Until Death Do Us Part?
The Marriage Model with Divorce

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`This Version: November 26, 2009`

Abstract

We extend the two-sided matching model with search frictions by allowing agents to continue searching for an upgrade while already matched. Numerous previous studies have shown that when matches are permanent and matched agents exit the market, the steady-state equilibrium exhibits a counterintuitive and inefficient phenomenon known as "block segregation." We show that this pathological equilibrium is not sustained when at least one side of the market can choose to dissolve the match. In particular, when both sides are allowed to upgrade, quite weak conditions on payoffs and search technology ensure that the steady state converges to perfect positively assortative matching, which is globally optimal with supermodular payoffs. When one side of the market has the ability to upgrade, higher quality agents match to sets of agents of strictly increasing quality. Allowing agents to separate increases the efficiency of matching.

As every newlywed couple is keenly aware, divorce is a real possibility in marriages today. In 2006, the divorce rate was 3.6 percent, only slightly less than one-half of the marriage rate of 7.5 percent.[1] Social and economic commentators across the political spectrum have pointed to divorce as the thread that unravels American family structure. In fact, even former President of the United States Bill Clinton has commented on divorce and its impact on society, at the National Prayer Breakfast in 1996:

> It is a rewarding thing to see the divorce rate leveling off and the teen pregnancy rate going down and the first indications that America may be coming back together around the values that made this a great nation. But we need to support those efforts. There may not be much we can do here as lawmakers. Hillary said in her book that, 'Til death do us part' has often become, 'Til the going gets tough.' It may be that it ought to be a little harder to get a divorce where children are involved.

But is divorce really all that bad? We use an economic model to show that divorce can, in fact, increase aggregate welfare. We start with the standard two-sided search and matching model, where agents on both sides of the market are heterogeneous in their quality and each agent must search for a partner from the other side of the market in order to obtain positive utility. Search is costly in that agents are impatient and prefer an immediate match to one that occurs later in life. Each agent's payoff is increasing in the partner's quality. Furthermore, the payoffs are supermodular: the increase in agent X's payoff that results from matching to a higher-quality partner is higher when agent X's own quality is higher. Utility is non-transferable: no payments can be made between agents. We add to this standard model that agents can continue to search while matched, divorce, and rematch if they find a better partner. Our main result is that the ability to separate results in a steady-state matching pattern that is more efficient than that observed in a model without divorce.

The logic is simple. Without divorce, agents are locked into a marriage forever. Because of this, they are naturally reluctant to match early because they are waiting for Mr. or Mrs. Right. But searching is costly, so agents trade off the desire to be with the right person against the deadweight loss of search. As previous papers in this literature have shown, this results in an equilibrium where agents match in blocks: each agent within an interval will match to any other agent within that interval. This is inefficient: the total of all agents' utilities is less than it would be if matches were instantly arranged by a benevolent social planner. Under complementarity, the efficient outcome occurs when a man of quality x marries a woman of quality x. Matching within intervals of non-zero length, however, results in most agents being matched to partners whose quality differs from their own.

Allowing divorce eliminates this. Now there is no need to match in blocks, because each party has the ability to upgrade later. So, agents match early, and if they find someone better, they divorce and rematch. Over time, the resulting equilibrium converges to the efficient equilibrium of each agent matching to someone of identical quality (positive assortative matching). Allowing separation gives each party an exit option, and that option has value because it can dissolve inefficient matches and replace them with more efficient ones. While there may be winners and losers in the dissolution of individual matches, society as a whole is better off because agents are eventually matched with their equal partner.

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[1] These figures are according to U.S. Census Bureau [2009]. The estimates for 2008 are 7.1 percent for the marriage rate and 3.5 percent for the divorce rate [Tejada-Vera and Sutton, 2009].

To see concretely how divorce improves matching and increases efficiency, first consider the world without divorce. Because matches are permanent and waiting is costly, even the highest-quality agents accept an entire range of partners, not just ones of the highest-quality. In the resulting equilibrium, agents of a given quality will therefore be matched to a set of partners of varying qualities. Consequently, there will be unequal matches. But if all parties can search while matched, such matches will not survive in equilibrium. Because the quality distribution and the total mass of agents is the same on both sides of the market, for each woman W_H of quality higher than x who is matched to a man of quality less than x, there must be a man M_H of quality higher than x who is matched to a woman of quality less than x. Eventually, our high-quality friends W_H and M_H will meet each other, and, since both prefer each other to their current partners, they will divorce and marry each other instead. Divorces and remarriages will continue until everyone is matched to their own type. At this point, no further divorces will occur: if a man of quality x (an x-man) is married to a y-woman, he would be willing to divorce her only for women of quality $y > x$, but all of these women are already happily married to spouses worthier than our x-man! Comparing the resulting matching pattern to the no-divorce equilibrium, we see that some people (the higher-quality ones previously married to lower-quality partners) are now better off, while others (the lower-quality ones who were previously lucky to have higher-quality spouses) are worse off. However, because of supermodularity in payoffs, the total utility gain of the higher-quality agents marrying up exceeds the utility loss of the lower-quality agents marrying down.

Our paper proceeds as follows. We begin by relating our work to two distinct sets of literature: the analysis of divorce in the social sciences (Section 1.1) and the economic study of matching (Section 1.2). In Section 2, we proceed to outline the common framework for our analysis and define various matching patterns that we will be identifying in the various submodels. Section 3 specializes the framework for the baseline case with no on-the-match search and summarizes the results for this case. Section 4, which forms the heart of the paper, analyzes the case when both sides can search for upgrades while matched and either side can divorce the current partner when a better match is found. We will label this *symmetric divorce*.[2] Section 5 discusses *asymmetric divorce* (only one side of the market can search to upgrade).[3]

1 Relation to Extant Literature

1.1 Divorce: A Historical Perspective

The debate on divorce has a long history in the social sciences, and has raged for over a hundred years. The early rounds of debate took place during the Victorian era, with critics arguing both for and against divorce. The first major figure to identify the "divorce problem" of the 19th century was Charles F. Thwing in "The Family" [Thwing and Thwing, 1913], who argued that individualism and secularity were leading to immorality and the demise of traditional family values. It only took four years for social science to provide the first counter-argument to Thwing, which came with Walter F. Willcox's "The Divorce Problem" [Willcox, 1891]. Inspired by the work on divorce of French statistician Bertillon, Willcox based his argument

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[2] Note that the key feature here is really the ability to search while matched—the ability to divorce without on-the-match search would make no difference in a steady-state model, since a match that was once acceptable in a static environment remains acceptable forever. Nonetheless, we use the word "divorce" instead of "on-the-match search and divorce" for brevity.

[3] The asymmetric model is also equivalent to one where agents on one side of the market can divorce only after obtaining their spouses' consent. This follows because in equilibrium such consent will never be given.

on statistical evidence as opposed to morality. He was the first to argue that the prohibition of divorce is an ineffective response to the problem, and divorcees should be allowed to remarry. The only way of prohibiting divorce effectively would be to make them tremendously expensive, which would come at the expense of discriminating against the poor.

Two sociologists afterward, however, were the major names of the debate for several years and together crushed the scientific opposition to divorce: George E. Howard and James P. Lichtenberger [Howard, 1904; Lichtenberger, 1909; O'Neill, 1967]. Indeed, Howard articulated to a changing Victorian society that marriage was a social institution "to be freely dealt with by men according to human needs." Lichentenberger disagreed with social scientists that attempted to reform or impact the pattern of divorce, saying the role of social science was to observe instead of affect the natural progression of society [Lichtenberger, 1909]. Additional social scientists defended divorce as the Victorian era wore on and oversaw a heated, albeit lopsided, debate on the divorce problem [O'Neill, 1967; Parsons, 1914; Ross, 1909, p.194]. Ross especially took a toll against the church but argued that divorce rates would decline with institutional changes such as sex and marriage education. Elsie Clews Parson, the only woman of her time to become known in the debate, argued that marriage was essentially contract of stability and that society opposed divorce not for moral reasons, but out of fear of the changing roles of men and women [Parsons, 1915].

The argument for and against divorce continued into the modern era, with the debate focusing on the effects of divorce on children.[4] In the Victorian era children had been seen in terms of potential wealth and support, but later these views changed, seeing children as "economically worthless, but emotionally priceless" [Zelizer, 1985]. This paradigm shift had tremendous impact on the debate over divorce; although the anti-divorce argument used to be framed upon religious grounds, today it tends to depict children as victims of their parents' whims, making no-fault divorces equal to crimes against children [Lacey, 1992]. Indeed, Judith Wallerstein, a prominent divorce critic today, concentrates on the effect of divorce on children. Her work began in 1971 with a study of 60 divorced families in Marin County, California. Over the years she has urged parents to stay together for their children, with the exception of cases of extreme emotional or physical abuse [Wallerstein and Kelly, 1980].

Yet this opposition to divorce within modern social science rested on moral and religious, rather than scientific, grounds [O'Neill, 1967]. Hetherington and Kelly [2002] have found evidence that divorce is not as harmful to children as has been argued. A long-term experiment begun in 1970 (also the time of Wallerstein's Marin County experiment) involved 1,400 divorced families and 2,500 children. In 2002, Hetherington and Kelly concluded that the negative effects of divorce on children were not nearly as grave as depicted by Wallerstein and, importantly, that results of divorce are highly individualized, making it difficult for divorce to be innately negative for children with the exception of high conflict families [Hetherington and Kelly, 2002].[5]

While the debate on divorce in the social sciences will continue for years, with much disagreement on the cost and benefits of divorce, there is little disagreement that divorce is an important feature of marriage. Figure 1 plots the marriage and divorce rates from 1860. The Census Bureau defines the marriage rate

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[4] One exception is Waite and Gallagner [2000], which argues for marriage primarily based on the welfare of husbands and wives, not of children.

