Order-Based Methods in Economics

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Preliminaries

Motivation: Occam's Razor

- Occam's Razor: We should never make more assumptions than strictly necessary.
- In economics, many assumptions are made simply for technical convenience:
  - Smoothness of functions required to define **complementarity** in terms of cross-derivatives;
  - Smoothness and local concavity needed to obtain IFT-based **comparative statics** for optimization problems;
  - Some notion of continuity called for by most **fixed-point theorems**.
- In many cases, we can do away with these assumptions by simply looking at the **order properties** of the sets and functions involved.

Order

Consider a set X with a binary relation $\succeq$, which satisfies:

- **Reflexivity:** $(\forall x \in X), x \succeq x$;
- **Transitivity:** $(x \succeq y, y \succeq z) \;\Rightarrow\; x \succeq z$;
- **Antisymmetry:** $(x \succeq y, y \succeq x) \;\Rightarrow\; x = y$.

Then $(X, \succeq)$ is called a **partially ordered set** or **poset**.
When $x \succeq y$ and $x \neq y$, we write $x \succ y$.

Examples:

- $(\mathbb{R}, \geq)$: the real numbers with the usual order
- $(\mathbb{R}^n, \geq)$: $\mathbb{R}^n$ with the product order, i.e., $\boldsymbol{x} \geq \boldsymbol{y}$ iff $x_i \geq y_i$ for all i
- $(\mathcal{P}(A), \subseteq)$: for any set A, the power set of A with set inclusion

If $\succeq$ is also complete (that is, $(\forall x, y \in X)$, $x \succeq y$ or $y \succeq x$), $(X, \succeq)$ is called a **totally ordered set** or **chain**.

Above, only $(\mathbb{R}, \geq)$ is a chain.

General Monotone Functions

The familiar concept of increasing functions generalizes naturally to general posets.

Definition. Let $(X, \succeq_X)$ and $(Y, \succeq_Y)$ be posets, and let $f : X \to Y$. Then f is **increasing** (resp., **decreasing**) if $x \succeq_X x' \Rightarrow f(x) \succeq_Y f(x')$ (resp., $x \succeq_X x' \Rightarrow f(x) \preceq_Y f(x')$). If $x \succ_X x' \Rightarrow f(x) \succ_Y f(x')$ (resp., $x \succ_X x' \Rightarrow f(x) \prec_Y f(x')$), then f is **strictly increasing** (resp., **strictly decreasing**).

Lattices: Definitions

- Total order provides too much structure for many applications.
- Yet partial order does not provide enough structure.
- Middle ground: lattices!

Definition. A poset $(X, \succeq)$ is a lattice if and only for each $x, y \in X$ there exist

- $x \vee y \in X$ such that $x \vee y \succeq x, y$ and $x \vee y \preceq z$ for each $z \in X$ such that $z \succeq x, y$. The element $x \vee y$ is called the **join** or **least upper bound** of x and y.
- $x \wedge y \in X$ such that $x \wedge y \preceq x, y$ and $x \wedge y \succeq z$ for each $z \in X$ such that $z \preceq x, y$. The element $x \wedge y$ is called the **meet** or **greatest lower bound** of x and y.

Mnemonic: $\vee \approx \cup \Rightarrow$ BIGGER; $\wedge \approx \cap \Rightarrow$ SMALLER

Lattices: Examples

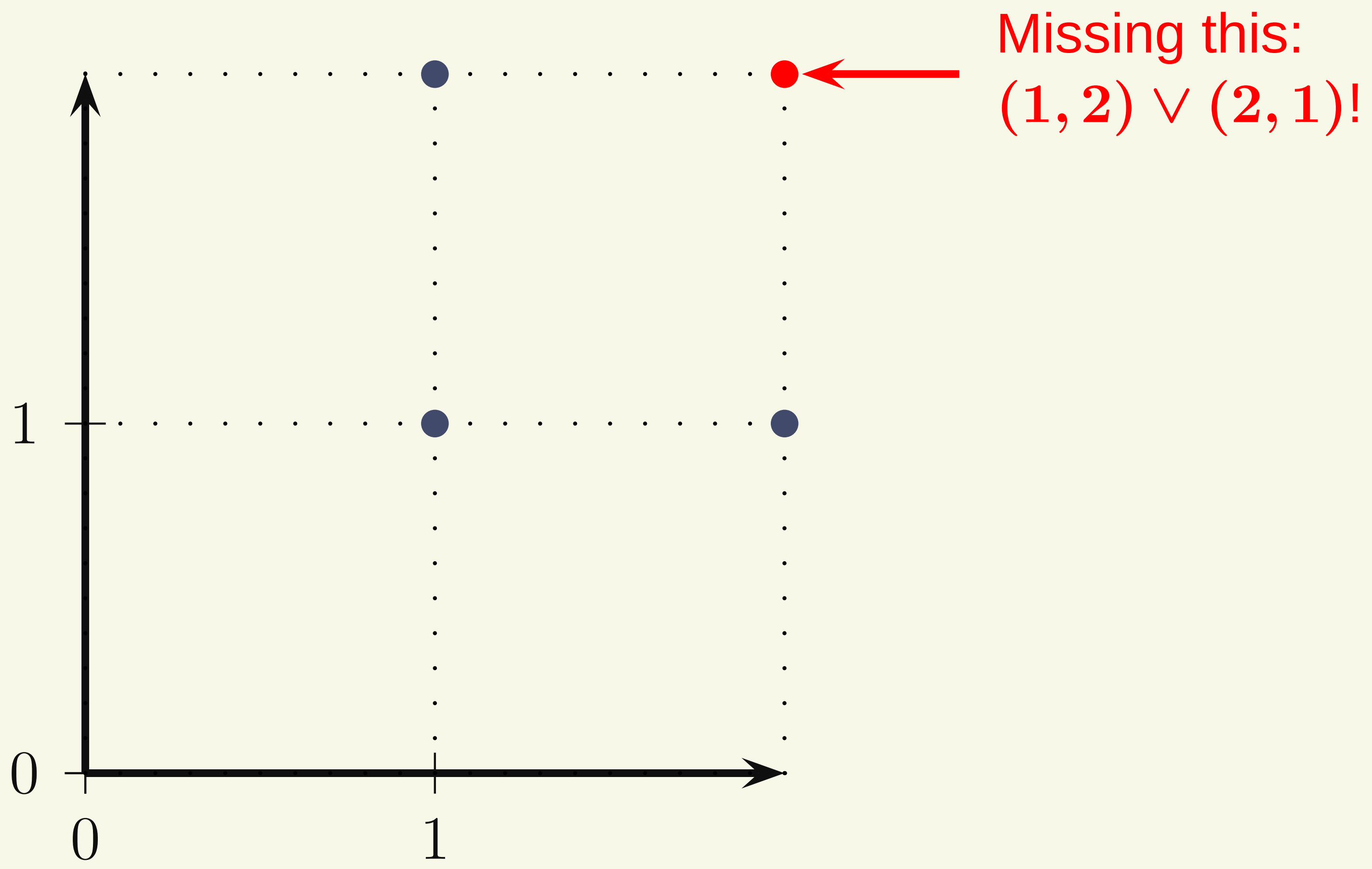
- Any ordered set: if $x \succeq y$, then $x \vee y = x$ and $x \wedge y = y$.
- $\mathbb{R}^n$ with the product order as before ($\boldsymbol{x} \geq \boldsymbol{y}$ iff $x_i \geq y_i$ for all i):
  $x \vee y = (\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_n, y_n))$;
  $x \wedge y = (\min(x_1, y_1), \min(x_2, y_2), \ldots, \min(x_n, y_n))$.
- Generalized **product lattice**. Let I be some indexing set, and let $(X_\alpha, \succeq_\alpha)$ be a lattice for each $\alpha \in I$. Define the order $\succeq$ on $X = \prod_{\alpha \in I} X_\alpha$ by ($x \succeq y$ iff $x_\alpha \succeq_\alpha y_\alpha$ for all $\alpha \in I$).
  It is easy to see that $(X, \succeq)$ is a lattice, because
  $x \vee y = (x_\alpha \vee y_\alpha)_{\alpha \in I}$ and $x \wedge y = (x_\alpha \wedge y_\alpha)_{\alpha \in I}$.

