

Computing randomized Nash equilibria for games that are larger than 2×2 can be difficult, but working a few examples can help you better understand Nash's subtle concept of equilibrium. At the top of page 138, Osborne describes a general procedure for finding randomized Nash equilibria for any finite game, based on the characterization in Proposition 116.2. Here, we describe this procedure in somewhat different terms, with an illustrative application.

We are given some game, including a given set of players N and, for each i in N , a given set of feasible actions A_i for player i and a given payoff function $u_i: A_1 \times \dots \times A_n \rightarrow \mathbb{R}$ for player i . The support of a randomized equilibrium is, for each player, the set of actions that have positive probability of being chosen in this equilibrium.

To find a Nash equilibrium, we can apply the following 5-step method:

(1) Guess a support for all players. That is, for each player i , let S_i be a subset of i 's actions A_i , and let us guess that S_i is the set of actions that player i will use with positive probability.

(2) Consider the smaller game where the action set for each player i is reduced to S_i , and try to find an equilibrium where all of these actions get positive probability.

To do this, we need to solve a system of equations for some unknown quantities.

The unknowns: For each player i in N and each action s_i in i 's support S_i , let $\sigma_i(s_i)$ denote i 's probability of choosing s_i , and let w_i denote player i 's expected payoff in the equilibrium.

The equations: For each player i , the sum of these probabilities $\sigma_i(s_i)$ must equal 1.

For each player i and each action s_i in S_i , player i 's expected payoff when he chooses s_i but all other players randomize independently according to their σ_j probabilities must be equal to w_i .

Let $Eu_i(a_i | \sigma_{-i})$ denote player i 's expected payoff when he chooses action a_i and all other players are expected to randomize independently according to their σ_j probabilities.

Then the equations can be written: $\sum_{s_i \in S_i} \sigma_i(s_i) = 1 \quad \forall i \in N$; and $Eu_i(s_i | \sigma_{-i}) = w_i \quad \forall i \in N \quad \forall s_i \in S_i$. (Here \forall means "for all", \in means "in".) We have as many equations as unknowns ($w_i, \sigma_i(s_i)$).

(3) If the equations in step 2 have no solution, then we guessed the wrong support, and so we must return to step 1 and guess a new support.

Assuming that we have a solution from step (2), continue to (4) and (5)

(4) The solution from (2) would be nonsense if any of the "probabilities" were negative.

That is, for every player i in N and every action s_i in i 's support S_i , we need $\sigma_i(s_i) \geq 0$.

If these nonnegativity conditions are not satisfied by a solution, then we have not found an equilibrium with the guessed support, and so we must return to step 1 and guess a new support.

If we have a solution that satisfies all these nonnegativity conditions, then it is a randomized equilibrium of the reduced game where each player must can only choose actions in S_i .

(5) A solution from (2) that satisfies the condition in (4) would still not be an equilibrium of the original game, however, if any player would prefer an action outside the guessed support.

So next we must ask, for each player i and for each action a_i that is in A_i but is not in the guessed support S_i , could player i do better than w_i by choosing a_i when all other players randomize independently according to their σ_j probabilities? Recall $Eu_i(s_i | \sigma_{-i}) = w_i$ for all s_i in S_i .

So now, for every action a_i that is in A_i but is not in S_i , we need $Eu_i(a_i | \sigma_{-i}) \leq w_i$.

If our solution satisfies all these inequalities then it is an equilibrium of the given game.

But if any of these inequalities is violated (some $Eu_i(a_i | \sigma_{-i}) > w_i$), then we have not found an equilibrium with the guessed support, and so we must return to step 1 and guess a new support.

In a finite game, there are only a finite number of possible supports to consider.

Example. Find all Nash equilibria (pure and mixed) of the following 2×3 game:

Player 1	Player 2		
	L	M	R
T	7, 2	2, 7	3, 6
B	2, 7	7, 2	4, 5

It is easy to see that this game has no pure-strategy equilibria (2's best response to T is M, but T is not 1's best response to M; and 2's best response to B is L, but B is not 1's best response to L).

This eliminates the six cases where each player's support is just one action.

Furthermore, when either player is restricted to just one action, the other player always has a unique best response, and so there are no equilibria where only one player randomizes.

That is, both players must have at least two actions in the support of any equilibrium.

Thus, we must search for equilibria where the support of player 1's randomized strategy is $\{T, B\}$, and the support of player 2's randomized strategy is $\{L, M, R\}$ or $\{M, R\}$ or $\{L, M\}$ or $\{L, R\}$.

We consider these alternative supports in this order.

Guess support is $\{T, B\}$ for 1 and $\{L, M, R\}$ for 2?

We may denote 1's strategy by $p[T] + (1-p)[B]$ and 2's strategy by $q[L] + (1-q-r)[M] + r[R]$, that is $p = \sigma_1(T)$, $1-p = \sigma_1(B)$, $q = \sigma_2(L)$, $r = \sigma_2(R)$, $1-q-r = \sigma_2(M)$.

Player 1 randomizing over $\{T, B\}$ requires $Eu_1(T|\sigma_2) = Eu_1(B|\sigma_2)$, and so

$$7q + 2(1-q-r) + 3r = 2q + 7(1-q-r) + 4r.$$

Player 2 randomizing over $\{L, M, R\}$ requires $Eu_2(L|\sigma_1) = Eu_2(M|\sigma_1) = Eu_2(R|\sigma_1)$, and so

$$2p + 7(1-p) = 7p + 2(1-p) = 6p + 5(1-p).$$

We have three equations for three unknowns (p, q, r), but they have no solution (as the two indifference equations for player 2 imply both $p=1/2$ and $p=3/4$, which is impossible).

Thus there is no equilibrium with this support.

Guess support is $\{T, B\}$ for 1 and $\{M, R\}$ for 2?

We may denote 1's strategy by $p[T] + (1-p)[B]$ and 2's strategy by $(1-r)[M] + r[R]$. ($q=0$)

Player 1 randomizing over $\{T, B\}$ requires $Eu_1(T|\sigma_2) = Eu_1(B|\sigma_2)$, so $2(1-r) + 3r = 7(1-r) + 4r$.

Player 2 randomizing over $\{M, R\}$ requires $Eu_2(M|\sigma_1) = Eu_2(R|\sigma_1)$, so $7p + 2(1-p) = 6p + 5(1-p)$.

These solution for these two equations in two unknowns is $p = 3/4$ and $r = 5/4$.

But this solution would yield $\sigma_2(M) = 1-r = -1/4 < 0$, and so there is no equilibrium with this support.

(Notice: if player 2 never chose L then T would be dominated by B for player 1.)

Guess support is $\{T, B\}$ for 1 and $\{L, M\}$ for 2?

We may denote 1's strategy by $p[T] + (1-p)[B]$ and 2's strategy by $q[L] + (1-q)[M]$. ($r=0$)

Player 1 randomizing over $\{T, B\}$ requires $Eu_1(T|\sigma_2) = Eu_1(B|\sigma_2)$, so $7q + 2(1-q) = 2q + 7(1-q)$.

Player 2 randomizing over $\{L, M\}$ requires $Eu_2(L|\sigma_1) = Eu_2(M|\sigma_1)$, so $2p + 7(1-p) = 7p + 2(1-p)$.

These solution for these two equations in two unknowns is $p = 1/2$ and $q = 1/2$.

This solution yields nonnegative probabilities for all actions.

But we also need to check that player 2 would not prefer deviating outside her support to R.

However $Eu_2(R|\sigma_1) = 6p + 5(1-p) = 6 \times 1/2 + 5 \times 1/2 = 5.5 > Eu_2(L|\sigma_1) = 2 \times 1/2 + 7 \times 1/2 = 4.5$.

So there is no equilibrium with this support.

Guess support is $\{T, B\}$ for 1 and $\{L, R\}$ for 2?

We may denote 1's strategy by $p[T] + (1-p)[B]$ and 2's strategy by $q[L] + (1-q)[R]$. ($q=1-r$)

Player 1 randomizing over $\{T, B\}$ requires $Eu_1(T|\sigma_2) = Eu_1(B|\sigma_2)$, so $7q + 3(1-q) = 2q + 4(1-q)$.

Player 2 randomizing over $\{L, R\}$ requires $Eu_2(L|\sigma_1) = Eu_2(R|\sigma_1)$, so $2p + 7(1-p) = 6p + 5(1-p)$.