[5] Interestingly, although Wallerstein receives more attention in the popular media, Hetherington receives more citations in scholarly articles [Coltrane and Adams, 2003, p.368].

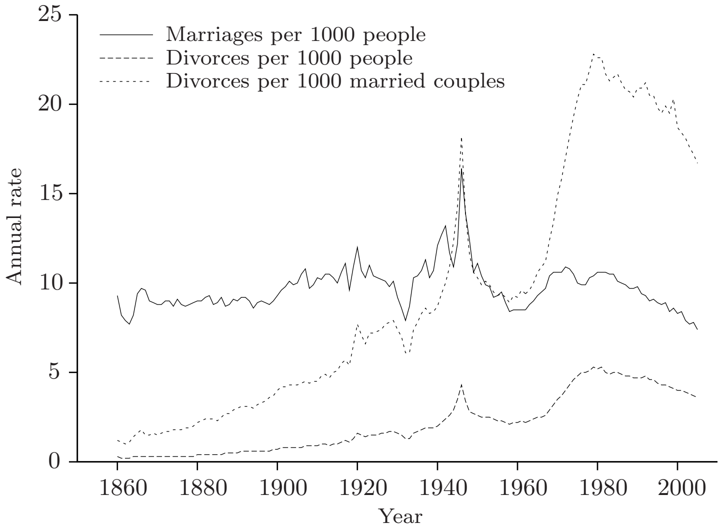

FIGURE 1. United States marriage and divorce rates, 1860–2005

Sources: 1860–1919: Jacobson [1959]; 1920–1998: Carter et al. [2006]; 1999–2005: U.S. Census Bureau [2009].

as the number of new marriages per thousand people, and the divorce rate as the number of new divorces per thousand people. The graph shows that the marriage and divorce rates generally move together, both peaking after World War II and slightly declining since 1980. The conventional wisdom that "half of all marriages end in divorce" comes from the rough calculation that the divorce rate is half the marriage rate. While this is approximately true in 2008, it is not true over the entire history since 1860. In other words, the marriage and divorce rates are not in steady state.[6]

A better way of examining the facts is to look at cohorts. Figure 2 portrays the proportion of marriages ending in divorce for all marriages within each decade since 1960. The most recent vintage from the 1990s has low divorce rates, partly because those couples have not had histories as long as the other cohorts. In particular, almost half of the marriages from the 1970s have ended in divorce 25 years out, which confirms the conventional wisdom. Whichever cohort you examine, however, there is no question that a nontrivial percentage of marriages end in divorce. This suggests that divorce is an important feature of marriage markets.

The economic literature on divorce has been mostly empirical; there have been few attempts at an economic model of the phenomenon. A notable exception is the recent work by Chiappori and Weiss [2000]. However, these authors operate with transferable utility, as opposed to our nontransferable-utility setting (more on this in the next section) and are more concerned with the determination of divorce rates than with equilibrium matching patterns. Nonetheless, an important parallel between their work and ours is the conclusion that the possibility of divorce can be welfare-improving.

The empirical literature has focused on the effects of changes in divorce law on the divorce rate. In the United States before 1969, states allowed divorce under evidence of marital fault, so that mutual consent of divorce was necessary to achieve a separation. This changed in 1969, when Governor Ronald Reagan of

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[6] Stevenson and Wolfers [2007] make exactly this point.

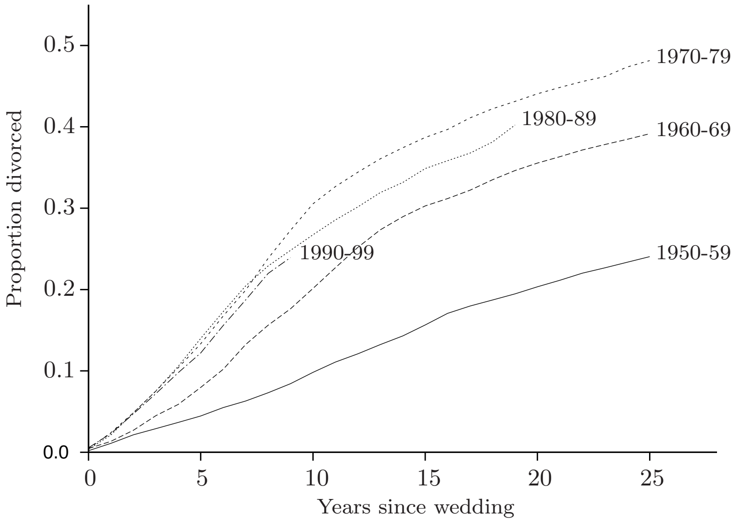

Figure 2. First marriages ending in divorce, by decade of marriage

The figure shows the probability of still being married n years after wedding, conditional on both spouses being alive. Source: Stevenson and Wolfers [2007].

California signed a law creating unilateral divorce, or "no-fault" divorce, in which either party by themselves can end the marriage. The empirical literature examines whether this change in divorce law, namely the introduction of unilateral divorce, increased the divorce rate. The findings remain mixed [Allen, 1992; Friedberg, 1999; Peters, 1986; Wolfers, 2006].

To our knowledge, neither empirical nor theoretical work has examined how divorce changes matching patterns. Has divorce changed who matches with whom? That is the question this paper seeks to answer.

1.2 Matching in Economics

The relation of our paper to most of the studies discussed above is mostly conceptual. What we borrow from that work is inspiration and research questions. In terms of the modeling approach, however, our work is much more closely related to the study of matching in economic theory. To understand both the roots of our model and the contributions that the model makes, it is necessary to take a short detour into the history of the economic study of matching.

Ever since the landmark papers of Gale and Shapley [1962] and Becker [1973], economists have explored the nuances of two-sided matching, commonly in applications of marriage, college admissions, labor markets, etc. In these models agents from one side of the market need to be matched to agents of the other side in order to be productive or derive personal utility. The payoffs from a match to both agents are functions of both agents' quality.

This literature began with a seminal paper by Gale and Shapley [1962], in which they demonstrated that certain marriage markets always allow "stable" matchings, in which no woman and man both prefer each other to their current spouses; this equilibrium concept is equivalent to the core in this situation (as noted by Shapley and Shubik [1972]).[7] One of the key insights in this class of matching models is due to Becker [1973]: when the payoff function is supermodular (i.e., there are complementarities in production), the only efficient allocation, and hence the unique core allocation, is the one where each agent matches only to her own type. We call this matching pattern *perfect positively assortative matching* (perfect PAM).

Models such as these typically examine the core, or perhaps competitive equilibria, of matching markets. To achieve these outcomes, the models tend to rely on a massive clearinghouse, which solicits preference reports from every agent on each side and constructs a matching according to some algorithm. The process is often framed as dynamic, but this is usually just for illumination of the algorithm itself. Even if no clearinghouse exists, each agent is assumed to be able to meet all potential partners without losing any time or exerting any effort. In sum, these early models depend crucially on the assumption of no search costs.

Though this setup may be reasonable for small populations or under certain circumstances, in many contexts it is grossly unrealistic. For example, with regard to marriage, men probably have not even seen all eligible women in the local area, much less the world. Furthermore, "matching opportunities" in most contexts take place over time. An agent meets a potential partner; decides (jointly with the potential mate) whether to "match," based on the "quality" of each agent; and, if a match does not occur, (hopefully!) meets a new potential partner.

The more complex (and arguably more realistic) setting of matching with search has been explored in a large number of papers in the 1990s, starting with the seminal paper of McNamara and Collins [1990] and culminating with the insightful papers by Shimer and Smith [2000, with transferable utility] and Smith [2006, with nontransferable utility]. Like our paper, these papers explore the steady state of the system, similar to the standard approach in evolutionary game theory, where the pivotal concept is that of an evolutionarily stable strategy (ESS), as introduced by Smith and Price [1973].[8]

When utility is nontransferable, papers in this literature commonly find the phenomenon of *block segregation* to which we alluded earlier: agents on each side of the market separate themselves into fixed bands or intervals of quality, such that an agent in a given band can be matched with any agent from the corresponding band on the other side, but to no agent from any other band. In particular, this implies that a small change in an agent's quality results either in no change or a discontinuous jump in the set of possible partners for that agent. This peculiar discontinuity in the otherwise continuous matching model was first discovered by McNamara and Collins [1990] and was subsequently revisited by authors such as Bloch and Ryder [2000]; Burdett and Coles [1997]; Chade [2001]; Eeckhout [1999]; Morgan [1998]. These papers show that the phenomenon is robust under continuous and discrete time, proportional discounting of the future and constant search costs, and various forms of the payoff function. Smith [2006] was the first to use a general payoff function to show that block segregation arises with any payoff function that is multiplicatively separable in the two partners' payoffs, and to show *why* this phenomenon occurs.

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[7] Marriage is a standard interpretation of the matching model. It considerably simplifies the terminology, without loss of generality. An alternative (but equivalent) interpretation is a labor market, in which firms and workers seek to match with one another. Such markets have been studied by Crawford and Knoer [1981]; Demange and Gale [1985]; Kelso and Crawford [1982]; Shapley and Shubik [1972], often with the additional element of a monetary transfer between firms and workers (namely, a salary). Roth and Sotomayor [1990] constructed a thorough discussion of two-sided matching theory, with particular attention to the empirical case of the National Resident Matching Program (NRMP).