  Note that $(\mathbb{R}^n, \geq)$, as defined above, is a product lattice:
  $I = \{1, \ldots, n\}$, $(X_\alpha, \succeq_\alpha) = (\mathbb{R}, \geq)$ for all α.
- $(\mathcal{P}(A), \subseteq)$: $C \vee D = C \cup D$ and $C \wedge D = C \cap D$.

Lattices: Illustration

Why the term "lattice"?

Consider the set $X_0 = \{(1,1),(2,1),(1,2)\} \subset \mathbb{R}^2$ with the product order. Is this a lattice?



Sublattices

Mathematicians are often interested in subsets of certain objects that themselves are objects of the same type as the larger object. E.g., subgroups, subrings, and subfields in algebra. By definition, these have to be closed with respect to the defining operation(s). Similarly, we define a **sublattice** as a subset of a lattice that is closed with respect to meet and join:

Definition. Let X be a lattice and let $A \subset X$. Then A is a **sublattice** of X if, for each $a, b \in A$, $a \wedge b \in A$ and $a \vee b \in A$. The set of all sublattices of X is denoted by $\mathcal{L}(X)$.

Obviously, a sublattice of any lattice is a lattice itself.

Less obviously, however, if we embed a lattice A into a larger lattice X (with the same order), A is not necessarily a sublattice of X.

Example: $A = \{(1,1),(1,2),(2,1),(10,10)\}$ by itself *is* a lattice with respect to the product order on $\mathbb{R}^2$ (for example, $(1,2) \vee_A (2,1) = (10,10)$). However, it is *not* a sublattice of $\mathbb{R}^2$, because $(1,2) \vee_{\mathbb{R}^2} (2,1) = (2,2) \notin A$.

Note that the example on the previous slide is *not* a lattice (with respect to the product order on $\mathbb{R}^2$) even when taken in isolation.

Complete Lattices

Definition. Let $(X, \succeq)$ be a lattice. $(X, \succeq)$ is **complete** if $\forall A \subseteq X, A \neq \emptyset$ there exist

- Least upper bound/ supremum of A:
  $\vee A \in X$ s.t. $\vee A \succeq a$ for all $a \in A$ and $\vee A \preceq b$ whenever $b \succeq a$ for all $a \in A$;
- Greatest lower bound/ infimum of A:
  $\wedge A \in X$ s.t. $\wedge A \preceq a$ for all $a \in A$ and $\wedge A \succeq b$ whenever $b \preceq a$ for all $a \in A$.

$\vee A$ is also denoted $\sup A$ and $\wedge A$ is also denoted $\inf A$.

Note that any finite lattice is complete (just iterate the definition of lattice!).

In $\mathbb{R}^n$, lattice completeness is equivalent to topological compactness:

Theorem. *Any lattice $X \subset \mathbb{R}^n$ (with the product order) is complete iff X is compact (with the product topology).*

This result is a special case of a theorem on interval topologies in general lattices due to Orrin Frink. For details and proof, check Theorem 20, p. 251 in Birkhoff (1967).

Example: $[0, 1]^2$ is complete, while $(0, 1)^2$ is not (can you check this directly?).

Tarski’s Fixed Point Theorem

Tarski's Fixed Point Theorem

We are now ready for our first major result.

Tarski's Fixed Point Theorem. *Let $(X, \succeq)$ be a nonempty, complete lattice, and let $f : X \to X$ be an increasing function. Then the set of fixed points of f, $E = \{x^* \in X \mid f(x^*) = x^*\}$, is a nonempty, complete lattice.*

This implies:

