde{Y})$ denote the certainty equivalent that the total income $\tilde{Y}$ would have for such a fictitious owner.
(a) Show that $CE_i(x_i(\tilde{Y}))$ can be expressed as a linear function of $CE^*(\tilde{Y})$.
(b) Show that $\sum_i CE_i(x_i(\tilde{Y})) = CE^*(\tilde{Y})$.

3. Individuals 1 and 2 are a brother and sister who have inherited equal shares of their mother's scattered real estate properties. They have constant risk tolerances T_1=\$20,000 and T_2=\$45,000 respectively. Appraisals of the property by independent real-estate experts suggest that the total returns from selling the properties individually over a period of months may be viewed as a draw from a Normal distribution with mean \$200,000 and standard deviation \$75,000.
But one real estate agent just offered to buy all the properties right now for \$150,000, provided that they accept the offer within 24 hours.
(a) Show that, if they were constrained to share equally all income from the sale of the properties, then 1 and 2 would disagree about whether the Normal(μ=\$200,000,σ=\$75,000) gamble was preferable to the certain \$150,000.
(b) Suppose that, because of their disagreement, the real-estate agent's 24-hour offer has been lost, and so their prospective returns now are defined by the Normal(μ=\$200,000,σ=\$75,000) distribution. Suppose 2 can make an offer acquire any part of her brother's 50% share, for any price now that would be acceptable to her brother. What share should 2 try to acquire, and what is the smallest payment that 1 would accept now in exchange for this reduction of his share of the risks? What is the maximum amount that 2 would be willing to pay to acquire this increased share (above the 50% that she already owns)?
(c) Show that, after their shares have been adjusted as in part (b), if the real-estate agent extended the \$150,000 offer for another 24 hours, both 1 and 2 would want to turn it down.
What is the smallest price for the entire estate that they would be willing to accept?