[8] A full dynamic analysis of the system is unfortunately exceedingly complex. For an idea of what happens out of steady state, see the biology paper by Alpern and Reyniers [2005] and the ongoing project by Smith [2002].

Block segregation has always raised criticism because it simply does not seem to fit with our sense of reality; Smith [2006, page 1134] writes that "there are no documented cases of block segregation." In addition, a discontinuous equilibrium in an entirely continuous model is in and of itself suspicious. It should also be noted that when payoffs are supermodular, block segregation is clearly inefficient. Both the implausibility and the inefficiency of the phenomenon prompt us to look for conditions that would eliminate this outcome.

Smith [2006] provides one set of conditions under which block segregation disappears and a more efficient matching pattern emerges. Because the set of potential partners for each agent in a model with costly search is a non-singleton set, the concept of perfect assortativeness is of no direct use as a measure of the degree of sorting in such a model. To extend the idea of "like matches to like" to this setting, Shimer and Smith [2000] introduce the concept of setwise positively assortative matching (setwise PAM): in setwise PAM, higher agents match to higher sets of agents. Smith [2006] shows that a sufficient condition for matching to be strictly assortative setwise is that the payoff function be strictly log-supermodular (note that strict supermodularity is not sufficient). When payoffs are multiplicatively separable (the borderline case between log-supermodularity and log-submodularity), block segregation obtains (which is only weakly positively assortative setwise).

Note, however, that the requirement on payoffs for strict setwise PAM is quite strong: the fact that strict supermodularity is not sufficient is quite troubling, and it is not entirely clear how to justify strict log-supermodularity conceptually or how to distinguish between log-supermodularity and simple supermodularity empirically. In addition, as we discuss in Section 3.2, the sorting resulting from strict setwise PAM can be very weak and thus still highly inefficient. This prompts us to look for a different set of conditions that could improve efficiency in the search model. Noting that the inefficiency arises from the need of agents to weigh the benefits of marrying someone immediately against the opportunity cost of giving up the search for someone better in the future, it is natural to suspect that the situation could be improved if agreeing to match with someone did not entail giving up search for better matches in the future. This points toward the possibility of continued on-the-match search and divorce as a vehicle for improvement.

Our model builds on the seminal work of Smith [2006]. There are two sides of the market, men and women, and each agent has a given level of quality.[9] Every man prefers higher quality women and every woman prefers higher quality men. Agents meet in continuous time according to a Poisson process generated by mutual search with quadratic search technology (see Section 2 for definitions). Each pair of agents who have met decide whether to marry. A married agent receives a positive flow of payoffs, which is a function of the agent's and his or her partner's qualities (increasing in the quality of the partner). Single agents receive no payoffs and must continue searching. Agents are impatient, which causes waiting for another match to be costly. All of these features have been used in existing models. Where we diverge from these models is that we allow agents to continue searching for better partners while married. Whereas in the standard model married agents exited the market immediately, in our model they remain on the market, and either one or both sides can continue searching. If a married agent finds a better match, he or she can divorce the current partner and remarry with the newly met partner instead. Divorce itself is costless.

Like Smith [2006] and most of the literature discussed above, we operate in a world of nontransferable utility (NTU), where wages are not available to equilibrate matches, and therefore payoffs are not transferable

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[9] There are no distinguishable sides in Smith [2006]; however, his model can easily be adapted to introduce such a distinction. Furthermore, the two sided-model can be reduced to a model with no sides under symmetry.

or quasi-linear in a transferable resource (see Shimer and Smith [2000] for the current state of the art in the transferable utility framework). There are two main reasons we concentrate on the NTU case. First, even though some intramatch transfers (such as making compromises within a marriage) do exist, the extent of these transfers in many contexts is limited in practice. Second, a key objective for our paper is to examine the robustness of block segregation to the possibility of on-the-match search, and block segregation is exclusively an NTU phenomenon.[10]

As mentioned earlier, we find that block segregation does in fact disappear when we introduce on-the-match search and divorce. In particular, if at least one side of the market has the ability to search while matched and to break an inferior match in favor of a better one, then the "banding" of block segregation is no longer a steady-state equilibrium, and *strict* setwise PAM obtains with *weakly* log-supermodular payoffs (and thus also with multiplicatively separable payoffs). Furthermore, when both sides are allowed to divorce, rather weak conditions on payoffs and search technology guarantee the existence of an equilibrium in which matching converges to *perfect* (not just setwise) PAM.

2 Model Setup

We now turn to the formal specification of the model we outlined above. The model we present encompasses three distinct submodels: no divorce, symmetric divorce, and asymmetric divorce. However, all three submodels share the same basic setup, the only difference being in upgrade search possibilities. Therefore we use this section to set up a common basic framework for all three models. This approach will necessarily make some definitions somewhat vague (to be made precise only in the sections dealing with the particular sub-models). It will, however, help avoid the redundancy that would result from defining each model from scratch.

The market has two distinguishable sides, or *supertypes*, of agents: men (M) and women (W). Both supertypes have equal mass, which we normalize to one.[11] Agents are heterogeneous in terms of their quality; each agent's type (quality) is distributed on the interval $[0, 1]$. We will use the terms "quality" and "type" interchangeably. Both supertypes of agents have the same type distribution, which is atomless and given by the p.d.f. l. This p.d.f. is everywhere positive and boundedly finite: $0 < \underline{l} < l(x) < \overline{l} < \infty$ for any $x \in [0, 1]$. For brevity, we will use the term "x-man" to refer to a man of quality x, and similarly for women.

Time is continuous and infinite. We will focus on the steady state of the model, where strategies, type distributions, and value functions are time-invariant. Agents meet potential partners randomly according to a Poisson process. The Poisson rate of the meeting process is determined according to quadratic search technology [Smith, 2002]; that is, the rate of meeting an individual in a given subset is proportional to the measure of individuals in that subset. The coefficient of proportionality, dubbed the "rendezvous rate" by

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[10] Chiappori and Weiss [2000] is the closest paper that conducts an equilibrium analysis of matching and divorce. That paper, however, focuses on the transferable utility case, i.e. examining the division of the surplus between partners. Because we operate in an NTU world, such intramatch welfare comparisons are not relevant. The NTU and TU matching models operate in parallel literatures; we proceed exclusively in the NTU world.

[11] The distinction between the two sides of the market (the supertypes) matters ultimately under asymmetric divorce. Men and women behave the same in equilibrium in the symmetric case. But we write the model with supertypes for full generality here, to ease exposition later.

Smith [2002], is determined by the search intensities of agents, which are determined exogenously. We will assume that the search intensity of all agents who are allowed to search is the same; we call the resulting rendezvous rate ρ.[12]

A single agent receives the flow payoff of zero. An agent of quality x who is married to a partner of quality y receives flow payoff $f(x, y) > 0$, with f a continuous function, such that partial derivatives with respect to both arguments and the cross-partial derivative exist everywhere.[13] (Note that this also implies that f is bounded above by $\bar{f} < \infty$ on $[0, 1]^2$.) The payoff is strictly increasing in the partner's quality (i.e., $f_2(x, y) > 0$ everywhere). Payoffs are discounted at interest rate $r > 0$. The present value to x of being married to y forever is thus $\int_0^\infty e^{-rt} f(x, y)\, dt = f(x, y)/r$.

We will be particularly interested in payoff functions that exhibit complementarities between the two agents' qualities (though we will not be assuming the existence of complementarities from the outset). The simplest way to capture the idea of complementarity is that of simple supermodularity.

Definition 1. Let $S \subset \mathbb{R}^2$ and let the cross-partial derivative of $\phi : S \to \mathbb{R}$ exist. Then $\phi(x, y)$ is *supermodular* if $\phi_{12}(x, y) > 0$ for all x and y. ϕ is *strictly supermodular* if the inequality is strict everywhere.

While the above definition neatly captures the everyday understanding of complementarity, in some cases a stronger form of the concept, log-supermodularity, will be necessary:

Definition 2. Let $S \subset \mathbb{R}^2$ and let the first partial derivative of $\phi : S \to \mathbb{R}_+$ with respect to the first argument exist. Then $\phi(x, y)$ is *log-supermodular* if $\phi_1(x, y_2)/\phi(x, y_2) \geq \phi_1(x, y_1)/\phi(x, y_1)$ for all x and $y_2 > y_1$, ϕ is *strictly log-supermodular* if the inequality is strict everywhere.

Observe that the weak inequality holds with equality everywhere if and only if ϕ is multiplicatively separable. This is the standard special case of log-supermodularity.

In addition to the possibility of voluntary divorce, a small fraction of matches is randomly dissolved by Nature. Dissolution of any given match follows a Poisson process with rate δ. The random dissolution of matches is necessary to ensure a steady supply of single agents. Random dissolution of matches was introduced as an instrument for this purpose by Shimer and Smith [2000] and Smith [2006]. Other approaches include a steady inflow of new agents [Burdett and Coles, 1997] and replacement of newly matched agents by clones [for example, Bloch and Ryder, 2000; McNamara and Collins, 1990; Morgan, 1998]. Both the cloning and inflow approaches, although fine for the no-divorce case, are awkward in our divorce setup, because matched agents in our model do not exit the market. The population would not remain constant unless inflows were balanced by outflows (deaths), whose rates would have to be matched to inflow, marriage, and divorce rates.

Strategies and Value Functions

An agent's strategy consists of:

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[12] It can be shown (details available upon request) that the results continue to hold when married agents' search intensities are lower than single agents', provided that the difference is sufficiently small.