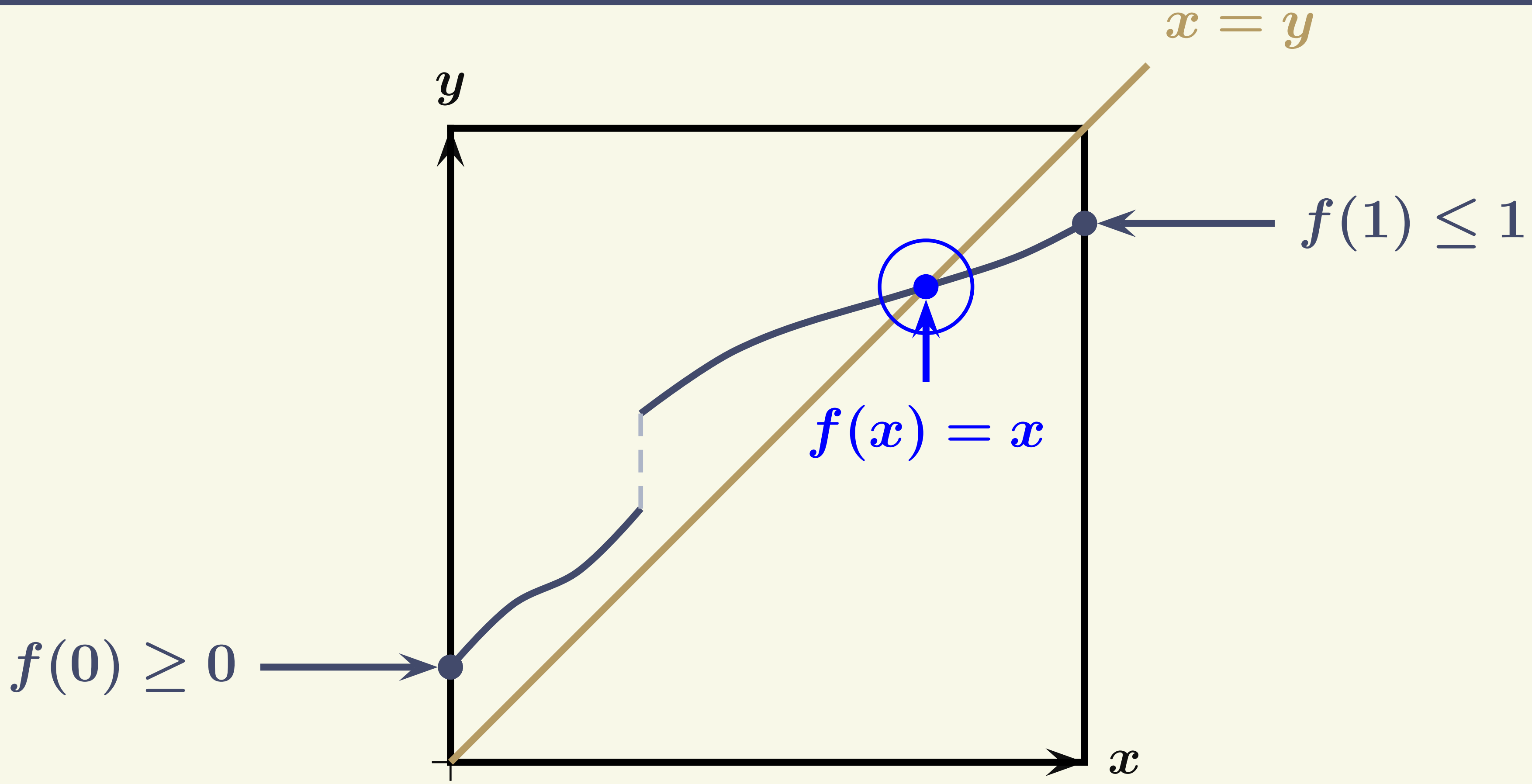
- At least one fixed point exists (even when f has jumps/ discontinuities!)
- The set of fixed points has a largest and a smallest element.
  In fact, the largest fixed point is $\vee\{x \in X \mid f(x) \succeq x\}$ and the smallest fixed point is $\wedge\{x \in X \mid f(x) \preceq x\}$.

Note that the theorem does *not* imply that E is a sublattice of X.

The requirement that X be complete is essential. Consider the incomplete lattice $X = (0, 1)$ and the increasing function $f(x) = x^2$.

There is no Tarski equivalent for decreasing functions!

Illustration of Tarski's Theorem



f must cross the diagonal!

Proof of Tarski's Theorem

Let $A = \{x \in X \mid f(x) \succeq x\}$; $B = \{x \in X \mid f(x) \preceq x\}$. Note that $E = A \cap B$.

1. We begin by showing that $\vee A$ is the greatest fixed point of f.
   (a) Since X is complete, it has a lowest element, $\wedge X$. Since $f(\wedge X) \in X$, we know that $f(\wedge X) \succeq \wedge X$, so that $\wedge X \in A$, i.e., A is nonempty.
   (b) Since A is nonempty and X complete, there exists $\vee A \in X$.
   (c) Take any $x \in A$:
      i. By def. of $\vee$, $x \preceq \vee A$.
      ii. Thus, since f is increasing, $f(x) \preceq f(\vee A)$.
      iii. By def. of A, $x \preceq f(x)$.
      iv. Putting it all together, $x \preceq f(x) \preceq f(\vee A)$.
      v. Since x was arbitrary, we see that $f(\vee A)$ is an upper bound for A.
   (d) Thus, by definition of $\vee$, $f(\vee A) \succeq \vee A$.
   (e) Since f is increasing, this also implies that $f(f(\vee A)) \succeq f(\vee A)$, i.e., $f(\vee A) \in A$.
   (f) But then, by def. of $\vee$, $f(\vee A) \preceq \vee A$.
   (g) By d and f (and antisymmetry of $\succeq$), $f(\vee A) = \vee A$, i.e., $\vee A \in E$. Thus E is nonempty.
   (h) Since $E \subseteq A$, $x^* \preceq \vee A$ for any $x^* \in E$, i.e., $\vee A = \max E$.
2. By virtually identical arguments, $\wedge B = \min E$.
3. All that remains to be shown is that E is a complete lattice. Take any $P \subseteq E$. We must show that $\vee_E P$ and $\wedge_E P$ exist.
   (a) Let $\Pi = \{x \in X \mid x \succeq \vee_X P\}$.
   (b) Take any nonempty $Q \subseteq \Pi$. Since X is complete, $\exists \vee_X Q, \wedge_X Q$. Since $\vee_X P$ is a lower bound for $Q \subseteq \Pi$, $\vee_X P \preceq \wedge_X Q$, so that $\vee_X Q, \wedge_X Q \in \Pi$. Thus Π is a complete sublattice of X.
   (c) Since f is increasing, it maps Π to itself. Then, since Π is a complete lattice, the results in 1–2 above imply that the set of fixed points of the restriction of f to Π, $E' = E \cap \Pi$, is nonempty and that $\min_\Pi E'$ exists. (In fact, since Π is a sublattice of X, $\min_\Pi E' = \min_X E'$.)
   (d) Finally, note that $E' = \Pi \cap E$ is precisely the set of upper bounds of P in E. Thus, $\min_\Pi E'$ is also the least upper bound of P in E, i.e., $\min_\Pi E' = \vee_E P$. (Note, however, that $\min_\Pi E'$ need not be the least upper bound of P in X!)
   (e) The existence of $\wedge_E P$ is proved analogously (use $\Pi' = \{x \in X \mid x \preceq \wedge_X P\}$).