These solution for these two equations in two unknowns is $p = 1/3$ and $q = 1/6$.

This solution yields nonnegative probabilities for all actions.

We also need to check that player 2 would not prefer deviating outside her support to M;

$Eu_2(M|\sigma_1) = 7p + 2(1-p) = 7 \times 1/3 + 2 \times 2/3 = 11/3 < Eu_2(L|\sigma_1) = 2 \times 1/3 + 7 \times 2/3 = 16/3$.

Thus, we have an equilibrium with this support: $((1/3)[T] + (2/3)[B], (1/6)[L] + (5/6)[R])$.

The expected payoffs in this equilibrium are $Eu_1 = 7 \times 1/6 + 3 \times 5/6 = 2 \times 1/6 + 4 \times 5/6 = 11/3 = 3.667$

and $Eu_2 = 2 \times 1/3 + 7 \times 2/3 = 6 \times 1/3 + 5 \times 2/3 = 16/3 = 5.333$.

Increasing differences and increasing strategies in Bayesian games

We may consider Bayesian games where first each player i first learns his type \tilde{t}_i , and then each player i chooses his action a_i . We will assume that each player i 's type is drawn from some probability distribution p_i , independently of all other players' types, so that $p_i(\tilde{t}_i) = \text{Prob}(\tilde{t}_i = t_i)$. The payoffs of each player i may depend on all players' types and actions according to some utility payoff function $U_i(a_1, \dots, a_n, \tilde{t}_1, \dots, \tilde{t}_n)$.

Consider a two-player Bayesian game where player 1 has two possible actions, T and B. Player 1 has several possible types, and each possible type is represented by a number t_1 . Player 2 may have many possible actions a_2 and many possible types t_2 . Suppose that player 2's type t_2 is independent of player 1's type t_1 .

The difference in player 1's payoff when he switches from B to T is

$$U_1(T, a_2, t_1, t_2) - U_1(B, a_2, t_1, t_2)$$

This difference depends on player 1's type t_1 , player 2's action a_2 , and player 2's type t_2 .

We say that player 1's payoffs satisfy increasing differences if this difference

$$U_1(T, a_2, t_1, t_2) - U_1(B, a_2, t_1, t_2)$$
 would increase (weakly, increase or stay constant)

whenever player 1's type t_1 is increased, no matter what player 2's action a_2 and type t_2 may be.

That is, increasing differences means that, for every r_1, t_1, a_2 , and t_2 :

$$\text{if } r_1 \geq t_1 \text{ then } U_1(T, a_2, r_1, t_2) - U_1(B, a_2, r_1, t_2) \geq U_1(T, a_2, t_1, t_2) - U_1(B, a_2, t_1, t_2).$$

With increasing differences, 1's higher types find T relatively more attractive than lower types do.

Player 1 is using a cutoff strategy if there is some number θ (the cutoff) such that, for each possible type t_1 of player 1: if $t_1 > \theta$ then type t_1 would choose [T] for sure in this strategy, if $t_1 < \theta$ then type t_1 would choose [B] for sure in this strategy, if $t_1 = \theta$ then type t_1 may choose T or B or may randomize in this strategy.

So in a cutoff strategy, there is at most one type which is randomizing between T and B, and all higher types must be choosing T for sure, and all lower types must be choosing B for sure.

Fact. If player 1's payoffs satisfy increasing differences then, no matter what strategy player 2 may use, player 1 will always want to use a cutoff strategy.

Thus, when we are looking for equilibria, the increasing-differences property assures us that player 1 must be using a cutoff strategy.

Notice that the probability of player 1 choosing T decreases as the cutoff θ increases.

More generally, in games where player 1's action can be any number in some range, we say that player 1's payoffs satisfy increasing differences if, for every pair of possible actions c_1 and d_1 such that $c_1 > d_1$, the difference $U_1(c_1, a_2, t_1, t_2) - U_1(d_1, a_2, t_1, t_2)$ would increase whenever player 1's type t_1 is increased, no matter what player 2's action a_2 and type t_2 may be.

If 1's payoffs satisfy increasing differences then, for any strategy of player 2, player 1 will have a best-response strategy that is increasing, in the sense that 1's action would increase (weakly) as his type increases.

("Weakly" increasing here means that two different types might use the same action, or the higher type might use a higher action, but the higher type would never use a lower action.)

Example: Player 1's types possible are $\{0, .1, .2, .3\}$, each with probability $1/4$.
 Player 2 has no private information. 1's actions are $\{T,B\}$, 2's actions are $\{L,R\}$.
 Given 1's type t_1 , the payoff matrix is

	L	R
T	$t_1, 0$	$t_1, -1$
B	$1, 0$	$-1, 3$

So 1's utility difference in switching from B to T depends on 2's action and 1's type as follows:
 $U_1(T,L,t_1) - U_1(B,L,t_1) = t_1 - 1$, $U_1(T,R,t_1) - U_1(B,R,t_1) = t_1 + 1$.

Notice that these differences increase in t_1 . So higher types t_1 always find T relatively more attractive than lower types, and player 1 will use a cutoff strategy.

The possible cutoff strategies are:

- $(\theta > .3)$ every type would choose [B], so 2 thinks the probability of T is $P(T)=0$;
- $(\theta = .3)$ $\{0, .1, .2\}$ would choose [B], but $.3$ would randomize in some way, so $P(T)$ is between 0 and $1/4$;
- $(.2 < \theta < .3)$ $\{0, .1, .2\}$ would choose [B], but $.3$ would choose [T], so $P(T) = 1/4$;
- $(\theta = .2)$ $\{0, .1\}$ would choose [B], $.2$ would randomize in some way, $.3$ would choose [T], and so $P(T)$ is between $1/4$ and $1/2$;
- $(.1 < \theta < .2)$ $\{0, .1\}$ would choose [B], $\{.2, .3\}$ would choose [T], so $P(T) = 1/2$;
- $(\theta = .1)$ 0 would choose [B], $.1$ would randomize in some way, $\{.2, .3\}$ would choose [T], and so $P(T)$ is between $1/2$ and $3/4$;
- $(0 < \theta < .1)$ 0 would choose [B], $\{.1, .2, .3\}$ would choose [T], so $P(T) = 3/4$;
- $(\theta = 0)$ 0 would randomize in some way, $\{.1, .2, .3\}$ would choose [T], and so $P(T)$ is between $3/4$ and 1 ;
- $(\theta < 0)$ every type would choose [T], and so $P(T) = 1$.

There is obviously no equilibrium in which player 2 chooses L for sure or R for sure. (check!)

To make player 2 willing to randomize, we must have $EU_2(L) = EU_2(R)$, that is,

$$P(T)(0) + (1 - P(T))(0) = P(T)(-1) + (1 - P(T))(3), \text{ and so } P(T) = 3/4.$$

So the cutoff θ must be between 0 and $.1$ (0 would choose [B], $\{.1, .2, .3\}$ would choose [T]).

Let q denote the probability that 2 chooses L.

To make 1's cutoff strategy optimal for him, 2's randomized strategy $q[L] + (1 - q)[R]$ must make player 1 prefer B when $t_1 = 0$, but must make 1 prefer T when $t_1 = .1$.

$$EU_1(T|t_1=0) \leq EU_1(B|t_1=0) \text{ implies } (q)(0) + (1-q)(0) \leq (q)(1) + (1-q)(-1), \text{ and so } 1/2 \leq q.$$

$$EU_1(T|t_1=.1) \geq EU_1(B|t_1=.1) \text{ implies } (q)(.1) + (1-q)(.1) \geq (q)(1) + (1-q)(-1), \text{ and so } q \leq 11/20.$$

So in equilibrium, 1 chooses B if $t_1 = 0$, 1 chooses T if $t_1 \geq .1$, and 2 randomizes, choosing L with some probability q that is between $1/2$ and $11/20$.

Now suppose instead player 1 has five possible types $\{0, .1, .2, .3, .4\}$, each with probability $1/5$.

To make player 2 willing to randomize, player 1 must use a strategy such that $P(T) = 3/4$.