[13] Note that this setup imposes symmetry: an x-man who is matched to a y-woman gets the same payoff as an x-woman who is matched to a y-man. An x-man who marries a y-woman receives, therefore, receives a payoff $f(x, y)$, while a y-woman who marries and x-man receives a payoff of $f(y, x)$. The total surplus is the sum of these two payoffs: $f(x, y) + f(y, x)$. Because utility is nontransferable, we focus on individual flow payoffs rather than a division of the joint surplus. The TU matching literature focuses more on division of surplus and transfers between the two sides of the market.

1. A single agent's acceptance set, i.e., the set of partners to which this agent is willing to match when single. Let $A^i(x)$ be the acceptance set of a single agent of supertype $i \in \{M, W\}$ and type x. Let $\alpha^i(x, y)$ be the indicator function for $y \in A^i(x)$. Thus, for example, $\alpha^M(x, y) = 1$ if and only if an x-man is willing to marry a y-woman, and $\alpha^W(x, y) = 0$ otherwise.
2. A married agent's acceptance set, i.e., the set of partners to which this agent is willing to upgrade when already married. Let $A^i(x \,|\, y)$ be the acceptance set of an agent of supertype $i \in \{M, W\}$ and type x when married to an agent of type y. Let $\alpha^i(x, z \,|\, y)$ be the indicator function for $z \in A^i(x \,|\, y)$. For example, $\alpha^M(x, z \,|\, y) = 1$ if and only if an x-man who is currently married to a y-woman is willing to divorce her and marry a z-woman instead, and $\alpha^M(x, z \,|\, y) = 0$ otherwise.

The expected average present value to an x-agent of supertype i who is single is $V^i(x)$. The expected average present value to an x-agent who is married to y is $V^i(x \,|\, y)$. Note that these are *average* values.[14] The actual expected present values are therefore $V^i(x)/r$ and $V^i(x \,|\, y)/r$. However, for brevity, we will use the term "value function" throughout the paper to refer to the expected *average* present value.

The density of men of type x who are married to women of type y is given by $\mu(x, y)$. It will also be convenient to denote $\mu^M(x, y) \equiv \mu(x, y)$ and $\mu^W(x, y) \equiv \mu(y, x)$. Note that the mass of married agents of a given type cannot exceed the total mass of agents of that type, so that a given function $\mu : [0, 1]^2 \to \mathbb{R}_+$ is an *admissible* match density function if and only if $\int_0^1 \mu(x, y)\, dy \leq l(x)$ for all x and $\int_0^1 \mu(x, y)\, dx \leq l(y)$ for all y.

The match density function defines also the densities of *unmatched* agents of supertype i and type x, which we denote $u^i(x)$. Note that $u^i(x) = l(x) - \int_0^1 \mu^i(x, y)\, dy$ and that $\int_0^1 u^M(x)\, dx = \int_0^1 u^W(x)\, dx = 1 - \int_0^1 \int_0^1 \mu(x, y)\, dxdy$. Finally, given a supertype i and a type x, the *matching set* of an x-agent of supertype i will be defined as the set of agents of supertype $-i$ that this agent can be matched to in equilibrium:

$$\mathcal{M}^i(x) = \{y \,|\, \mu^i(x, y) > 0\}.$$

Steady state

A steady state is defined as a situation where all relevant elements of the model are time-invariant. In fact, this is equivalent to saying that match densities are time invariant, since stationary strategies and stationary value functions arise naturally in a stationary environment. Thus, a steady state is given by the condition that the relevant match creation rate is everywhere equal to the match separation rate.

More precisely, in models with divorce the steady state will be defined by the condition that the rate at which men of type x get married to women of type y is equal to the total dissolution rate of (x, y) matches (for each x and y). In the model without divorce, the steady state will be defined by the condition that the rate at which agents of type x get married equals the rate at which single agents of type x enter the model (through random match dissolution). Note that the definition of steady state in the no-divorce model refers only to unmatched densities, instead of match densities. This is because the exact match density is irrelevant for agents' strategies in this case, since an agent's opportunity set does not depend on who is married to whom: a married person is an unavailable person, no matter whom she is married to.

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[14] Suppose agent x has expected present value $\nu^i(x)$. Then the *average* present value $V^i(x)$ is the constant flow payoff that this agent would have to receive from now to infinity in order to get the same average present value $\nu^i(x)$: $\nu^i(x) = \int_0^\infty e^{-rt} V^i(x)\, dt = V^i(x)/r$.

Search equilibrium

A *search equilibrium* of the model is given by an *admissible* match density function μ, acceptance sets A^i, and value functions V^i satisfying the following conditions:

1. Steady state;
2. Value functions consistent with the expected payoffs actually obtained, as given by value function equations;
3. Rational strategies, i.e., $\alpha^i(x, y) = 1$ if and only if $V^i(x \,|\, y) \geq V^i(x)$ and $\alpha^i(x, z \,|\, y) = 1$ if and only if $V^i(x \,|\, z) > V^i(x \,|\, y)$.

2.1 Types of Matching Patterns

The key question in any steady-state matching model is who matches with whom. In particular, when do higher quality agents match to higher quality partners? The most straightforward version of like-to-like matching is perfect positively assortative matching, whereby each agent matches to his or her own type:

Definition 3. There is *perfect positively assortative matching (perfect PAM)* if and only if $\mathcal{M}^i(x) = \{x\}$ for all $x \in [0, 1]$ and $i \in M, W$.

This concept is appealing not only because of its simplicity, but also because perfect PAM is the unique matching pattern that maximizes total surplus (see Becker [1973]) when payoffs are strictly supermodular.

However, perfect PAM generally cannot be achieved in our model, because the search cost imposed by time discounting results in non-singleton acceptance and matching sets.[15] This forces us to look for weaker definitions of assortative matching: i.e., ones where higher quality agents match to higher *sets* of agents. The appropriate standard concept, due to Shimer and Smith [2000] and Smith [2006], is setwise PAM:

Definition 4. There is *setwise positively assortative matching (setwise PAM)* if, for each $i \in \{M, W\}$, $x_1 < x_2$, and $y_1 < y_2$ such that $y_1 \in \mathcal{M}^i(x_2)$ and $y_2 \in \mathcal{M}^i(x_1)$, it is also true that $y_1 \in \mathcal{M}^i(x_1)$ and $y_2 \in \mathcal{M}^i(x_2)$. There is *strict* setwise PAM if, for each $x_1 < x_2$ and $y_1 < y_2$ such that $y_1 \in \mathcal{M}^i(x_2)$ and $y_2 \in \mathcal{M}^i(x_1)$, it is also true that $y_1 \in \operatorname{int} \mathcal{M}^i(x_1)$ and $y_2 \in \operatorname{int} \mathcal{M}^i(x_2)$.

As Shimer and Smith [2000] show, when men's matching sets are nonempty, setwise PAM obtains if and only if matching sets are intervals with weakly increasing upper and lower bounds. Strict setwise PAM obtains if and only if the bounds are strictly increasing, except possibly when they are equal to zero or one. Finally, note that, since men's and women's matching correspondences are inverses of each other, it is sufficient to check the conditions of the definition for just one of $i \in \{M, W\}$. This clearly shows how setwise PAM corresponds to the idea of higher-quality agents matching with higher-quality sets of partners. However, it should also be noted that setwise PAM is a much weaker condition than perfect PAM. For example, a situation where $\mathcal{M}^M(x) = [0.001x, 1]$ for all x constitutes *strict* setwise PAM, even though all agents' matching sets are virtually identical. In fact, $\mathcal{M}^M(x) = [0, 1]$ also implies strict setwise PAM, despite the fact that matching sets are identical and there is no actual sorting.

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[15] In addition, note that a perfect PAM distribution is degenerate, and hence the density μ does not exist (the "density" of the distribution is a Dirac delta function). Thus perfect PAM is not consistent with search equilibrium as defined above.

Block segregation is a particular form of *weak* setwise PAM, whereby agents separate themselves into disjoint matching classes, such that no matching occurs across classes:

Definition 5. There is *block segregation* (*banding*) if the type space partitions into at least two (and possibly infinitely many) disjoint classes $[\theta_1, \theta_0] \cup [\theta_2, \theta_1) \cup \ldots$ with $1 = \theta_0 > \theta_1 > \theta_2 > \ldots$ such that, for each $i \in \{M, W\}$, $x \in \mathcal{M}^i(y)$ if and only if x and y are in the same class.

Note that the upper and lower bounds of matching sets are constant almost everywhere and discontinuously increasing at a finite or countably infinite set of points. In particular, this matching pattern is not strict setwise PAM.

The definitions of the matching patterns described above are now standard in the matching literature. Initially defined in the context of the no-divorce model, they are based on matching sets alone. In the context of a model with search and upgrade, however, the matching sets do not come close to telling the full story: because of the possibility of divorce, different matches within the matching set have different expected durations and different equilibrium densities. In addition to asking which matches are possible in principle, we now must also ask which matches are likely to last. We therefore need a new characterization of assortative matching in the context of our modified models. Notice also that when the exogenous match dissolution rate is very high, few agents are able to upgrade from one match to another before they find themselves single again. In this situation, matching patterns are determined mostly by initial matching decisions, and there are only small efficiency gains from the possibility of on-the-match search. Since we are interested in the matching patterns arising from agents' ability to upgrade matches, which creates a dynamic sorting mechanism, we want to focus on situations when the exogenous dissolution rate is negligible. This leads us to the following definition:

Definition 6. The matching pattern *converges in δ to perfect PAM* if for all $R > 0$

$$\lim_{\delta \to 0_+} \sup_{x \in [0,1]} \left| l(x) - \int_{x-R}^{x+R} \mu(x, y)\, dy \right| = 0.$$

The definition says that the model converges in δ to perfect PAM if the mass of matches becomes increasingly concentrated around the diagonal ($\{(x, x) \,|\, x \in [0, 1]\}$) as the external match dissolution rate goes to zero. In other words, the matching pattern converges in δ to perfect PAM if the equilibrium matching measure weakly converges to the perfect PAM measure.