Complementarity

Increasing Differences I

Recall from basic micro that complementarity in, say, x_1 and x_2 is usually defined for smooth functions by

$$\frac{\partial f}{\partial x_1 \partial x_2} \geq 0.$$

- Smoothness does not have economic content: it is just for convenience.
- The actual content: first differences in one argument are increasing with respect to another argument.
- This leads to a natural redefinition for a two-argument function:

Definition. Let S and T be posets (not necessarily lattices), and let $f : S \times T \to \mathbb{R}$. Then f has **(strictly) increasing differences** if $f(s,t) - f(s,t')$ is (strictly) increasing in s for all $t \succ_T t'$.

Increasing Differences II

We can naturally extend the definition to any pair of variables in a multiple-argument function:

Definition. Let $X = \prod_{\alpha \in I} X_\alpha$ for a finite set I. Let $f : X \to \mathbb{R}$. Then f has **(strictly) increasing differences** in $(x_\alpha, x_{\alpha'})$ if

$$f(x_\alpha, x_{\alpha'}; x_{-(\alpha,\alpha')}) - f(x_\alpha, x'_{\alpha'}; x_{-(\alpha,\alpha')})$$

is (strictly) increasing in x_α for all $x_{\alpha'} \succ_{\alpha'} x'_{\alpha'}$ and all $x_{-(\alpha,\alpha')}$.

If a multiple-argument function has increasing differences in every pair of variables, we simply say that it has increasing differences:

Definition. Let $X = \prod_{\alpha \in I} X_\alpha$ for a finite set I. Let $f : X \to \mathbb{R}$. Then f has (strictly) increasing differences on X if f has (strictly) increasing differences in $(x_\alpha, x_{\alpha'})$ for every pair (α, α').

Increasing Differences for Smooth Functions

For smooth functions on a Euclidean space, increasing differences are equivalent to the familiar notion of complementarity:

Theorem. *If $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, then f has increasing differences in x_i, x_j if and only if $\frac{\partial f}{\partial x_i \partial x_j} \geq 0$ (everywhere).*

The proof follows directly from the definitions of derivatives and of increasing differences.

Problem: when the function is not smooth, increasing differences are cumbersome to handle mathematically. Let lattices come to rescue.

Supermodularity

Definition. Let X be a lattice, and let $f : X \to \mathbb{R}$. Then f is **supermodular** on X if, for any $x, y \in X$,

$$f(x \wedge y) + f(x \vee y) \geq f(x) + f(y).$$

If the inequality is strict for each incomparable pair (i.e., for each x, y such that $x \not\geq y$ and $y \not\geq x$), f is **strictly supermodular.**

Note that if X is a chain, then any function $f : X \to R$ is trivially supermodular (why?).

Loose intuition for why supermodularity relates to increasing differences:

$$f(x \wedge y) + f(x \vee y) \geq f(x) + f(y) \Leftrightarrow f(x \vee y) - f(x) \geq f(y) - f(x \wedge y):$$

The change from a medium element (x) to the largest element ($x \vee y$) is larger than the change from the smallest element ($x \wedge y$) to a medium element (y).

Relation between Supermodularity and Increasing Differences

The intuition is correct:

Theorem. *Let X be a lattice and $f : X \to R$. If f is supermodular on X, then it has increasing differences on X.*

In general, supermodularity is a *stronger* concept than increasing differences, i.e., the converse to the theorem above does not hold.

However, when X is the product of finitely many *chains* (totally ordered sets), the two concepts coincide.

Theorem. *Let $X = \prod_{\alpha \in I} X_\alpha$ for finite I, where X_α is a totally ordered set for each $\alpha \in I$, and let $f : X \to \mathbb{R}$. Then f is supermodular on X if and only if f has increasing differences on X.*

Note that $\mathbb{R}$ is a chain, so that **supermodularity and increasing differences are equivalent on** $\mathbb{R}^n$, which is the most relevant set for us!

Monotone Comparative Statics

Setup

Let X be a lattice, T a partially ordered set, and let $S : T \to \mathcal{L}(X) \subset \mathcal{P}(X)$ (where $\mathcal{P}(X)$ and $\mathcal{L}(X)$ denote the power set of X and the set of all sublattices of X, respectively). Thus $S(t) \subseteq X$ is a sublattice of X for each $t \in T$. Let $g : X \times T \to \mathbb{R}$. Let us analyze the family of constrained optimization problems (for all $t \in T$)

$$\max_{x \in S(t)} g(x, t).$$

We interpret T as a parameter space, X as the space of choice variables, and S as the parameter-dependent constraint set, which determines the feasible set $S(t)$ given t.

Define the correspondence $\phi : T \to \mathcal{P}(X)$by

$$\phi(t) = \arg\max_{x \in S(t)} g(x, t).$$

We are interested in conditions under which this correspondence is "increasing" (in a precise sense, to be defined below).

Ordering Sets

To speak of "increasing" correspondences, we need a way to order sets in a meaningful way. Consider the following ordering relation:

Definition. Let $(X, \succeq)$ be a lattice. Define the relation $\succeq_p$ on $\mathcal{P}(X)$ as follows: $A \succeq_p B$ iff $(a \in A, b \in B) \Rightarrow (a \vee b \in A, a \wedge b \in B)$.

- This relation reasonably extends the underlying order:
  - $\succeq_p$ is transitive and antisymmetric;
  - $A \succeq_p A$ is reflexive if and only if A is a sublattice. Thus, $\succeq_p$ is a partial order on $\mathcal{L}(X)$, the set of sublattices of X;
  - If $A = \{a\}$ and $B = \{b\}$, then $A \succeq_p B$ iff $a \succeq b$;
  - Similarly, if X is totally ordered (such as $\mathbb{R}$), then $A \succeq_p B$ iff $a \succeq b$ for each $a \in A \backslash B$ and $b \in B \backslash A$.
- Note, however, that $A \succeq_p B$ does *not* imply that $a \succeq b$ for each $a \in A$ and $b \in B$, even when a and b are comparable:
  - For example, in $\mathbb{R}$, $[1, 3] \succeq_p [0, 2]$, but $[1, 3] \ni 1.5 < 1.9 \in [0, 2]$;

Stronger Ordering of Sets

Following the last example, let us introduce a stronger set ordering notion:

Definition. Let $(X, \succeq)$ be a lattice. Define the relation $\succeq_{sp}$ on $\mathcal{P}(X)$ as follows: $A \succeq_{sp} B$ iff $(a \in A, b \in B) \Rightarrow (a \succeq b)$.