For that to occur in an increasing cutoff strategy, the cutoff must be at $t_1 = .1$. So $t_1 = 0$ chooses B; and when $t_1 > .1$ (which has probability $3/5$) player 1 chooses T. The remaining $3/4 - 3/5 = 0.15$ probability of T must come from 1 choosing T with probability $(0.15/0.2) = 0.75$ when $t_1 = .1$.

To make type $t_1 = .1$ willing to randomize, player 2's probability of choosing L must be $q = 11/20$.

Example. Player 1's type t_1 is drawn from a Uniform distribution on the interval from 0 to 1, and payoffs (u_1, u_2) depend on 1's type as follows, where ϵ is a number between 0 and 1 (say $\epsilon=0.1$):

	L	R
T	$\epsilon t_1, 0$	$\epsilon t_1, -1$
B	$1, 0$	$-1, 3$

Player 1's payoffs satisfy increasing differences, so player 1 should use a cutoff strategy, doing T if $t_1 > \theta_1$, doing B if $t_1 < \theta_1$, where θ_1 is some number between 0 and 1.

Then player 2 would think that the probability of 1 doing T is $\text{Prob}(t_1 > \theta) = 1 - \theta$.

You can easily verify that there is no equilibrium where player 2 is sure to choose either L or R.

For player 2 to be willing to randomize between L and R, both L and R must give her the same expected payoff, so $0 = (-1)(1 - \theta_1) + (3)\theta_1$, and so $\theta_1 = 0.25$.

So in equilibrium, player 1 must use the strategy: do T if $t_1 > 0.25$, do B if $t_1 < 0.25$.

For player 1 to be willing to implement this strategy, he must be indifferent between T and B when his type is exactly $t_1 = \theta_1 = 0.25$. Let q denote the probability of player 2 doing L.

Then to make type θ_1 indifferent between T and B, q must satisfy $\epsilon\theta_1 = (1)q + (-1)(1 - q)$, which implies $q = (1 + \epsilon\theta_1)/2 = (1 + 0.25\epsilon)/2$. (So as $\epsilon \rightarrow 0$, q approaches 0.5.)

Now consider a game with two-sided incomplete information.

Suppose player 1's type t_1 is drawn from a Uniform distribution on the interval from 0 to 1, player 2's type t_2 is drawn independently from a Uniform distribution on the interval from 0 to 1, and the payoffs depend on 1's type as follows, for some given number ϵ between 0 and 1:

	L	R
T	$\epsilon t_1, \epsilon t_2$	$\epsilon t_1, -1$
B	$1, \epsilon t_2$	$-1, 3$

With increasing differences, the action T becomes more attractive to higher types of player 1.

Similarly, the action L becomes more attractive to higher types of player 2.

So we should look for an equilibrium where each uses a cutoff strategy of the form

- player 1 does T if $t_1 > \theta_1$, player 1 does B if $t_1 < \theta_1$,
- player 2 does L if $t_2 > \theta_2$, player 2 does R if $t_2 < \theta_2$,

for some pair of cutoffs θ_1 and θ_2 .

It is easy to check that neither player's action can be certain to the other, and so these cutoffs θ_1 and θ_2 must be strictly between 0 and 1.

With t_1 Uniform on 0 to 1, the probability of player 1 doing T ($t_1 > \theta_1$) is $1 - \theta_1$.

Similarly, the probability of player 2 doing L ($t_2 > \theta_2$) is $1 - \theta_2$.

The cutoff types must be indifferent between the two actions. So we have the equations

$$\epsilon\theta_1 = (1)(1 - \theta_2) + (-1)\theta_2, \quad \epsilon\theta_2 = (-1)(1 - \theta_1) + (3)\theta_1.$$

The unique solution to these equations is $\theta_1 = (2 + \epsilon)/(8 + \epsilon^2)$, $\theta_2 = (4 - \epsilon)/(8 + \epsilon^2)$.

Unless a player's type exactly equals the cutoff (which has zero probability), he is not indifferent between his two actions, and he uses the action yielding a higher expected payoff given his type.

As $\epsilon \rightarrow 0$, these equilibria approach the randomized strategies $(.75[T] + .25[B], .5[L] + .5[R])$.

Sequential equilibria of signaling games:

- (1) Player 1 learns his type \tilde{t}_1 which is drawn from some set of possible types T_1 according to some probability distribution P . So $P(t_1)$ denotes the probability that $\tilde{t}_1=t_1$.
 - (2) Knowing his type \tilde{t}_1 , player 1 chooses an action a_1 in some set of possible actions A_1 .
 - (3) Player 2 observes 1's action a_1 , and then player 2 chooses her own action a_2 in some set of possible actions A_2 .
 - (4) Each player i gets a payoff that can depend on player 1's type \tilde{t}_1 , player 1's action a_1 , and player 2's action a_2 according to some function $U_i(\tilde{t}_1, a_1, a_2)$.
- These sets and probability distributions $(T_1, P, A_1, A_2, U_1, U_2)$ together define a signaling game.

A randomized strategy for player 1 specifies probabilities of actions in A_1 depending on 1's type. So a randomized strategy for player 1 may be denoted by a function σ_1 , where $\sigma_1(a_1|t_1)$ denotes the probability of player 1 choosing action a_1 if 1's type is t_1 .

A randomized strategy for player 2 specifies probabilities of actions in A_2 that can depend on 1's action. So a randomized strategy for player 2 may be denoted by a function σ_2 , where $\sigma_2(a_2|a_1)$ denotes the probability of player 2 choosing action a_2 if she observes that 1 chose action a_1 .

The expected payoff for each player i in the game is

$$EU_i(\sigma_1, \sigma_2) = \sum_{t_1 \in T_1} P(t_1) \sum_{a_1 \in A_1} \sigma_1(a_1|t_1) \sum_{a_2 \in A_2} \sigma_2(a_2|a_1) U_i(t_1, a_1, a_2).$$

In equilibrium, each player i 's strategy σ_i maximizes his EU_i given the other player's strategy.

When player 1 uses the strategy σ_1 , the marginal probability of his action a_1 is

$$Q(a_1|\sigma_1) = \sum_{t_1 \in T_1} P(t_1) \sigma_1(a_1|t_1)$$

Observing 1's action a_1 may give player 2 new information about player 1's type.

Let $q(t_1|a_1)$ denote the probability that player 2 would assign to player 1's type being t_1 if she (player 2) observed 1 choosing action a_1 . This function q describes 2's beliefs.

To be consistent with player 1's strategy σ_1 , player 2's beliefs q must satisfy Bayes's formula:

$$q(t_1|a_1) = P(t_1) \sigma_1(a_1|t_1) / Q(a_1|\sigma_1) \text{ for each action } a_1 \text{ such that } Q(a_1|\sigma_1) > 0.$$

For any action a_1 that has zero probability under the strategy σ_1 , in the sense that $Q(a_1|\sigma_1) = 0$, any probability distribution $q(\cdot|a_1)$ over T_1 could represent consistent beliefs for player 2.

When player 2 observes 1's action a_1 and gets beliefs $q(\cdot|a_1)$, her expected payoff from choosing her action a_2 would be $EU_2(a_2|a_1, q) = \sum_{t_1 \in T_1} q(t_1|a_1) U_2(t_1, a_1, a_2)$.

With beliefs q , the strategy σ_2 is sequentially rational for player 2 after observing a_1 if

$$\sum_{a_2 \in A_2} \sigma_2(a_2|a_1) EU_2(a_2|a_1, q) = \max_{a_2 \in A_2} EU_2(a_2|a_1, q).$$

When player 2 is expected to use the strategy σ_2 , player 1's expected payoff from choosing action a_1 given any type t_1 would be $EU_1(a_1|t_1, \sigma_2) = \sum_{a_2 \in A_2} \sigma_2(a_2|a_1) U_1(t_1, a_1, a_2)$.

Against σ_2 , the strategy σ_1 is sequentially rational for player 1 with type t_1 if

$$\sum_{a_1 \in A_1} \sigma_1(a_1|t_1) EU_1(a_1|t_1, \sigma_2) = \max_{a_1 \in A_1} EU_1(a_1|t_1, \sigma_2).$$

In a sequential equilibrium (σ_1, σ_2, q) : q is consistent with σ_1 , σ_1 is sequentially rational for 1 with every possible type t_1 , and σ_2 is sequentially rational for 2 after every possible action a_1 in A_1 .