3 Baseline Case: No Divorce

We are now ready to specify and solve each of the three sub-models: no divorce, symmetric divorce, and asymmetric divorce. We begin with the baseline case of no divorce. This is the model that has been widely studied in the literature. This section includes no new results: all the findings in this section are due to Smith [2006]. We state the results for this case as a benchmark for evaluating the effects of adding the possibility of divorce.

3.1 Model Specification

In this case, only single agents can search, and married agents' acceptance sets $A^i(x \mid y)$ are constrained to be empty. Furthermore, the exact matching density is strategy-irrelevant, and therefore, in the definition of equilibrium, we replace the match density function μ with the unmatched densities u^i. Smith [2006] proves that there is a unique equilibrium distribution of unmatched agents. (A continuum of matching densities is compatible with this equilibrium, but they all have the same matching sets, because $\mu(x,y) > 0$ if and only if $\alpha^M(x,y) = \alpha^W(y,x) = 1$.) The value functions are given by

$$\begin{aligned} V^i(x) &= \frac{\rho}{r} \int_0^1 \alpha^i(x,y) \left(V^i(x \mid y) - V^i(x) \right) u^{-i}(y) \alpha^{-i}(y,x)\, dy \\ &= \frac{\rho}{r} \int_0^1 \max \left(V^i(x \mid y) - V^i(x), 0 \right) u^{-i}(y) \alpha^{-i}(y,x)\, dy. \end{aligned} \tag{1}$$

and

$$V^i(x \mid y) = f(x,y) + \frac{\delta}{r} \left(V^i(x) - V^i(x \mid y) \right). \tag{2}$$

The steady state equation is

$$\delta \left(l(x) - \int_0^1 u^i(x)\, dx \right) = u^i(x) \rho \int_0^1 u^{-i}(y) \alpha^i(x,y) \alpha^{-i}(y,x)\, dy. \tag{3}$$

When symmetry of the strategies of the two supertypes is imposed, this no-divorce model reduces to that of Smith [2006], and thus the results in that paper hold. Furthermore, Smith also notes that the results extend to the non-symmetric case. We therefore turn to the description of the matching patterns.

3.2 Matching Patterns

The first key observation in Smith's model is that block segregation (see Definition 5) obtains whenever payoffs are multiplicatively separable in the two partners' qualities. This is a common observation in the literature, though Smith [2006] was the first to show it with general multiplicatively separable payoffs (other papers used particular functional forms, such as $f(x,y) = y$ or $f(x,y) = xy$). The logic of Smith's proof consists of two steps. First, he observes that multiplicative payoffs yield identical von Neumann-Morgenstern preferences over matches. Second, search frictions create a highest band of agents, who are accepted by everybody, because their quality exceeds the time cost of waiting for another meeting, even for an agent who can be sure that nobody will reject her. Thus agents in the highest band have not only the same preferences, but also the same opportunities. Consequently, they must make the same choices. Proceeding recursively to ever lower bands yields the overall block segregation result.

Proposition 1. *Assume $f(x,y) = \varphi_1(x)\varphi_2(y)$ for functions φ_1 φ_2, with $\varphi_1 > 0$. Then there is block segregation in the no-divorce model. If $\varphi_2(0) = 0$, there are infinitely many segregation classes.*

Proof. See Proposition 2 and Lemma 7 in Smith [2006]. □

Note that block segregation relies heavily on the presence of search frictions and the permanence of matches. Waiting is costly, so even the highest-type agent will accept some non-top-quality agents. In addition, since a match is permanent, initial matching decisions fully determine lifetime utility, so all agents who are accepted by the highest-type agent (including the highest type agent herself) must have the same opportunity set. Since multiplicative payoffs imply identical cardinal preferences over matches, these agents, who face the same opportunities, make the same decisions.

Intuitively, this outcome can be avoided if the marginal payoffs from a better partner increase sharply with an agent's type, so that higher-quality agents become relatively more patient in waiting for better partners. In this case, the cardinal preferences over matches are no longer the same for all agents (they increase faster for higher-quality agents), which causes the banding result to break down. Instead, strict setwise PAM obtains: agents of strictly higher quality match to sets of partners of strictly higher quality (see Definition 4). This is the central result of Smith's paper:

Proposition 2. *If the payoff function is strictly log-supermodular, there is strict setwise positively assortative matching.*

Proof. See Smith [2006], Proposition 3. □

The proof proceeds by mathematical manipulation of the value function, but its intuition follows the path outlined above. Under the strong form of complementarity exhibited by log-supermodular payoffs (see Definition 2), agents of strictly higher quality find waiting for better matches strictly more desirable, so their acceptance thresholds are strictly higher. While search frictions still imply that matching sets are intervals instead of points, these intervals are now strictly increasing in an agent's type, and the constant bands from above unravel.

4 Symmetric Divorce

The previous section shows that matching patterns in a world with no on-the-match search can be quite unsatisfying. First, with multiplicatively separable functions, we obtain an unintuitively discontinuous matching correspondence (banding). Second, even when no banding occurs, the steady-state equilibrium pattern does not typically achieve efficiency. In particular, inefficient matching is observed when payoffs are supermodular, so that the unique total-surplus maximizing matching pattern is perfect PAM, which is never reached in equilibrium. Three levels of inefficiency can be observed in this case: if payoffs are log-supermodular, the equilibrium is strict *setwise* PAM; when they are multiplicatively separable, the equilibrium is *weak setwise* PAM (block segregation); but, when payoffs are supermodular, but not log-supermodular, even setwise PAM does not obtain (in fact, *negative* setwise assortative matching could occur). We therefore introduce the possibility for agents to continue to search while already matched to someone and ask ourselves whether this possibility leads to more efficient outcomes.

4.1 Model Specification

Now, all agents are allowed to search while matched and to divorce their current partners when a more desirable match is found. Before writing down the value functions and the steady-state equation, it will be convenient to define a few auxiliary functions. First, for each $i \in \{M, W\}$, let $\Omega^i(x, y)$ be the rate at which an x-agent of supertype i meets y-agents that are willing to marry him or her (the *opportunity rate*):

$$\Omega^i(x,y) = \rho\left(u^{-i}(y)\alpha^{-i}(y,x) + \int_0^1 \alpha^{-i}(y,x\,|\,x')\mu^{-i}(y,x')\,dx'\right). \tag{4}$$

The first summand corresponds to x meeting a single y, whereas the second stands for x meeting an already-married y who is willing to divorce and marry x instead. Similarly, let $D^i(x,y)$ be the rate at which an x-agent of supertype i divorces his or her current partner, given that the current partner's quality is y (the *divorce rate*):

$$\begin{aligned} D^i(x,y) = \rho\int_0^1 \alpha^i(x,y'\,|\,y)\Big[&u^{-i}(y')\alpha^{-i}(y',x)+ \\ &\int_0^1 \alpha^{-i}(y',x\,|\,x')\mu^{-i}(y',x')\,dx'\Big]\,dy'. \end{aligned} \tag{5}$$

The first term in the brackets represents x meeting a single y', whereas the second corresponds to x meeting a y' who is already married and willing to divorce and marry x instead.

We can now write down the value functions and the steady-state equation. The value function of a single agent of supertype $i \in \{M,W\}$ is

$$\begin{aligned} V^i(x) &= \frac{1}{r}\int_0^1 \alpha^i(x,y)\left(V^i(x\,|\,y) - V^i(x)\right)\Omega^i(x,y)\,dy \\ &= \frac{1}{r}\int_0^1 \max\left(V^i(x\,|\,y) - V^i(x), 0\right)\Omega^i(x,y)\,dy. \end{aligned} \tag{6}$$

Since the flow payoff from not being married is zero, the value comes only from expected future marriages. In particular, the expected value is the integral over all possible partner quality levels y of the product of three terms: an indicator whether an agent of type x is willing to marry a y-partner at all, the value gain from being married to a y-partner, and the rate at which x will meet *available* partners of this type. The second line of the equation above follows from the fact that x-agents choose $\alpha^i(x,y)$ rationally.

Similarly, the value of a married agent is

$$\begin{aligned} V^i(x\,|\,y) = f(x,y) &+ \frac{1}{r}(V^i(x) - V^i(x\,|\,y))(\delta + D^{-i}(y,x)) \\ &+ \frac{1}{r}\int_0^1 \max\left(V^i(x\,|\,y') - V^i(x\,|\,y), 0\right)\Omega^i(x,y')\,dy'. \end{aligned} \tag{7}$$

The first term represents the payoff from being married to the current partner; the second term represents the value loss when the marriage is dissolved by Nature or due to divorce by the current partner; the final term stands for the possibility of upgrade to a more desirable partner.

The steady-state equation is

$$\begin{aligned} &\mu(x,y)\left[\delta + D^M(x,y) + D^W(y,x)\right] = \\ &\qquad u^M(x)\alpha^M(x,y)\Omega^M(x,y) + \int_0^1 \mu(x,y')\alpha^M(x,y\,|\,y')\Omega^M(x,y)\,dy'. \end{aligned} \tag{8}$$

The left-hand side is the match dissolution rate at (x,y), while the right-hand side is the match formation rate at (x,y). The first term on the right represents new (x,y)-marriages involving a single x-man, while the second term stands for new marriages involving an x-man who was married to someone else when he met his y-woman (note that this side of the equation could equivalently be expressed by integrating over women rather than men).