- Clearly, $\succeq_{sp}$ is *stronger* than $\succeq_p$, i.e., $A \succeq_{sp} B \Rightarrow A \succeq_p B$.
- $\succeq_{sp}$ is antisymmetric and transitive, but *not* reflexive (except on singleton sets).

We can now define monotonicity for correspondences.

Ascending Correspondences: Definitions

Definition. Let $(T, \succeq_T)$ be a poset and $(U, \succeq_U)$ a lattice. Let $\phi : T \to \mathcal{P}(U)$.
Then ϕ is **ascending** if $\phi(t) \succeq_p \phi(t')$ for each $t \succ_T t'$.
ϕ is **strongly ascending** if $\phi(t) \succeq_{sp} \phi(t')$ for each $t \succ_T t'$.

We use the term "ascending" instead of "increasing" because the condition is $t \succ_T t'$ instead of $t \succeq_T t'$.

- This change is necessary because $\succeq_p$ and $\succeq_{sp}$ are generally not reflexive, so that $\phi(t) \succeq_p \phi(t)$ (or $\phi(t) \succeq_{sp} \phi(t)$) may not hold for ascending correspondences.
- However, when $\phi(t)$ is a sublattice of U for each t, i.e., $\phi : T \to \mathcal{L}(U)$, ϕ is an ascending correspondence iff it is an increasing function from $(T, \succeq_T)$ to $(\mathcal{L}(U), \succeq_p)$ (why?).
- When ϕ is singleton-valued, the concepts of ascending and strongly ascending correspondences are both equivalent to the concept of increasing functions (why?).

We are now ready for the central result of the theory, due to Topkis (1979).

Ascending Correspondences and Bounds

As before, let $(T, \succeq_T)$ be a poset, $(U, \succeq_U)$ a lattice, and let $\phi : T \to \mathcal{P}(U)$. In addition, suppose that U is complete (so that upper and lower bounds $\vee A$ and $\wedge A$ exist for each $A \in \mathcal{P}(U)$). Then ascending correspondences have monotonic upper and lower bounds as follows:

Theorem. *Let U be complete.*
If ϕ is ascending, then $\wedge\phi(t)$ and $\vee\phi(t)$ are increasing.
If ϕ is strongly ascending, then $\wedge\phi(s) \succeq_U \vee\phi(t)$ for each $s \succ_T t$.

Proof for ascending ϕ.

Take $s \succeq_T t$. We must show $\wedge\phi(s) \succeq_U \wedge\phi(t)$ and $\vee\phi(s) \succeq_U \vee\phi(t)$.
Take $x \in \phi(s)$ and $y \in \phi(t)$. Since ϕ is ascending, $x \wedge y \in \phi(t)$ and $x \vee y \in \phi(s)$.
Because $x \wedge y \in \phi(t)$, $\wedge\phi(t) \preceq_U x \wedge y \preceq_U x$.
Because this holds for any $x \in \phi(s)$, $\wedge\phi(t)$ is a lower bound for $\phi(s)$ and therefore $\wedge\phi(t) \preceq_U \wedge\phi(s)$, QED 1.
Similarly, because $x \vee y \in \phi(s)$, $\vee\phi(s) \succeq_U x \vee y \succeq_U y$.
Because this holds for any $y \in \phi(t)$, $\vee\phi(s)$ is an upper bound for $\phi(t)$ and therefore $\vee\phi(s) \succeq_U \vee\phi(t)$, QED 2.

Proof for strongly ascending ϕ.

Take $s \succeq_T t$, $x \in \phi(s)$.
Because ϕ is strongly ascending, $x \succeq_U y'$ for any $y' \in \phi(t)$, i.e., x is an upper bound for $\phi(t)$ and thus $x \succeq_U \vee\phi(t)$.
Since $x \in \phi(s)$ was chosen arbitrarily, this shows that $\vee\phi(t)$ is a lower bound for $\phi(s)$ and thus $\vee\phi(t) \preceq_U \wedge\phi(s)$, QED.

Topkis' Monotonicity Theorem

Recall that we have defined $\phi(t) = \arg\max_{x \in S(t)} g(x,t)$, where $g : X \times T \to \mathbb{R}$ for a lattice X and a partially ordered set T, and $S : T \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the set of sublattices of X.

Topkis' Monotonicity Theorem. *Given the setup above,*

i. *If g is supermodular on X for any $t_0 \in T$, then $\phi(t_0)$ is a sublattice of $S(t_0)$.*

ii. *If g is* strictly *supermodular on X for any $t_0 \in T$, then $\phi(t_0)$ is a chain.*

iii. *If (1) g is supermodular on X for* each *$t \in T$ and has increasing differences in (x,t), and (2) S is ascending, then ϕ is ascending.*

iv. *If, in addition to the conditions in iii, g has* strictly *increasing differences in (x,t), then ϕ is* strongly *ascending.*

Proof of Topkis' Theorem

First note that all the statements hold when $\phi(t)$ is empty. Thus, for the rest of the proof we will assume that $\phi(t) \neq \emptyset$, all t.

i.:
Because $\phi(t_0) \neq \emptyset$, we can take $x, y \in \phi(t_0) \subset S(t_0)$. We must show $x \wedge y, x \vee y \in \phi(t_0)$.
Since $S(t_0)$ is a sublattice of X, $x \wedge y, x \vee y \in S(t_0)$ and thus, by def. of ϕ, $g(x \wedge y, t_0), g(x \vee y, t_0) \leq g(x, t_0) = g(y, t_0)$.
By supermodularity, however, $g(x \wedge y, t_0) + g(x \vee y, t_0) \geq g(x, t_0) + g(y, t_0)$.
But this is possible only if $g(x \wedge y, t_0) = g(x \vee y, t_0) = g(x, t_0) = g(y, t_0)$, i.e., $x \wedge y, x \vee y \in \phi(t_0)$. QED.

ii:
Take $x, y \in \phi(t_0)$. We must show x and y are comparable.
By i, $g(x \wedge y, t_0) = g(x \vee y, t_0) = g(x, t_0) = g(y, t_0)$.
Now, suppose x and y are not comparable. Then, by the definition of strict supermodularity,
$g(x \wedge y, t_0) + g(x \vee y, t_0) > g(x, t_0) + g(y, t_0)$. RAA!
Thus x and y must be comparable, i.e., $\phi(t_0)$ is a chain. QED.

iii:
Take $s \succ t \in T$. We must show $\phi(s) \succeq_p \phi(t)$, i.e., taking any $x \in \phi(s), y \in \phi(t)$, we must show that $x \vee y \in \phi(s)$ and $x \wedge y \in \phi(t)$.