Trading between a buyer and a seller, who knows more about the object being sold

Facts about Uniform distributions. Suppose that \tilde{X} is a random variable drawn from a Uniform distribution on the interval from A to B, for some given numbers A and B such that $A < B$.

Then $E(\tilde{X}) = (A+B)/2$. Furthermore, for any number θ between A and B:

$$\Pr(\tilde{X} < \theta) = \Pr(\tilde{X} \leq \theta) = (\theta - A)/(B - A),$$

$$E(\tilde{X} | \tilde{X} \leq \theta) = E(\tilde{X} | \tilde{X} < \theta) = (A + \theta)/2,$$

$$E(\tilde{X} | \tilde{X} \geq \theta) = E(\tilde{X} | \tilde{X} > \theta) = (\theta + B)/2.$$

Example. To illustrate the problems of trading between individuals who have different information, consider the following simple situation, involving two players.

Player 1 is the seller of some unique object which he owns.

Player 2 is the only possible buyer of this object.

Depending on the object's quality, it may be worth as little as \$40 to player 1 and \$60 to player 2 (if its quality is low) or as much as \$100 to player 1 and \$120 to player 2 (if its quality is high).

Player 1 knows the quality of the object. Let 1's type \tilde{t}_1 denote his value of keeping the object.

With any quality, the object would be worth \$20 more to player 2 than to player 1.

That is, given 1's type \tilde{t}_1 , the value of the object to player 2 would be $V_2(\tilde{t}_1) = \tilde{t}_1 + 20$.

Player 2's belief about \tilde{t}_1 is described by a Uniform distribution on the interval \$40 to \$100.

Game where buyer bids Suppose first that player 2 can offer to buy for any positive price r , and then player 1 will accept or reject the offer. If the offer is rejected then they each get profit 0.

If the offer is accepted then 1's profit is $r - \tilde{t}_1$ and 2's profit is $V_2(\tilde{t}_1) - r$.

In a subgame-perfect equilibrium, player 1 will accept if $\tilde{t}_1 < r$, but player 1 will reject if $\tilde{t}_1 > r$.

Player 2's expected profit from offering any price r is $\Pr(\tilde{t}_1 < r) * (E(V_2(\tilde{t}_1) | \tilde{t}_1 < r) - r)$.

For any number r between 40 and 100, this expected profit is

$$\Pr(\tilde{t}_1 < r) * (E(\tilde{t}_1 + 20 | \tilde{t}_1 < r) - r) = \Pr(\tilde{t}_1 < r) * (E(\tilde{t}_1 | \tilde{t}_1 < r) + 20 - r) =$$

$$((r - 40)/(100 - 40)) * ((40 + r)/2 + 20 - r) = (r - 40) * (80 - r) / 120 = (-3200 + 40r - r^2) / 120.$$

This quadratic formula is maximized by letting $r = 60$.

(The buyer cannot gain by bidding less than 40 or more than 100, because a bid below 40 would be surely rejected, and a bid above 100 would be worse than the surely-accepted bid of 100.)

So in the unique subgame-perfect equilibrium of this game, player 2 offers to buy for \$60, and player 1 accepts if $\tilde{t}_1 < 60$. The probability of trade is $\Pr(\text{trade}) = (60 - 40)/(100 - 40) = 1/3$.

Game where seller bids Suppose now that player 1 can offer to buy for any positive price r , and then player 2 will accept or reject the offer. If the offer is rejected then they each get profit 0. If the offer is accepted, then 1's profit is $r - \tilde{t}_1$ and 2's profit is $V_2(\tilde{t}_1) - r$. In this game, the price is named by the player who has private information, and so signaling effects will give us many equilibria.

Let's look first for an equilibrium where there is some price \bar{r} such that player 2 would surely accept an offer to sell for \bar{r} but would surely reject an offer to sell for any price higher than \bar{r} . In this equilibrium, player 1 will offer \bar{r} if $\tilde{t}_1 < \bar{r}$.

For player 2 to accept the offer \bar{r} , 2's expected profit from accepting \bar{r} must not be negative, so $0 \leq E(V_2(\tilde{t}_1) | \tilde{t}_1 < \bar{r}) - \bar{r} = E(\tilde{t}_1 + 20 | \tilde{t}_1 < \bar{r}) - \bar{r} = (40 + \bar{r})/2 + 20 - \bar{r}$, which implies $\bar{r} \leq 80$.

Player 2 can be expected to reject any offer to sell at a price greater than \bar{r} , because such a trade would be unprofitable for player 2 if she made the worst inference about player 1, which is that his type is $\tilde{t}_1 = 40$, in which case the object is worth only $40 + 20 = \$60$ to player 2.

So we can construct such an equilibrium for any \bar{r} such that $60 \leq \bar{r} \leq 80$.

In such an equilibrium, types higher than \bar{r} may be expected to make some offer higher than 120, which player 2 could never profitably accept.

An offer between \bar{r} and 120 may be rejected by player 2 because this surprise offer may lead player 2 to believe that 1's type is 40, in which case the object is only worth 60 to player 2.

Among these almost-pooling equilibria, player 1 most prefers the equilibrium with $\bar{r} = 80$.

In this equilibrium, the probability of trade is $\Pr(\text{trade}) = \Pr(\tilde{t}_1 < 80) = (80 - 40)/(100 - 40) = 2/3$.

There are many other equilibria where 1's types make more offers.

Let's look for an equilibrium in which some types of player 1 would offer to sell for \$70, but all higher types would offer to sell for \$100, and player 2 would be sure to accept \$70 but her probability of accepting \$100 would be between 0 and 1.

To find this equilibrium, there are unknowns that we must find:

let q denote the probability that player 2 would accept an offer of \$100,

and let θ denote the highest type of player 1 that would offer \$70.

For player 2 to be willing to randomize between accepting and rejecting \$100, her expected profit from accepting it must be 0, and so

$$0 = E(V_2(\tilde{t}_1) | \tilde{t}_1 > \theta) - 100 = E(\tilde{t}_1 + 20 | \tilde{t}_1 > \theta) - 100 = (\theta + 100)/2 + 20 - 100, \text{ and so } \theta = 60.$$

For player 1 to offer \$70 below when his type is below θ but \$100 when his type is above θ , we need that $70 - t_1 \geq q(100 - t_1)$ when $t_1 < \theta$, and $70 - t_1 \leq q(100 - t_1)$ when $t_1 > \theta$.

These inequalities imply $70 - \theta = q(100 - \theta)$, and so $q = (70 - 60)/(100 - 60) = 1/4$.

In this equilibrium, $\Pr(\text{trade}) = \Pr(\tilde{t}_1 < \theta) + \Pr(\tilde{t}_1 > \theta)q = (20/60) + (40/60)(1/4) = 1/2$.

(Advanced result: There is a separating equilibrium in which each possible type t_1 of player 1 would offer to sell for $r(t_1) = t_1 + 20$, and the probability q of player 2 accepting would depend on the offer r according to the formula $q(r) = e^{-(r-60)/20}$, for any $r \geq 60$.)

Generalizing the basic signaling game, let's allow now that the number of players n may be greater than 2, and that player 1's action might not be perfectly observed by the other players.

So the structure of a general signaling game with n players is as follows:

- (1) First, player 1 learns his type \tilde{t}_1 , which is drawn from some set of possible types T_1 according to some probability distribution P . So $P(t_1)$ denotes the probability that $\tilde{t}_1 = t_1$.
- (2) Knowing \tilde{t}_1 , player 1 chooses an action a_1 in some set of possible actions A_1 .
- (3) A signal $\tilde{s}(a_1)$ that depends (perhaps randomly) on player 1's action a_1 is publicly observable to all the other players in the game. Let S denote the set of possible values of this signal. Let $\rho(s|a_1)$ denote the probability of the signal being $\tilde{s}=s$ given that player 1 chose the action a_1 .
- (4) Knowing the signal \tilde{s} , each player j in $\{2, \dots, n\}$ independently chooses an action a_j in some set of possible actions A_j .
- (5) Finally, each player i in $\{1, 2, \dots, n\}$ gets a payoff that can depend on player 1's type \tilde{t}_1 , the signal \tilde{s} , and the players' actions (a_1, a_2, \dots, a_n) according to some function $U_i(\tilde{t}_1, a_1, \tilde{s}, a_2, \dots, a_n)$.