Finally, we also require that

$$\alpha^i(x,y) = 0 \;\Rightarrow\; \mu^i(x,y) = 0. \tag{9}$$

This is because $\alpha^i(x,y) = 0$ implies $V^i(x \,|\, y) < V^i(x)$ (by rationality of strategies), and, since all agents are free to divorce, all such matches would instantly dissolve.

4.2 Equilibrium and Convergence

We will use a constructive approach to equilibrium analysis: we will explicitly construct an equilibrium, which will simultaneously prove that an equilibrium exists and characterize that equilibrium. We begin by assuming that a search equilibrium exists and establish some basic facts that must be true in any equilibrium. We then conjecture that a particular strategy profile gives rise to a search equilibrium. We prove that this is indeed the case by explicitly constructing the unique match densities and value functions that arise from the conjectured equilibrium strategies and then showing that the constructed value functions do in fact imply that these strategies are optimal. Finally, we use the construction from the existence proof to show that the resulting equilibrium matching pattern converges in δ to perfect PAM.

Suppose a search equilibrium exists. The first observation we make is that every single agent initially accepts any partner he or she meets. This should not be surprising; because matched agents can continue searching for upgrades, single agents do not give anything up by accepting a match, and they gain some immediate payoff from being matched. Thus, being married to *anyone* is strictly preferable to remaining single.

Lemma 1. *Everyone always accepts everyone when single:* $\alpha^i(x,y) = 1$ *for all* $x, y \in [0,1]$ *and* $i \in \{M, W\}$*. Furthermore, the preference for marrying over remaining single is strict for all agents.*

Proof. See Appendix. □

Another useful observation is that $V^i(x \,|\, y) - V^i(x)$ is everywhere less than $f(x,y)$. The intuition for this result is clear: $f(x,y)$ is the value to an x-agent of being in a *perpetual* match with a y-agent. Thus, $f(x,y)$ would be the difference between the value of having the right to stay married to a y-agent forever and the value of being single. Since an actual match with a y-agent is less valuable than the right to stay married with that agent forever (because the spouse can initiate a divorce), the difference of the value of an *actual* match with a y-agent and the value of being single is less than $f(x,y)$.

Lemma 2. $V^i(x \,|\, y) - V^i(x) < f(x,y)$ *for all* $x, y \in [0,1]$ *and* $i \in \{M, W\}$*.*

Proof. See Appendix. □

Next, observe that the direction of change in $V^i(x \,|\, y)$ in response to changes in y is determined entirely by changes in the terms corresponding to current payoff and the possibility of divorce, i.e., by changes in the expression $f(x,y) + \frac{1}{r}(V^i(x) - V^i(x \,|\, y))(\delta + D^{-i}(y,x))$ in (7). This is because the upgrade possibilities do not depend on the quality of the current partner; since $V^i(x) - V^i(x \,|\, y) < 0$, the expression above represents a trade-off between higher current payoffs and higher possibility of divorce initiated by the partner. It also follows that where the partial derivatives $V_2^i(x\,|\,y)$ and $D_1^{-i}(y,x)$ exist, the sign of $V_2^i(x\,|\,y)$ is determined by $f_2(x,y)$ and $D_1^{-i}(y,x)$. More precisely,

Lemma 3. *Wherever $V^i(x\,|\,y)$ and $D^{-i}(y,x)$ are differentiable with respect to y, the sign of $V_2^i(x\,|\,y)$ is determined by the following identity:*

$$\operatorname{sgn}\left(V_2^i(x\,|\,y)\right) = \operatorname{sgn}\left(f_2(x,y) + \frac{1}{r}(V^i(x) - V^i(x\,|\,y))D_1^{-i}(y,x)\right).$$

Proof. See Appendix. □

The result of Lemma 3 does not immediately imply monotonicity of $V^i(x\,|\,y)$ and is thus in principle compatible with many strategies. However, we consider it natural to look for an equilibrium in simple, symmetric, and intuitive strategies. In particular, we conjecture that there is an equilibrium in which all married agents upgrade when possible. Consider the following conjecture:

Conjecture 1. There is a search equilibrium in which $\alpha^M(x,y) = \alpha^W(x,y) = 1$ for all x and y, and, for any $i \in \{M,W\}$, $\alpha^i(x,z\,|\,y) = 1$ if and only if $z > y$.

This conjectured strategy ("accept when single, upgrade when possible") gives rise to a well-defined, symmetric match density μ and continuous value functions $V(x)$ and $V(x\,|\,y)$, which are the same for men and for women. In addition, $V(x\,|\,y)$ is differentiable with respect to y for any x (these results are proven in the Appendix). To establish an equilibrium, the only remaining step is to show that it is optimal for every agent to follow this strategy, given these match densities and value functions and given that all other agents are following this strategy. Lemma 1 already showed that it is optimal for every agent to accept any other agent when single. Thus, we only need to find sufficient conditions under which it is strictly optimal for every agent to always upgrade when possible. That is, we need to find conditions under which the $V(x\,|\,y)$ derived above is strictly increasing in y.

If everybody uses the conjectured strategy, the divorce rate is given by

$$D(x,y) = \rho \int_y^1 \left[u(y') + \int_0^x \mu(y',x')\,dx'\right] dy',$$

which is increasing in the first argument and differentiable with respect to that argument, with

$$D_1(x,y) = \rho \int_y^1 \mu(y',x)\,dy'. \tag{10}$$

To interpret this, suppose that the agent of quality x in the expression is a man (who is married to a y-woman). Call him Mr. X. Then the expression equals the rate at which Mr. X meets women who are better than his current wife and are currently married to x-men. Suppose Mr. X's quality increases slightly. Then the expression for $D_1(x,y)$ is precisely the additional pool of desirable partners to whom the x-man can now upgrade. Before the quality increase, the y-women who were married to other x-men would not have divorced their partners for Mr. X, but they are happy to do so after the slight increase in his quality.

Since $V(x\,|\,y)$ is differentiable with respect to its second argument and $D(x,y)$ is differentiable with respect to its first argument, Lemma 3 applies. That is, $V(x\,|\,y)$ is everywhere increasing in y if and only if

$$f_2(x,y) + \frac{1}{r}(V(x) - V(x\,|\,y))D_1(y,x) > 0. \tag{11}$$

This condition holds if and only if the payoffs $f(x,y)$ are increasing in y sufficiently quickly, relative to the increase in the divorce rate due to y. When everybody else is using the conjectured strategy, $D_1(y,x)$ is proportional to the density of y-agents whose spouses are better than x (by (10)), which is in turn bounded above by the total density of y-agents, $l(y)$. Noting that the difference $V(x\,|\,y) - V(x)$ is less than $f(x,y)$ (by Lemma 2), we can now replace the condition (11) by a simple condition on the primitives of the model: the relative increase in the payoffs at (x,y) due to an increase in y must be at least as large as the density of agents at y times ρ/r, for any x and y.

Lemma 4. *Let*

$$\frac{f_2(x,y)}{f(x,y)} \geq \frac{\rho}{r} l(y) \quad \textit{for all } x \textit{ and } y.$$

Then the strategy in Conjecture 1 is the unique best response to itself.

Proof. Let the conditions in the statement of the lemma hold, and let everybody (except a given x-agent) use the conjectured strategy. If the agent is single, the conjectured strategy is uniquely optimal by Lemma 1. If the agent is married to some y-agent, Lemma 3 implies, as explained above, that the conjectured strategy is the unique optimal response if and only if the condition (11) holds. By (10) and the admissibility of μ, $D_1(y,x) \leq \rho l(y)$. By Lemmas 2 and 1, $0 < V(x\,|\,y) - V(x) < f(x,y)$. Consequently, the second summand in condition (11) is greater than $-(\rho/r) f(x,y) l(y)$, which by the assumption of Lemma 4 is no less than $-f_2(x,y)$. But then (11) holds. □

We now arrive at our first main result.

Theorem 1. *If*

$$\frac{f_2(x,y)}{f(x,y)} \geq \frac{\rho}{r} l(y) \quad \textit{for all } x \textit{ and } y,$$

then there exists a search equilibrium in which all agents follow the strategy given by $\alpha(x,y) = 1$ *for all* x *and* y*;* $\alpha^i(x,z\,|\,y) = 1$ *if and only if* $z > x$*. Furthermore, the corresponding equilibrium match density is symmetric,* $\mu(x,y) = \mu(y,x)$ *for all* x *and* y*.*

Proof. See Appendix. □

Note that the condition in the hypothesis always holds for a payoff function with sufficiently high relative marginal benefit from being married to a better partner ($f_2(x,y)/f(x,y)$): when there are large gains from upgrades, these upgrades will take place. Second, for any payoff function, the condition holds for sufficiently high interest rate r. As individuals become more impatient, the immediate gain from an upgrade outweighs the possible future loss from an increased risk of being divorced. In fact, this theorem is stronger than the claim that "accept everyone initially and upgrade whenever possible" is part of a search equilibrium. We prove that this strategy is the *unique* best response to itself in the stationary environment that it generates. That is, it is an evolutionarily stable strategy (ESS), as defined by Smith and Price [1973].[16]

The proof of Theorem 1 takes place in several steps. The first task is to show that the conjectured strategy "accept when single, upgrade when possible" induces a well defined steady state match distribution. This

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[16] The idea here is that this strategy maximizes one's payoff or, using the original biological term, *evolutionary fitness* in a population where everyone plays this strategy, and that no alternative strategies can do equally well. Thus if subpopulations using an alternative strategy were to arise, they would lose out in the *survival of the fittest*.

involves defining a best response function that has a unique fixed point. While ordinarily this approach is straightforward, the operator defining the steady-state matching density is not a contraction, which requires a more complicated solution technique. In particular, we discretize the problem and solve recursively, and then show that the unique solutions of a sequence of discretized problems converge to the unique solution of the original continuous problem. This establishes that the conjectured equilibrium strategy gives rise to a well defined, symmetric match density function. The final step involves solving for the value functions. It follows that the search equilibrium exists and the corresponding equilibrium match densities are symmetric.