1. By def., $x \in S(s)$ and $y \in S(t)$. Since S is ascending, this implies $x \vee y \in S(s)$ and $x \wedge y \in S(t)$.
2. Since $x \in \phi(s)$ and $x \vee y \in S(s)$, the def. of ϕ implies $0 \geq g(x \vee y, s) - g(x, s)$.
3. Since g is supermodular in x, $g(x \vee y, s) - g(x, s) \geq g(y, s) - g(x \wedge y, s)$.
4. Since $y \succeq x \wedge y$ and $s \succeq t$, increasing differences of g in its two arguments imply $g(y, s) - g(x \wedge y, s) \geq g(y, t) - g(x \wedge y, t)$.
5. Finally, since $y \in \phi(t)$ and $x \wedge y \in S(t)$, the def. of ϕ implies $g(y, t) - g(x \wedge y, t) \geq 0$.

The inequalities in 2–5 above form the following chain:

$$0 \geq g(x \vee y, s) - g(x, s) \geq g(y, s) - g(x \wedge y, s) \geq g(y, t) - g(x \wedge y, t) \geq 0.$$

But then all the expressions in the chain must equal zero.
In particular, $g(x \vee y, s) = g(x, s)$, so that $x \vee y \in \phi(s)$ and $g(y, t) = g(x \wedge y, t)$, so that $x \wedge y \in \phi(t)$, QED.

iv:
Take $s \succ t$ in T and $x \in \phi(s)$ and $y \in \phi(t)$. We must show $x \succeq y$.
Suppose not. Then $y \succ x \wedge y$. Thus, by strictly increasing differences, the second-to last inequality in the chain from iii is strict, which is a contradiction. RAA. QED.

Existence?

- Is there a solution?

  The results obtained so far, elegant though they are, do not tell us whether a solution exists at all. $\phi(t) = \emptyset$ for all t satisfies all the conclusions of Topkis' monotonicity theorem!

- Order properties are not enough to answer this.
- We will need *topological* properties.
  - In particular, we will appeal to a generalization of the well-known extreme-value theorem from basic calculus: **continuous functions** achieve minima and maxima on **compact** sets.
  - Since we only need maxima, we'll use a weaker notion: **upper semi-continuity.**

Continuity

Recall the basic topological definition of continuity:

Definition. Let X and Y be two topological spaces, and let $f : X \to Y$. Then f is **continuous** iff the preimage of any open set in Y is open in X. That is, for each open $U \subset Y$, $f^{-1}(U)$ is open in X.

Also recall that for metric spaces this is equivalent to the ϵ-δ definition from calculus (which is easier to check):

Theorem. *Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f : X \to Y$. Then f is continuous (in the topologies induced by the metrics) iff $(\forall x_0 \in X, \epsilon > 0)(\exists \delta > 0)$ such that $(d_X(x_0, x) < \delta \Rightarrow d_Y(f(x_0), f(x)) < \epsilon)$.*

For $Y = \mathbb{R}$ (with the usual metric), the condition becomes $(\forall x_0 \in X, \epsilon > 0)(\exists \delta > 0)$ such that $(d_X(x_0, x) < \delta \Rightarrow -\epsilon < f(x_0) - f(x) < \epsilon)$.

Note that this can be broken up into two conditions:

1. **"No jumps up":**
   $(\forall x_0 \in X, \epsilon > 0)(\exists \delta > 0)$ such that $(d_X(x_0, x) < \delta \Rightarrow f(x) < f(x_0) + \epsilon)$.
2. **"No jumps down":**
   $(\forall x_0 \in X, \epsilon > 0)(\exists \delta > 0)$ such that $(d_X(x_0, x) < \delta \Rightarrow f(x) > f(x_0) - \epsilon)$.

Semi-Continuity I

Since each of the conditions at the end of the previous slide gives us one half of the definition of continuity, we will call them *semi-continuity* conditions. In particular, the "no jumps up" condition will be called upper semi-continuity, and the "no jumps down" condition will be called lower semi-continuity.

Definition. Let (X, d_X) be a metric space and let $f : X \to \mathbb{R}$. Then f is **upper semi-continuous** iff $(\forall x_0 \in X, \epsilon > 0)(\exists \delta > 0)$ such that $(d_X(x_0, x) < \delta \Rightarrow f(x) < f(x_0) + \epsilon)$.
f is **lower semi-continuous** iff $-f$ is upper semi-continuous.

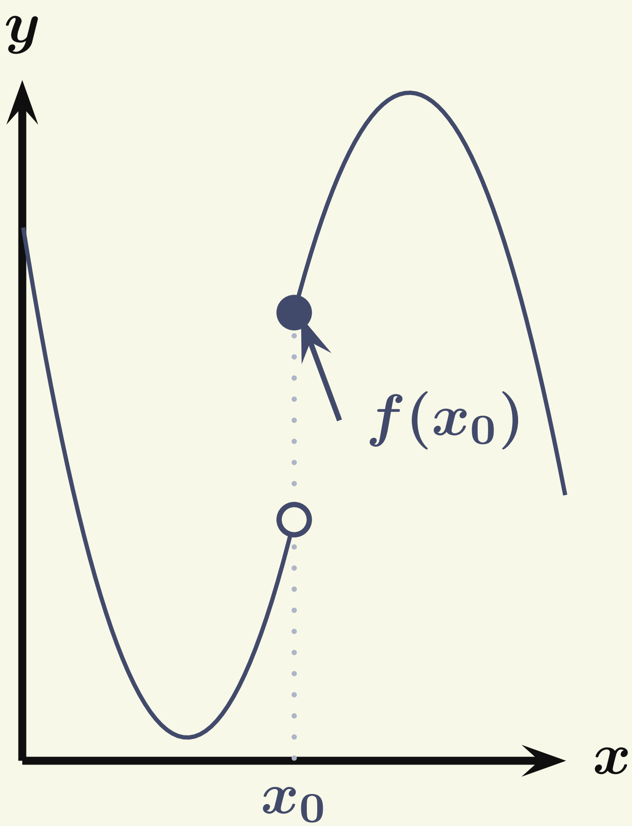
Note that the $-f$ condition for lower semi-continuity is exactly equivalent to condition 2 on the previous slide.