Player 1's action in A_1 can depend on 1's type in T_1 . To describe 1's strategy, for each a_1 in A_1 and each t_1 in T_1 , let $\sigma_1(a_1|t_1)$ denote 1's probability of choosing action a_1 when 1's type is t_1 .

For any $j \geq 2$, player j 's action in A_j can depend on the publicly observed signal in S .

To describe j 's strategy, for each a_j in A_j and each s in S , let $\sigma_j(a_j|s)$ denote j 's probability of choosing action a_j when the signal is s . We may also let $\sigma_j(s) = \sigma_j(\bullet|s)$ denote the probability distribution over A_j that player j uses to determine his action when the signal is s .

Given the strategies $(\sigma_2, \dots, \sigma_n)$, player 1's expected payoff from action a_1 when his type is t_1 is

$$EU_1(a_1|t_1) = \sum_{s \in S} \sum_{a_2 \in A_2} \dots \sum_{a_n \in A_n} \rho(s|a_1) \sigma_2(a_2|s) \dots \sigma_n(a_n|s) U_1(t_1, a_1, s, a_2, \dots, a_n).$$

The strategy σ_1 is rational for player 1 against $(\sigma_2, \dots, \sigma_n)$ if, for every type t_1 in T_1 , $\sigma_1(\bullet|t_1)$ assigns positive probability $\sigma_1(a_1|t_1) > 0$ only to actions that maximize $EU_1(a_1|t_1)$ over a_1 in A_1 .

When player 1 uses the strategy σ_1 , the probability of any possible signal s is

$$Q(s) = \sum_{t_1 \in T_1} \sum_{a_1 \in A_1} P(t_1) \sigma_1(a_1|t_1) \rho(s|a_1).$$

The signal gives the other players new information about player 1's action and type.

Let $q(t_1, a_1|s)$ denote the probability that other players would assign to 1's type being t_1 and 1's action being a_1 if they observed the signal s . This function q describes their beliefs.

To be consistent with player 1's strategy σ_1 , the beliefs q must satisfy Bayes's formula:

$$q(t_1, a_1|s) = P(t_1) \sigma_1(a_1|t_1) \rho(s|a_1) / Q(s) \text{ for every possible signal } s \text{ such that } Q(s) > 0.$$

For a signal s that has zero probability under σ_1 (in that $Q(s) = 0$), any distribution of probabilities $q(t_1, a_1|s)$ over the possible (t_1, a_1) pairs may be called (weakly) consistent.

For any signal s in S , the subgame after s is played by $\{2, \dots, n\}$, and each player j 's expected payoff in this subgame depends on the actions (a_2, \dots, a_n) and the beliefs q by the formula:

$$EU_j(a_2, \dots, a_n|s) = \sum_{t_1 \in T_1} \sum_{a_1 \in A_1} q(t_1, a_1|s) U_j(t_1, a_1, s, a_2, \dots, a_n).$$

The strategies $(\sigma_2, \dots, \sigma_n)$ are sequentially rational with beliefs q if, for every possible signal s in S , the profile of randomized actions $(\sigma_2(s), \dots, \sigma_n(s))$ is an equilibrium of this subgame after s .

The profile of actions and beliefs $(\sigma_1, q, \sigma_2, \dots, \sigma_n)$ is a sequential equilibrium if the strategy σ_1 is rational for 1 against $(\sigma_2, \dots, \sigma_n)$, the beliefs q are consistent with σ_1 , and the strategies $(\sigma_2, \dots, \sigma_n)$ are sequentially rational with the beliefs q .

Analysis of the general War of Attrition game.

The game has two given parameters, T and K , where T is the largest number of days that the two players can fight, and K is the value of the prize.

There are two players numbered 1 and 2.

Each player i must choose a number a_i in the set $\{0, 1, \dots, T\}$.

Here i 's decision s_i represents the number of days that player i is prepared to fight for the prize.

A player wins the prize only if he is prepared to fight strictly longer than the other player.

They will fight for as many days as both are prepared to fight.

Each day of fighting costs each player one dollar, and the prize is worth K dollars.

Assume utility is money, and so the utility payoffs for players 1 and 2 are as follows:

Player 1's payoff is $u_1(a_1, a_2) = K - a_2$ if $a_1 > a_2$, but $u_1(a_1, a_2) = -a_1$ if $a_1 \leq a_2$.

Player 2's payoff is $u_2(a_1, a_2) = K - a_1$ if $a_2 > a_1$, but $u_2(a_1, a_2) = -a_2$ if $a_2 \leq a_1$.

Let us look for a symmetric mixed-strategy equilibrium where each player uses a mixed strategy $p = (p(0), p(1), \dots, p(T))$ that assigns positive probability $p(c) > 0$ to every c in $\{0, 1, \dots, T\}$.

So player 1 must get the same expected payoff $Eu_1(c, \tilde{a}_2)$ from choosing any pure strategy $a_1 = c$ when player 2 uses the mixed strategy p to randomly determine \tilde{a}_2 .

Notice first that $Eu_1(0, \tilde{a}_2) = 0$.

More generally,

$$Eu_1(c, \tilde{a}_2) = (K-0)*p(0) + (K-1)*p(1) + \dots + (K-(c-1))*p(c-1) - c*(p(c) + \dots + p(T)).$$

Compare the results for player 1 of preparing to fight an additional day:

$$Eu_1(c+1, \tilde{a}_2) = (K-0)*p(0) + (K-1)*p(1) + \dots + (K-c)*p(c) - (c+1)*(p(c+1) + \dots + p(T)).$$

$$\begin{aligned} \text{So } Eu_1(c+1, \tilde{a}_2) &= Eu_1(c, \tilde{a}_2) + K*p(c) - (p(c+1) + \dots + p(T)) \\ &= Eu_1(c, \tilde{a}_2) + (K+1)*p(c) - (p(c) + \dots + p(T)) \end{aligned}$$

To make player 1 indifferent among all pure strategies, we must have

$$Eu_1(c+1, \tilde{a}_2) = Eu_1(c, \tilde{a}_2) = Eu_1(0, \tilde{a}_2) = 0 \text{ for all } c \text{ in } \{0, 1, 2, \dots, T-1\}.$$

So for all $c < T$, we must have $p(c) = (p(c) + \dots + p(T))/(K+1)$.

But $p(0) + p(1) + \dots + p(T) = 1$. So $p(c) + \dots + p(T) = 1 - (p(0) + \dots + p(c-1))$.

Thus $p(c) = (1 - p(0) - \dots - p(c-1))/(K+1)$ for all $c < T$.

So we can compute this symmetric randomized-strategy equilibrium by the formulas

$$p(0) = 1/(K+1),$$

$$p(1) = (1 - p(0))/(K+1),$$

$$p(2) = (1 - p(0) - p(1))/(K+1),$$

and so on, up to $p(T-1) = (1 - p(0) - p(1) - \dots - p(T-2))/(K+1)$,

The remaining probability is in $p(T) = 1 - p(0) - \dots - p(T-1)$.

It can be shown that these formulas yield the general solution:

$$p(c) = K^c / (K+1)^{c+1} \text{ for } c = 0, 1, \dots, T-1,$$

and $p(T) = (K/(K+1))^T$. As T goes to infinity, this terminal probability $p(T)$ goes to 0.

Introduction to repeated games Players 1 and 2 will meet on $\tau+1$ days, numbered $0,1,2,\dots,\tau$.

On each day, each player i must choose to be generous (g_i) or selfish (f_i).

On each day k , they get payoffs (u_{1k}, u_{2k}) that depend on their actions (c_{1k}, c_{2k}) as follows:

Player 1: \	Player 2:	g_2	f_2	
g_1		3, 3	0, 5	(Prisoners' dilemma)
f_1		5, 0	2, 2	

except on the last day τ their payoffs will be:

Player 1: \	Player 2:	g_2	f_2	
g_1		5, 5	0, 4	(Trust game)
f_1		4, 0	2, 2	

On each day, each player knows what both players did on all previous days.

Each player wants to maximize the expected discounted sum of his payoffs $V_i = \sum_{k=0}^{\tau} \delta^k u_{ik}$ for some given discount factor δ between 0 and 1.

If the first payoff matrix (the prisoners' dilemma) were played once, (f_1, f_2) would be the unique equilibrium, yielding the Pareto-dominated payoff allocation $(2, 2)$.