Now that we have established the existence of an equilibrium and derived an algorithm for deriving its match density, we can turn to investigating the properties of the emerging matching pattern. We will use the discretized model developed in the proof of the main theorem above to help us establish the equilibrium matching pattern under the equilibrium found in the previous section.

Consider the highest-type agents in the discrete model with $\delta = 0$. Because there is no external dissolution and nobody divorces a highest-type agent, we know that no highest-highest matches are ever dissolved, once they are formed. Thus, there is no outflow from this type of match. By the definition of steady state, there must therefore be no inflow to such matches either. But this implies that there cannot be any highest-type agents that are single or matched to a lower-type agent, because whenever two such agents meet, they will marry each other and create inflow into highest-highest matches. Thus, all highest-type agents are matched to highest-type agents. We can repeat this argument recursively, proceeding to ever lower types, to conclude that all agents are matched to agents of their own type.

Note that $\delta = 0$ does not yield any search equilibrium (as defined above) in the original, continuous model. The existence proof in the previous section breaks down. The matching "density" is a Dirac δ function. Nonetheless, the case with $\delta = 0$ in the discrete case suggests that the matching pattern in the continuous model should converge in δ to perfect positively assortative matching (recall Definition 6). This intuition turns out to be correct:

Theorem 2. *When the equilibrium established above exists, the corresponding matching pattern converges in δ to perfect PAM.*

Proof. See Appendix. □

The key criterion in this theorem is that δ/ρ converges to zero (therefore δ converging to zero is sufficient). In words, the external dissolution rate relative to the rendezvous rate must vanish, so the rate at which matches dissolve shrinks relative to the rate of agents are meeting other agents. Recalling that the role of δ is to provide a steady supply of new agents, we can somewhat loosely interpret the result of the theorem as saying that divorce leads to assortative matching when agent turnover is low, so that matching patterns are determined by the long-term interactions of agents, rather than by the behavior of newly arriving market participants.

The proof of Theorem 2 proceeds by first constructing a discrete equilibrium and then applying convergence results to show that the equilibrium of the continuous model is close to that of the discrete model. When formation of entirely new matches by single agents (either new arrivals or agents left single as the result of match dissolution) is dominated by long-term upgrade dynamics among already-matched individuals (i.e., the dissolution rate δ is low), the matching pattern corresponding to the equilibrium we have identified approaches perfect positively assortative matching.

This is reassuring for many reasons. First, it suggests that divorce, an important feature of real-life marriage markets, has "good" results for the model. The matching literature has identified positively assortative matching as the gold standard, and for years block segregation has plagued the equilibrium of two sided search and matching models. While Smith [2006] varied the payoff functions to show under what conditions setwise PAM obtains, we keep the payoff functions fixed, but rather add a new institutional feature to the model, namely separation. Allowing agents to search while matched and separate if desired both captures an important feature of reality, and eliminates this pathological equilibrium of block segregation. Thus we see our result as complementary to that of Smith [2006]; both papers give conditions under which the two-sided matching model with search can arrive at positive assortative matching.

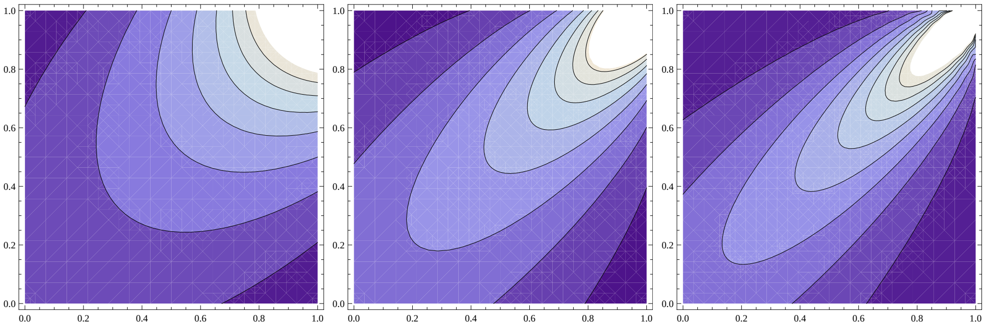

FIGURE 3. Contour maps of steady-state equilibrium matching densities μ for uniform type densities and three different values of $\psi = \delta/\rho$: $\psi_H = 0.5$ (left), $\psi_M = 0.1$ (middle), and $\psi_L = 0.025$ (right)

Figure 3 shows the match density on the path to convergence. The figure plots the contours of the match density function, which is a joint density over the two type spaces, integrating to the total mass of married agents. The lighter-colored regions denote higher density. From left to right, the three pictures represent the steady states for decreasing values of the ratio of the external dissolution rate δ to the rendezvous rate ρ. As δ/ρ shrinks toward zero, the matching density becomes more concentrated along the major diagonal, which represents perfect assortative matching. Also note that the higher types converge faster, as the density is higher-peaked in the upper-right corner of each picture.

Figure 4 provides some summary measures of the speed of the convergence to perfect PAM as δ approaches zero. We see that as the external dissolution rate decreases, both the total mass of matched agents and the percentage of agents matched to partners close to their type increases as δ/ρ decreases. As a result, the total surplus increases.

5 Asymmetric Divorce

In this final extension of the model, we'll investigate whether allowing just *one* side of the market to search while matched is sufficient to induce stricter assortative matching than in the absence of on-the-match search.

5.1 Model Specification

Let us assume that women (without loss of generality) can continue searching while matched. If they find a better match, the previous one is dissolved and a new one is formed instead. Men, on the other side, cannot

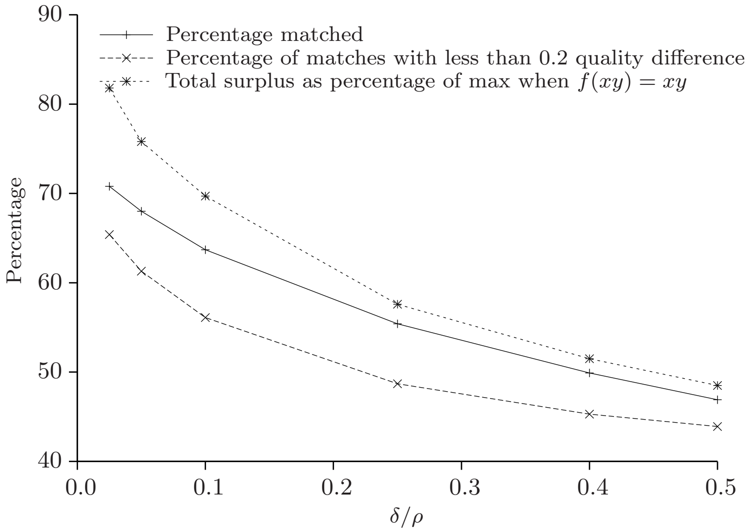

FIGURE 4. Convergence to PAM as $\delta/\rho \to 0$

search while matched and cannot initiate a divorce. The value function for a single woman is

$$\begin{aligned} V^W(x) &= \frac{1}{r}\int_0^1 \alpha^W(x,y)\left(V^W(x \,|\, y) - V^W(x)\right)\rho u^M(y)\alpha^M(y,x)\,dy \\ &= \frac{\rho}{r}\int_0^1 \max\left(V^W(x \,|\, y) - V^W(x), 0\right) u^M(y)\alpha^M(y,x)\,dy. \end{aligned} \tag{12}$$

For each acceptable type y of men, the contribution to the expected payoff is the value to the x-woman of being married to a y-man times the rate at which the x-woman can expect to meet available y-men. The value function for a married woman is

$$\begin{aligned} V^W(x \,|\, y) = f(x,y) &+ \frac{\delta}{r}(V^W(x) - V^W(x \,|\, y)) \\ &+ \frac{\rho}{r}\int_0^1 \max\left(V^W(x \,|\, y') - V^W(x \,|\, y), 0\right) u^M(y')\alpha^M(y',x)\,dy'. \end{aligned} \tag{13}$$

The first term represents the payoff from being married to the current partner; the second term represents the payoff when the marriage is dissolved by Nature; the third stands for the possibility of upgrade to a more desirable partner.

Before we write down the value functions for men and the steady-state equations, it will be useful to introduce two more pieces of notation. First, let us denote the rate at which an x-man meets y-women who are willing to marry him by $\Omega(x,y)$ (where Ω stands for "opportunity rate"):

$$\Omega(x,y) = \rho\left(u^W(y)\alpha^W(y,x) + \int_0^1 \alpha^W(y,x \,|\, x')\mu^W(y,x')\,dx'\right). \tag{14}$$

The first term in parentheses represents our x-man meeting a single y-woman who would accept him, while the second represents him meeting an already-married y-woman who is willing to divorce her current partner for him. Note that when no y-women are willing to accept x, $\Omega(y,x) = 0$. Second, let us denote the dissolution rate of a marriage between an x-man and a y-woman by $D(x, y)$:

$$D(x,y) = \delta + \rho \int_0^1 \alpha^W(y, x' \mid x) u^M(x') \alpha^M(x', y)\, dx'. \tag{15}$$

The first term stands for dissolution by Nature, while the second stands for dissolution due to the woman meeting a better match and divorcing her partner.