Also note that f is continuous iff it is both upper and lower semi-continuous.

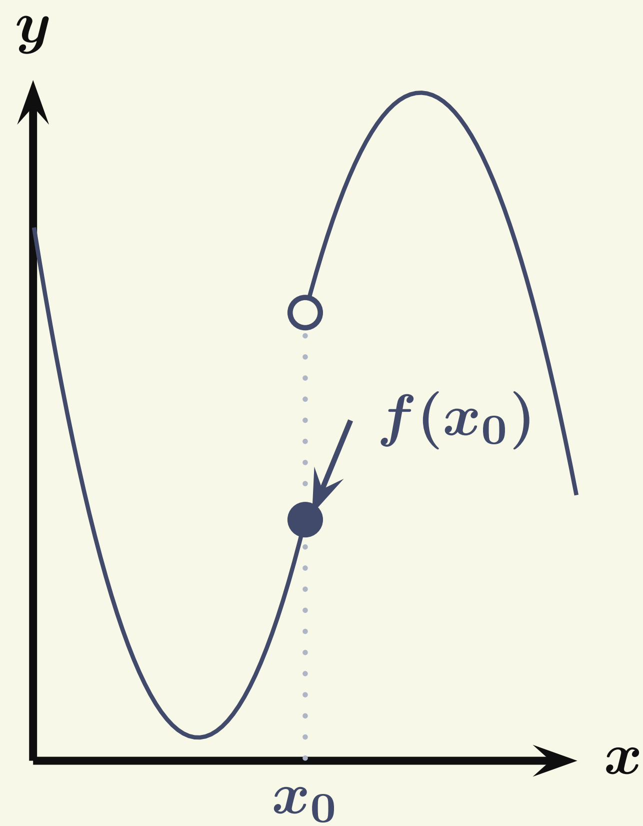
The "no jumps" conditions are tricky; think carefully about them (see next slide for illustration). For example, "the no jumps up" condition does *not* mean that $f(x)$ cannot be higher to the right than to the the left of x!

Semi-Continuity II: Illustration

Upper semi-continuous



Lower semi-continuous



Semi-Continuity III: Alternate Definitions

Just like continuity, semi-continuity can be defined in terms of several equivalent conditions. In different situations, different definitions may be easier to work with. We give two such conditions here.

Theorem. *Let X be a metric space and $f : X \to \mathbb{R}$. Then f is upper semi-continuous (resp., lower semi-continuous) iff $\limsup_{x \to x_0} f(x) \leq f(x_0)$ (resp., $\liminf_{x \to x_0} f(x) \geq f(x_0)$) for all $x \in X$.*

The proof is immediate from the definitions.

Theorem. *Let X be a metric space and $f : X \to \mathbb{R}$. Then f is upper semi-continuous (resp., lower semi-continuous) iff $f^{-1}((-\infty, \alpha))$ (resp., $f^{-1}((\alpha, \infty))$) is an open set in X for each $\alpha \in \mathbb{R}$.*

Again, the proof follows straightforwardly from the definitions.

Note that the last condition can be applied to any partially order topological space in place of $\mathbb{R}$. In general topology, this would be used as the definition.

Existence of Extreme Values

We are now ready for the existence theorem that we need to complete our analysis of the maximization problem.

Theorem. *Let X be a topological space and let $g : Z \to \mathbb{R}$, where $Z \subset X$ is a nonempty and compact set in X. If g is upper semi-continuous (resp., lower semi-continuous), then $\arg\max_{x \in Z} g(x)$ (resp., $\arg\min_{x \in Z} g(x)$) is nonempty and compact.*

Proof (for the upper semi-continuous case).
Consider the collection of open intervals $\mathcal{I} = \{(-\infty, m) \mid m \in \mathbb{N}\}$, which covers R.
Let $\mathcal{J} = \{g^{-1}((-\infty, m)) \mid m \in \mathbb{N}\}$. Since g is u.s.c., the sets $g^{-1}((-\infty, m))$ are open in X.
Since $\mathcal{I}$ covers $\mathbb{R} \supseteq g(Z)$, $\mathcal{J}$ covers Z.
Thus $\mathcal{J}$ is an open cover of Z and, since Z is compact, has a finite subcover.
Since finitely many sets in $\mathcal{J}$ cover Z, finitely many sets in $\mathcal{I}$ must cover $g(Z)$.
Therefore $g(Z)$ is bounded above by some $m \in \mathbb{N}$ and thus has a least upper bound.
Let $\bar{y}$ be the least upper bound of $g(Z)$.
Assume that $\bar{y} \notin g(Z)$.
Since $\bar{y}$ is the l.u.b., there exists a sequence (y_n) in $g(Z)$ that converges to $\bar{y}$ from below.
Observe that the collection $\{(-\infty, y_n) \mid n \in \mathbb{N}\}$ covers $g(Z)$.
By the same argument as before for $\mathcal{I}$, this cover must have a finite subcover.
Thus (y_n) is bounded above by some $y_i \in g(Z)$, and thus $y_i \geq \bar{y}$.
But, since $\bar{y} = \sup(g(Z))$, this is impossible if $\bar{y} \notin g(Z)$. RAA.
We conclude that $\bar{y} \in g(Z)$, i.e., $g^{-1}(\bar{y}) = \arg\max_{x \in Z} g(x) \neq \emptyset$.
Finally, observe that $\arg\max_{x \in Z} g(x) = g^{-1}(\bar{y}) = \left[g^{-1}((\infty, \bar{y}))\right]^C \cap Z$,which is compact as the intersection of a closed set and a compact set ($\left[g^{-1}((\infty, \bar{y}))\right]^C$ is closed because g is u.s.c.).

Putting It All Together: Introduction

- We can now put together the results of Topkis' Monotonicity Theorem and the existence result we just proved to obtain a complete characterization of the solution to the monotone comparative statics problem.

- To simplify the link between order (lattice) and topological properties, we will assume $X \subset \mathbb{R}^n$ for some n (with the product order and usual topology).

  Recall that a lattice in $\mathbb{R}^n$ is complete (in the product order) iff it is compact (in the usual topology).