But in multi-period games, opportunities to respond later can enlarge the set of equilibria.

For example, consider the strategy "choose g_i until f_1 or f_2 is chosen, and thereafter choose f_i ."

We can show that it is an equilibrium here for both players to choose this strategy, if $\delta \geq 2/3$.

Consider first the case of $\tau=1$, where the prisoners' dilemma is played once, followed by one play of the trust game at the end. Under the strategies described here, on the last day,

they will play the good (g_1, g_2) equilibrium of the "trust game" if both were previously generous, but they will play the bad (f_1, f_2) equilibrium if either player was previously selfish.

So the overall payoffs will depend on their first-day choices as follows:

Player 1: \	Player 2:	g_2	f_2
g_1		$3+5\delta, 3+5\delta$	$0+2\delta, 5+2\delta$
f_1		$5+2\delta, 0+2\delta$	$2+2\delta, 2+2\delta$

Then (g_1, g_2) is an equilibrium at the first day if $3+5\delta \geq 5+2\delta$, that is, if $\delta \geq 2/3$.

A similar calculation can be made for any number τ of repetitions of the prisoners' dilemma.

Assuming that the strategies described above will be played after the first stage, the players' overall payoffs will depend on their first-day choices as follows:

		g_2	f_2
g_1		$3(1-\delta^\tau)/(1-\delta)+5\delta^\tau, 3(1-\delta^\tau)/(1-\delta)+5\delta^\tau$	$0+2\delta(1-\delta^\tau)/(1-\delta), 5+2\delta(1-\delta^\tau)/(1-\delta)$
f_1		$5+2\delta(1-\delta^\tau)/(1-\delta), 0+2\delta(1-\delta^\tau)/(1-\delta)$	$2+2\delta(1-\delta^\tau)/(1-\delta), 2+2\delta(1-\delta^\tau)/(1-\delta)$

(Here we use the fact that $w+w\delta+w\delta^2+\dots+w\delta^{s-1} = w(1-\delta^s)/(1-\delta)$.)

If $1 > \delta \geq 2/3$ then $3(1-\delta^\tau)/(1-\delta)+5\delta^\tau \geq 5+2\delta(1-\delta^\tau)/(1-\delta)$.

As $\tau \rightarrow \infty$, overall payoffs in this equilibrium depend on first-day actions as follows:

Player 1: \	Player 2:	g_2	f_2
g_1		$3/(1-\delta), 3/(1-\delta)$	$0+2\delta/(1-\delta), 5+2\delta/(1-\delta)$
f_1		$5+2\delta/(1-\delta), 0+2\delta/(1-\delta)$	$2/(1-\delta), 2/(1-\delta)$

Repeated games

Infinitely repeated games can be used as simple models of long-term relationships.

The game will be played at an infinite sequence of time periods numbered 1,2,3,...

Suppose that the set of players is $\{1,2\}$. In each period k , each player i must choose an action a_{ik} in some set A_i . In period k , each player i 's payoff u_{ik} will depend on both players' actions according to some utility function $U_i: A_1 \times A_2 \rightarrow \mathbb{R}$; that is $u_{ik} = U_i(a_{1k}, a_{2k})$.

We assume here that the actions at each period are publicly observable, and so each player's action in each period may depend on the history of actions by both players at all past periods.

Given any discount factor δ such that $0 \leq \delta < 1$, the δ -discounted average value of player i 's payoffs is $DAV(u_{i1}, u_{i2}, u_{i3}, \dots) = (1-\delta)u_{i1} + \delta u_{i2} + \delta^2 u_{i3} + \dots + \delta^{k-1} u_{ik} + \dots$.

(For any x , $DAV(x, x, x, \dots) = x$. If δ is slightly less than 1 then the players are very patient.)

The objective of each player i in the repeated game is to maximize the expected discounted average value of his payoffs, with respect to some discount factor δ , where $0 < \delta < 1$.

Fact. (Recursion formula) $DAV(u_{i1}, u_{i2}, u_{i3}, \dots) = (1-\delta)u_{i1} + \delta DAV(u_{i2}, u_{i3}, u_{i4}, \dots)$.

We may describe equilibria of repeated games in terms of a various social states.

At each period of the game, the players will understand that their current relationship is described by one of these social states, and their expectations about each others' behavior will be determined by this state. This state may be called the state of play in the game at this period.

(These social states are a characteristic of the equilibrium, not of the game, as they describe the different kinds of expectations that the players may have about each others' future behavior.)

To describe an equilibrium or scenario in terms of social states, we must specify the following:

(1) Social states We must list the set of social states in this equilibrium. (States may denoted by numbers or may be named for the kinds of interpersonal relationships that they represent.)

(2) State-dependent strategies. For each state θ , we must specify a profile of (possibly randomized) actions $(\tilde{s}_1(\theta), \tilde{s}_2(\theta))$ describing the predicted behavior of the players in any period when this θ is the state of play.

(3) Transitions. For each social state θ , we must specify the profiles of players' actions that would cause the state of play in the next period to change from this state to another state. We may let $\Theta(a_1, a_2; \theta)$ denote the state of play in the next period after a period when the state of play was θ and the players chose actions (a_1, a_2) (possibly deviating from the prediction $(\tilde{s}_1(\theta), \tilde{s}_2(\theta))$).

(4) Initial state. We must specify which social state is initial state of play in the first period of the game. Here we will generally let state "0" denote this initial state.

Given any scenario as in (1)-(3) above, and given any discount factor δ , let $V_i(\theta)$ denote the expected δ -discounted average value of player i 's payoffs in this scenario when (ignoring (4)) the state of play begins in state θ . Given $\delta < 1$, these numbers $V_i(\theta)$ can be computed (with algebra) from the equations: $V_i(\theta) = E[(1-\delta)U_i(\tilde{s}_1(\theta), \tilde{s}_2(\theta))] + \delta V_i(\Theta(\tilde{s}_1(\theta), \tilde{s}_2(\theta); \theta))$.

Fact. A scenario as in (1)-(3) above is a subgame-perfect equilibrium if, for every player i and every state θ , player i could not expect to gain by unilaterally deviating from the prediction $\tilde{s}_i(\theta)$ in a period when the state of play is θ . That is, we have an equilibrium if, for every state θ ,

$V_1(\theta) \geq E[(1-\delta)U_1(a_1, \tilde{s}_2(\theta))] + \delta V_1(\Theta(a_1, \tilde{s}_2(\theta); \theta))$, for all a_1 in A_1 ,

$V_2(\theta) \geq E[(1-\delta)U_2(\tilde{s}_1(\theta), a_2)] + \delta V_2(\Theta(\tilde{s}_1(\theta), a_2; \theta))$, for all a_2 in A_2 .

Example. Consider a repeated game where, in each period, the players play the following "Prisoners' dilemma" game in which each must decide whether to "cooperate" or "defect".

	c_2	d_2
c_1	5, 5	0, 6
d_1	6, 0	1, 1

We first consider a version of the "grim trigger" equilibrium:

The states are $\{0, 1\}$. (State 0 represents "trust" or "friendship"; state 1 represents "distrust".)

The predicted behavior in state 0 is (c_1, c_2) . The predicted behavior in state 1 is (d_1, d_2) .

In any period when the current state of play is 0, if the players' action profile is (c_1, d_2) or (d_1, c_2) then the state of play next period will switch to state 1, otherwise it will remain state 0.

When the state of play is 1, the future state of play always remains state 1.

The expected discounted average values for the players in the states satisfy the equations:

$$V_1(0) = (1-\delta)U_1(c_1, c_2) + \delta V_1(0), \quad V_1(1) = (1-\delta)U_1(d_1, d_2) + \delta V_1(1),$$

$$V_2(0) = (1-\delta)U_2(c_1, c_2) + \delta V_2(0), \quad V_2(1) = (1-\delta)U_2(d_1, d_2) + \delta V_2(1).$$

$$\text{So } V_1(0) = (1-\delta)5 + \delta V_1(0), \quad V_1(1) = (1-\delta)1 + \delta V_1(1), \quad \text{and so } V_1(0) = 5, \quad V_1(1) = 1.$$

$$\text{Similarly, } V_2(0) = 5, \quad V_2(1) = 1.$$

For this scenario to be an equilibrium, we need:

$$V_1(0) \geq (1-\delta)U_1(d_1, c_2) + \delta V_1(1), \quad V_1(1) \geq (1-\delta)U_1(c_1, d_2) + \delta V_1(1),$$

$$V_2(0) \geq (1-\delta)U_2(c_1, d_2) + \delta V_2(1), \quad V_2(1) \geq (1-\delta)U_2(d_1, c_2) + \delta V_2(1).$$

That is, we need: $5 \geq (1-\delta)6 + \delta 1$ and $1 \geq (1-\delta)0 + \delta 1$, which are satisfied when $1 \geq \delta \geq 1/5$.

Now lets consider another (more forgiving) equilibrium:

The states are $\{0, 1, 2\}$. (state 0 is "friendship"; state 1 is "punishing 1"; state 2 is "punishing 2".)

The predicted behavior in state 0 is (c_1, c_2) . The predicted behavior in state 1 is (c_1, d_2) .

The predicted behavior in state 2 is (d_1, c_2) .

When the state of play is 0, if the players choose (d_1, c_2) then the next state of play will be 1, if the players choose (c_1, d_2) then the state of play next period will be 2, otherwise it will remain 0.

When the state of play is 1, if the players choose (c_1, d_2) then the next state of play will be 0, otherwise it will remain 1. When the state of play is 2, if the players choose (d_1, c_2) then the next state of play will be 0, otherwise it will remain 2.

The expected discounted average values $V_1(\theta)$ for player 1 in each state θ satisfy the equations:

$$V_1(0) = (1-\delta)U_1(c_1, c_2) + \delta V_1(0), \quad V_1(1) = (1-\delta)U_1(c_1, d_2) + \delta V_1(0),$$

$$V_1(2) = (1-\delta)U_1(d_1, c_2) + \delta V_1(0).$$

$$\text{So } V_1(0) = (1-\delta)5 + \delta V_1(0), \quad \text{and } V_1(0) = 5. \quad \text{So } V_1(1) = (1-\delta)0 + \delta 5, \quad \text{and } V_1(1) = 5\delta.$$

$$\text{So } V_1(2) = (1-\delta)6 + \delta 5, \quad \text{and } V_1(2) = 6 - \delta. \quad \text{Similarly, } V_2(0) = 5, \quad V_2(1) = 6 - \delta, \quad V_2(2) = 5\delta.$$

$$\text{To have an equilibrium, we need: } V_1(0) \geq (1-\delta)U_1(d_1, c_2) + \delta V_1(1),$$

$$V_1(1) \geq (1-\delta)U_1(d_1, d_2) + \delta V_1(1), \quad V_1(2) \geq (1-\delta)U_1(c_1, c_2) + \delta V_1(2),$$

and similar conditions for player 2.

These inequalities become $5 \geq (1-\delta)6 + \delta 5\delta$, $5\delta \geq (1-\delta)1 + \delta 5\delta$, $6 - \delta \geq (1-\delta)5 + \delta(6 - \delta)$,

which are satisfied when $1 \geq \delta \geq 1/5$. (Algebraic fact used: $(1-\delta^2) = (1-\delta)(1+\delta)$.)

Example. Consider a repeated game where, in each period, the players play the following "War of Attrition" game in which each must decide whether to "Fight" or "NotFight," and payoffs

	f_2	n_2
f_1	-1, -1	9, 0
n_1	0, 9	0, 0

If played only **once**, this game has three equilibria: (f_1, n_2) yielding $(9, 0)$, (n_1, f_2) yielding $(0, 9)$, and $(0.9[f_1]+0.1[n_1], 0.9[f_2]+0.1[n_2])$ yielding expected payoffs $(0, 0)$.

Suppose this is played **twice**, and period-2 behavior can depend on the outcome in period 1. The overall goal of each player i is to maximize $u_i(1)+\delta u_i(2)$, where $u_i(t)$ is i 's payoff in period t . In a subgame perfect equilibrium of the overall two-period game, the players' anticipated behavior in the final period 2 must look like an equilibrium of the one-period game, given whatever happened in period 1. But the players' understanding of which equilibrium they will play in the second period may depend on the outcome of their play in the first period.

We may say that the state of the players' shared understanding in period 2 will be "state 1" if they expect to play the (f_1, n_2) equilibrium in period 2, "state 2" if they expect to play the (n_1, f_2) equilibrium in period 2, and "state 0" if they expect the randomized equilibrium in period 2.

Consider a subgame-perfect eqm where period-2 depends on period-1 play as follows:

- if (f_1, n_2) is played in period 1 then they anticipate state 1 ((f_1, n_2) again) in period 2;
- if (n_1, f_2) is played in period 1 then they anticipate state 2 ((n_1, f_2) again) in period 2;
- and otherwise they anticipate state 0 (the randomized equilibrium) in period 2.

When the first-period influences second-period behavior in this way, total discounted payoffs for the two players depend on the first-period moves as follows:

	f_2	n_2
f_1	$-1+0\delta, -1+0\delta$	$9+9\delta, 0+0\delta$
n_1	$0+0\delta, 9+9\delta$	$0+0\delta, 0+0\delta$

(For discounted average value over two periods, we would divide all these payoffs by $1+\delta$.)

So there are three possible equilibria in the first period: (f_1, n_2) yielding total expected payoffs $(9+9\delta, 0)$, (n_1, f_2) yielding total expected payoffs $(0, 9+9\delta)$, and a symmetric randomized equilibrium where each player fights with probability $p = (9+9\delta)/(10+9\delta)$ and each player's expected total payoff is just 0.

But there are also other subgame-perfect equilibria. For example, the anticipated second-period equilibrium might depend on first-period play as follows:

- if (f_1, n_2) is played in period 1 then they anticipate state 2 (switch to (n_1, f_2)) in period 2;
- if (n_1, f_2) is played in period 1 then they anticipate state 1 (switch to (f_1, n_2)) in period 2;
- and otherwise they anticipate state 0 (the randomized equilibrium) in period 2.

Then total discounted payoffs for the two players depend on the first-period moves as follows:

	f_2	n_2
f_1	$-1+0\delta, -1+0\delta$	$9+0\delta, 0+9\delta$
n_1	$0+9\delta, 9+0\delta$	$0+0\delta, 0+0\delta$

So there are three possible equilibria in period 1: (f_1, n_2) yielding total expected payoffs $(9, 9\delta)$, (n_1, f_2) yielding total expected payoffs $(9\delta, 9)$, and a symmetric randomized equilibrium where each player fights with probability $p = 9/(10 + 9\delta)$ and each player's expected total payoff is $81\delta/(10 + 9\delta)$, which is about 4.26 when δ is close to 1.

Consider a repeated game where players 1 and 2 repeatedly play the game below infinitely often. The players want to maximize their δ -discount average value of payoffs, for some $0 < \delta < 1$.

	f_2	n_2
f_1	-1, -1	9, 0
n_1	0, 9	0, 0

A subgame-perfect equilibrium:

States: there are three states, numbered 0,1,2. The initial state in period 1 is state 0.

(State 1 may be interpreted as "1 has ownership", state 2 may be interpreted as "2 has ownership" and state 0 may be interpreted as "fighting for ownership".)

Strategies: Let $s_i(\theta)$ denote the move that player i would choose in state θ .

Player 1's strategy is $s_1(1) = f_1$, $s_1(2) = n_1$, $s_1(0) = q[f_1] + (1-q)[n_1]$ for some q between 0 and 1.

Player 2's strategy is $s_2(1) = n_1$, $s_2(2) = f_1$, $s_2(0) = q[f_2] + (1-q)[n_2]$ for the same q .

We will need to find what q makes this an equilibrium.

Transitions: When the current state is state 0, the state next period would be:

state 1 if (f_1, n_2) is played now, state 2 if (n_1, f_2) is played now, and state 0 if (f_1, f_2) or (n_1, n_2) is played now. Once the game is in state 1 or 2, it stays in the same state forever.

Values Let $V_i(\theta)$ denote the expected discounted average value of payoffs for player i in state θ .