Putting It All Together: Theorem

Theorem. *Let $X \subseteq \mathbb{R}^n$ be a lattice and T a partially ordered set. Let $S : T \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the set of sublattices of X. Let $S(t)$ be* ***nonempty and compact*** *for each $t \in T$. Let $g : X \times T \to \mathbb{R}$ be a function that is* ***upper semi-continuous*** *on X for each $t \in T$. Define $\phi(t) = \arg\max_{x \in S(t)} g(x, t)$. Then:*

i. *If g is supermodular on X for any $t_0 \in T$, then $\phi(t_0)$ is a nonempty, compact, and complete sublattice of $S(t_0)$. In particular, $\phi(t_0)$ has a greatest element $\bar{\phi}(t_0)$ and a least element $\underline{\phi}(t_0)$.*

ii. *If g is* strictly *supermodular on X for any $t_0 \in T$, then $\phi(t_0)$ is a chain.*

iii. *If (1) g is supermodular on X for* each *$t \in T$ and has increasing differences in (x, t), and (2) S is ascending, then ϕ is ascending. In particular, the greatest and least elements, $\bar{\phi}$ and $\underline{\phi}$, are increasing.*

iv. *If, in addition to the conditions in iii, g has* strictly *increasing differences in (x, t), then ϕ is* strongly *ascending. In particular, $\underline{\phi}(t) \geq \bar{\phi}(t')$ for each $t \succeq t'$.*

Putting It All Together: Proof

i.: $\phi(t_0)$ is a sublattice of X by Topkis' Monotonicity Theorem.
It is a nonempty and compact set by the existence theorem for upper semi-continuous functions.
Since every compact lattice in $\mathbb{R}^n$ is complete, $\phi(t_0)$ is a complete sublattice of X and thus has a greatest and a least element.

ii is just part ii of Topkis' Monotonicity Theorem.

The first parts of iii and iv are just the respective parts of Topkis' Monotonicity Theorem. The results about $\bar{\phi}$ and $\underline{\phi}$ follow directly from the characterization of the upper and lower bounds of ascending correspondences, stated and proved immediately after the definitions of ascending correspondences in these notes.

Putting It All Together: Example

Consider a **monopolist** deciding production quantity $q \in [a, b] \subset \mathbb{R}$.

- The revenue function is $R(q)$, which we will assume to be upper semi-continuous.
- The cost function is $C(q, t)$, where t is a cost-reducing efficiency parameter. We will assume C to be lower semi-continuous in q.
- In addition, we will assume that $C(q, t)$ has decreasing differences in (q, t).

  That is, when efficiency t is higher, an increase in quantity results in a smaller increase in costs. Note that when C is smooth, this is equivalent to assuming $C_{qt} \leq 0$, i.e., marginal costs that are decreasing in t.
- The monopolist's problem is $q^*(t) = \arg\max_{t \in [a,b]} \pi(q, t)$, where $\pi(q, t) = R(q) - C(q, t)$.
- We see that the conditions of part iii of the theorem are satisfied:

  $\pi(q, t) = R(q) - C(q, t)$ is upper semi-continuous and has increasing differences in (q, t). Since $q \in \mathbb{R}$, which is a chain, π is (trivially) supermodular in q. $S(t) = [a, b]$ is nonempty, compact, and constant (and hence ascending).
- Thus, the set of optimal choices $q^*(t)$ is a nonempty and compact sublattice of $\mathbb{R}$; the lowest optimal choice $\bar{q}^*(t)$ and the highest optimal choice $\underline{q}^*(t)$ are increasing in t.

  If C has strictly decreasing differences, then the lowest optimal choice for a higher t is at least as high as the highest optimal choice for a lower t, i.e., $\forall t > t'$, $\underline{q}^*(t) \geq \bar{q}^*(t')$.

More Examples

Numerous other simple examples can be found in various areas of economics (see Amir (2003) for details).

1. **Consumer's problem:** under what circumstances is a good in a two-good world normal?
2. **Price-setting monopolist** with constant marginal cost c: when is optimal price increasing in c?
3. **Assortative matching:** when do likes match with likes?
4. **Growth theory:** when do savings increase in current production?

Supermodular Games

Overview

The results of Tarski and Topkis can be fruitfully applied to a large class of games exhibiting complementarities in the players' actions.

A game is supermodular if all players' action sets are complete lattices, and payoff functions are supermodular in own action and have increasing differences in own action and any opponent's action.

As a direct consequence of Tarski's and Topkis' theorems applied to best response functions, all supermodular games have (ordered) pure strategy equilibria and monotone comparative statics.

Classic example: oligopoly games.

The topic of supermodular games is beyond the scope of these introductory notes. However, I provide references below, and am open to developing an additional set of notes on supermodular games, if there is enough interest.

References

Textbooks and Survey Articles

- The **main textbook** for lattice-theory results in economics is the following:
  Topkis, D. M. (1998). *Supermodularity and Complementarity*. Princeton University Press.

- A very good and concise **introduction to the theory** is found in **Chapter 2** of
  Vives, X. (2000). *Oligopoly Pricing: Old Ideas and New Tools*. MIT Press.

- An early (and highly mathematical) **classic text** is
  Birkhoff, G. (1967). *Lattice Theory*. American Mathematical Society Colloquium Publications, third edition.

- A **simplified overview** of the theory and its main applications, working in the reals (for the most part in one dimension):
  Amir, R. (2003). Supermodularity and complementarity in economics: An elementary survey. *CORE Discussion Paper No. 2003/104*. Available at SSRN: http://ssrn.com/abstract=981364.

Original Articles: Part I

- Tarski's theorem:
  Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309.

- General monotone comparative statics:

  - Topkis' theorem:
    Topkis, D. M. (1978). Minimizing a submodular function on a lattice. *Operations Research*, 26(2):305–321.

  - Generalization of Topkis' theorem, replacing supermodularity with quasisupermodularity and increasing differences with the single-crossing property:
    Milgrom, P. and Shannon, C. (1994). Monotone comparative statics. *Econometrica*, 62(1):157–180.

  - Application to stochastic problems, including Bayesian games (such as auctions):
    Athey, S. C. (2002). Monotone comparative statics under uncertainty. *Quarterly Journal of Economics*, 117:187–223.

Original Articles: Part II

Supermodular games:

- **Topkis, D. M. (1979). Equilibrium points in nonzero-sum n-person submodular games. *SIAM Journal of Control and Optimization*, 17(6):773–787.**
- **Vives, X. (1990). Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics*, 19:305–321.**
- **Milgrom, P. and Roberts, J. (1990). Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica*, 58:1255–1278.**