The recursion equations for states 1 and 2 are

$$V_i(1) = (1-\delta)U_i(f_1, n_2) + \delta V_i(1), \text{ for } i=1,2, \text{ and so } V_1(1) = 9 \text{ and } V_2(1) = 0;$$

$$V_i(2) = (1-\delta)U_i(n_1, f_2) + \delta V_i(2), \text{ for } i=1,2, \text{ and so } V_1(2) = 0 \text{ and } V_2(2) = 9.$$

To check the equilibrium condition in state 1, notice that

$$9 = V_1(1) \geq (1-\delta)U_1(n_1, n_2) + \delta V_1(1) = (1-\delta)(0) + \delta(9) = 9\delta,$$

$$0 = V_2(1) \geq (1-\delta)U_2(f_1, f_2) + \delta V_2(1) = (1-\delta)(-1) + \delta(0) = -(1-\delta).$$

The equilibrium conditions in state 2 are similar.

In state 0, for player 1 to be willing to randomize between f_1 and n_1 , he must expect the same discounted average value $V_1(0)$ from choosing f_1 or n_1 this period, and so we must have

$$V_1(0) = q((1-\delta)U_1(f_1, f_2) + \delta V_1(0)) + (1-q)((1-\delta)U_1(f_1, n_2) + \delta V_1(1)), \text{ and}$$

$$V_1(0) = q((1-\delta)U_1(n_1, f_2) + \delta V_1(2)) + (1-q)((1-\delta)U_1(n_1, n_2) + \delta V_1(0)).$$

The latter is $V_1(0) = q(1-\delta)0 + q\delta 0 + (1-q)(1-\delta)0 + (1-q)\delta V_1(0)$, implying $V_1(0) = 0$.

Then $V_1(0) = q(1-\delta)(-1) + q\delta V_1(0) + (1-q)(1-\delta)9 + (1-q)\delta 9$, implies $q = 9/(10-\delta)$.

Similarly, $V_2(0) = 0$.

Another subgame-perfect equilibrium for this repeated game, with δ -discounting.

	f_2	n_2
f_1	-1, -1	9, 0
n_1	0, 9	0, 0

States: there are three states, numbered 0,1,2. The initial state in period 1 is state 0.

(In this equilibrium, state 1 may be interpreted as "1's turn", state 2 may be interpreted as "2's turn" and state 0 may be interpreted as "confused about whose turn it is".)

Strategies: Let $s_i(\theta)$ denote the move that player i would choose in state θ .

Player 1's strategy is $s_1(1) = f_1$, $s_1(2) = n_1$, $s_1(0) = q[f_1] + (1-q)[n_1]$ for some q between 0 and 1.

Player 2's strategy is $s_2(1) = n_1$, $s_2(2) = f_1$, $s_2(0) = q[f_2] + (1-q)[n_2]$ for the same q as player 1.

We will need to find what q makes this an equilibrium.

Transitions: When the current state is state 0, the state next period would be: state 2 if (f_1, n_2) is played now, state 1 if (n_1, f_2) is played now, and state 0 if (f_1, f_2) or (n_1, n_2) is played now.

When the current state is 1, the next state is always 2. When the current state is 2, the next state is always 1. (So from state 1 or 2, the state of play alternates between states 1 and 2 forever.)

Values Let $V_i(\theta)$ denote the expected discounted average value of payoffs for player i in state θ .

The recursion equations for states 1 and 2 are

$$V_i(1) = (1-\delta)U_i(f_1, n_2) + \delta V_i(2) \text{ and } V_i(2) = (1-\delta)U_i(n_1, f_2) + \delta V_i(1) \text{ for } i=1,2.$$

$$\text{So } V_1(1) = (1-\delta)9 + \delta(1-\delta)0 + \delta^2 V_1(0), \text{ and so } V_1(1) = 9(1-\delta)/(1-\delta^2) = 9/(1+\delta)$$

$$\text{and } V_1(2) = (1-\delta)0 + \delta V_1(1) = 9\delta/(1+\delta).$$

$$\text{Similarly, } V_2(2) = 9/(1+\delta) \text{ and } V_2(1) = 9\delta/(1+\delta).$$

To check the equilibrium condition in state 1, notice that

$$9/(1+\delta) = V_1(1) \geq (1-\delta)U_1(n_1, n_2) + \delta V_1(2) = \delta^2 9/(1+\delta),$$

$$9\delta/(1+\delta) = V_2(1) \geq (1-\delta)U_2(f_1, f_2) + \delta V_2(2) = (1-\delta)(-1) + \delta 9/(1+\delta).$$

The equilibrium conditions in state 2 are similar.

In state 0, for player 1 to be willing to randomize between f_1 and n_1 , he must expect the same discounted average value $V_1(0)$ from choosing f_1 or n_1 this period, and so we must have

$$V_1(0) = q((1-\delta)U_1(f_1, f_2) + \delta V_1(0)) + (1-q)((1-\delta)U_1(f_1, n_2) + \delta V_1(2)), \text{ and}$$

$$V_1(0) = q((1-\delta)U_1(n_1, f_2) + \delta V_1(1)) + (1-q)((1-\delta)U_1(n_1, n_2) + \delta V_1(0)).$$

$$\text{So } V_1(0) = q(1-\delta)(-1) + q\delta V_1(0) + (1-q)(1-\delta)9 + (1-q)\delta^2 9/(1+\delta),$$

$$\text{and } V_1(0) = q(1-\delta)0 + q\delta 9/(1+\delta) + (1-q)(1-\delta)0 + (1-q)\delta V_1(0).$$

These two equations can be solved for the two unknowns $V_1(0)$ and q . The explicit formula is hard to derive, but the equations can be solved numerically on a computer. (I used Excel.)

The results with $\delta = 0.99$ are $V_1(0) = 4.446$ and $q = 0.585$

The value for player 2 in state 0 is of course the same, $V_2(0) = V_1(0)$, because everything is symmetric as long as they are in state 0.

Action-probabilities and Belief Probabilities (from Section 10.4)

Suppose that we are given some extensive game with imperfect information.

With imperfect information, each decision node now needs two labels, one indicating which player makes the decision here, and another indicating what is the state of the player's information at this node. In a game tree, I indicate these by a label of the form "Player.InformationState", where a player cannot distinguish his nodes that have the same information state. Nodes with the same player and information state form an information set, which is also commonly indicated by a dashed line or curve in the game tree. Two nodes with the same player and information label must be followed by the same set of feasible moves.

Given any randomized strategy for any player i , at any information set of player i that could occur with positive probability when he plays this strategy, we can compute a probability distribution over the set of possible actions for player i at this information set.

These probabilities are called action probabilities or move probabilities.

That is, the action-probability for any action c at any information state s of any player i , which we may denote by $\sigma_i(c|s)$, denotes the conditional probability that player i will choose action c if information set s occurs in the game. (I usually put action probabilities in parentheses "(•)".)

A behavioral strategy for player i is a list of an action-probability distribution for each of player i 's information sets.

A behavioral-strategy profile is a list of a behavioral strategy for each players, specifying an action probability for every possible action at every possible information set of every players.

Given a profile of behavioral or randomized strategies for all players in the game, the prior probability of any node in the tree is the multiplicative product of all chance-probabilities and action-probabilities on the path that leads to this node from the starting node.

(The chance probabilities on all branches that follow chance nodes are part of the given structure of the extensive game, from Section 7.6.)

By Bayes's formula, when player i moves at his information set s , the belief probability that player i should assign to any node x in this information set s should be:

(the prior probability of x)/(the sum of prior probabilities of all nodes in this information set s)
whenever this formula is well-defined (not $0/0$).

So the belief-probability for any node x in any information set s of any player i , which we may denote by $\mu_i(x|s)$, is the conditional probability of node x being true that player i would believe if the information state s occurred in the game. (I put belief probabilities in angle brackets "<•>".)

A beliefs system is a list of such belief probability distributions over the nodes of each information set of each player in the game.

A beliefs system is consistent (in the weak sense) with a behavioral-strategy profile if the beliefs satisfy all Bayes's formula, as above, whenever this formula is well-defined (not $0/0$).

A behavioral-strategy profile is sequentially rational given a beliefs system if, at every information set, the player is assigning positive probability only to actions that maximize his expected payoff, given his beliefs about the current node in his information set and given what the behavioral-strategy profile specifies about players' behavior after this information set.

A sequential equilibrium is a behavioral-strategy profile and a beliefs system such that the behavioral-strategy profile is sequentially rational given the beliefs system and the beliefs system is consistent with the behavioral-strategy profile.