Now, we can concisely write down the value functions of men and the steady-state equations. The value function for a single man is

$$\begin{aligned} V^M(x) &= \frac{1}{r} \int_0^1 \alpha^M(x,y) \left(V^M(x \mid y) - V^M(x) \right) \Omega(x,y)\, dy \\ &= \frac{1}{r} \int_0^1 \max \left(V^M(x \mid y) - V^M(x), 0 \right) \Omega(x,y)\, dy. \end{aligned} \tag{16}$$

The value function for a married man is

$$V^M(x \mid y) = f(x,y) + \frac{1}{r}(V^M(x) - V^M(x \mid y)) D(x,y). \tag{17}$$

The steady state equation is (for each x and y)

$$\mu(x,y) D(x,y) = \alpha^M(x,y) u^M(x) \Omega(x,y). \tag{18}$$

The left-hand side is the match dissolution rate at (x, y), while the right-hand side is the rate at which new (x, y)-couples are formed. Note that the right-hand side is zero when an x-man does not accept a y-woman. If the man accepts, the formation rate is proportional to the mass of single men (since only *single* men are allowed to form new matches) and the rate at which a given single man meets y-women who accept him.

In addition to the steady-state equation above, we also require that

$$\alpha^W(y,x) = 0 \;\Rightarrow\; \mu(x,y) = 0. \tag{19}$$

This is because $\alpha^W(y, x) = 0$ implies $V^W(x \mid y) < V^W(x)$ (by rationality of strategies), and, since women are free to divorce, all such matches would instantly dissolve.

5.2 Strategies

The first observation is that men act as if divorce were not a possibility. That is, in deciding whether to accept a match with a woman, they simply compare the flow value of being married to that woman forever to the value of being single. This is immediate from the value-function equation for married men: reordering the terms in (17), we get

$$V^M(x \mid y) - V^M(x) = \frac{r}{r + D(x,y)} (f(x,y) - V^M(x)). \tag{20}$$

Since r and $D(x, y)$ are positive, it follows that the sign of $V^M(x \mid y) - V^M(x)$ equals the sign of $f(x, y) - V^M(x)$. Since $f(x, y)$ is strictly increasing in y, we know that if $f(x, y_0) - V^M(x) > 0$, then $f(x, y) - V^M(x) > 0$ for all $y > y_0$. Therefore, if $V^M(x \mid y_0) - V^M(x)$ is positive, then $V^M(x \mid y) - V^M(x)$ is positive for all $y > y_0$. But then the rationality of strategies requires that if an x-man accepts a y-woman, he must also accept all women of quality higher than y. We have thus proven the following results.

Lemma 5. *Men's value functions and strategies satisfy the following monotonicity conditions:*

1. *Men make marriage decisions by comparing the value of being matched to someone forever to the value of remaining single:*
$$\operatorname{sgn}(V^M(x \,|\, y) - V^M(x)) = \operatorname{sgn}(f(x,y) - V^M(x));$$
2. *Men's acceptance strategies are monotonically increasing in partners' quality:*
$$\alpha^M(x,y) = 1 \;\Rightarrow\; for\ all\ y' > y, \alpha^M(x,y') = 1.$$

The result above already shows that men will employ cutoff strategies; that is, their acceptance sets will be intervals with an upper bound of 1. Combined with the continuity of f and the fact that $f(x,y) > 0$, we can strengthen this result by showing that these intervals are nonempty, nondegenerate and closed, and that the lower limits of the intervals are defined by an indifference condition:

Lemma 6. *Men's acceptance sets are closed, nonempty, and nondegenerate intervals: for all x, there exists an $a^M(x) \in [0,1]$ such that $(A^M(x)) = [a^M(x), 1]$). Men who do not accept all women are indifferent between marrying their marginal partner and remaining single:*

$$a^M(x) \neq 0 \;\Rightarrow\; V^M(x) = V^M(x \,|\, a^M(x)).$$

Proof. See Appendix. □

We now turn to the women's side. The first observation here is that a married woman's value function is strictly increasing in the partner's quality. This is straightforward, since a higher-quality partner increases the immediate payoff from being married without affecting the opportunities for future upgrade. Since men cannot divorce women, this means that marrying a higher-quality man is an unambiguous improvement for a woman. Stating this formally:

Lemma 7. *The married women's value function $V^W(x \,|\, y)$ is strictly increasing in the partner's quality y.*

Proof. Suppose $V^W(x \,|\, y_1) \geq V^W(x \,|\, y_2)$ for $y_1 < y_2$. Then $f(x, y_1) < f(x, y_2)$. Together with $V^W(x \,|\, y_1) \geq V^W(x \,|\, y_2)$ this implies that the right-hand side of (13) is greater for y_2 than for y_1, so that the left-hand side must be too, i.e., $V^W(x \,|\, y_2) > V^W(x \,|\, y_1)$. Contradiction. □

An immediate corollary from this lemma is that any woman will divorce her current partner whenever she meets one with higher quality:

Lemma 8. *Married women always upgrade when possible: $A^W(x \,|\, y) = (y, 1]$ for each x and y.*

Proof. Follows immediately from Lemma 7 and the rationality of strategies (condition 3 in the definition of equilibrium). □

Next, note that single women do not lose anything by accepting a match with anyone: they obtain the immediate rewards of being married to someone, and their options for future matches are in no way affected. It therefore follows that single women will always accept a match with anyone they meet:

Lemma 9. *Single women always accept everybody: $A^W(x) = [0,1]$ for all x.*

Proof. See Appendix. □

We now have a complete characterization of all agents' strategies: men use cutoff acceptance strategies, with the lower limit determined by an indifference condition, whereas women accept everyone when single and upgrade whenever possible.

5.3 Matching Patterns

We are now prepared to tackle the main question: who matches with whom in the asymmetric-divorce model? To do so, we first make the simple observation that, just as in the baseline model, the matching sets are fully determined by single agents' acceptance sets. The matching density for the pair (x, y) is positive if and only if x and y are acceptable to each other when single. The intuition is straightforward: there is positive inflow to (x, y) matches if and only if x and y accept each other, and there is positive outflow if and only if $\mu(x, y) > 0$. The two must be equal in steady state.

Lemma 10. *The matching set for each agent x equals the set of agents y such that x and y are mutually acceptable to each other when single: $\mu(x, y) > 0$ if and only if $\alpha^M(x, y) = \alpha^W(y, x) = 1$.*

Proof. Let $\mu(x, y) > 0$. Equation (19) immediately implies that $\alpha^W(y, x) = 1$. Furthermore, since $D(x, y) \geq \delta > 0$ for all x and y, the left-hand side of equation (18) is positive. For the right-hand side to be positive, we require $\alpha^M(x, y) = 1$.

Now let $\alpha^M(x, y) = \alpha^W(y, x) = 1$. Notice that since the type density is everywhere positive ($l > 0$) and all matches are dissolved at a positive rate $\delta > 0$, u^M and u^L are also everywhere positive. Thus $W(x, y) \geq \rho u^W(y)\alpha^W(y, x) > 0$, and the right-hand side of (18) is positive. For the left-hand side to be positive, we need $\mu(x, y) > 0$. □

The problem of describing the matching sets therefore reduces to describing the acceptance sets of single agents. Recalling the results from the strategy section above, we see that the only missing piece in the puzzle is a characterization of the acceptance thresholds of men, $a^M(x)$. We begin by rewriting the value functions using the findings in the previous section. In particular, plugging the surplus equation (20) into the single men's value function (16), applying Lemma 6, integrating, collecting terms, and noting that $a^M(x)$ must be chosen optimally, yields

$$V^M(x) = \max_{a^M(x)} \frac{\int_{a^M(x)}^1 H(x, y) f(x, y)\, dy}{1 + \int_{a^M(x)}^1 H(x, y)\, dy}, \quad \text{where} \quad H(x, y) = \frac{\Omega(x, y)}{r + D(x, y)}. \tag{21}$$

Applying Lemmas 8 and 9 to (14) and (15), the match dissolution and opportunity rates simplify to

$$\Omega(x, y) = \rho \left(u^W(y) + \int_0^x \mu^W(y, x')\, dx' \right); \tag{22}$$

$$D(x, y) = \delta + \rho \int_x^1 u^M(x')\alpha^M(x', y)\, dx'. \tag{23}$$

It is easy to see that the opportunity rate Ω increases in x, while the dissolution rate D decreases. Consequently, the ratio H is monotonically increasing, and hence almost everywhere differentiable as a function of x. It then follows that the value function V is also almost everywhere differentiable. More precisely, these observations give us the following result:

Lemma 11. *The single men's value function $V^M(x)$ is almost everywhere differentiable. Furthermore, $H_1(x, y)$ is defined almost everywhere, with $H_1(x, y) > 0$ whenever $y > a^M(x)$.*

Proof. Because the integrands in (22) and (23) are non-negative, $\Omega(x, y)$ is weakly increasing in x and $D(x, y)$ is weakly decreasing in x. Therefore, for each y, these functions are almost everywhere differentiable as functions of x, with $\Omega_1(x, y) \geq 0$ and $D_1(x, y) \leq 0$. Consequently, $H(x, y)$ is weakly increasing in x, and $H_1(x, y)$ exists almost everywhere, with $H_1 \geq 0$. This, in turn, implies that $V(x)$ is also differentiable almost everywhere (by the Envelope Theorem).

For the final statement, note that $D_1(x, y) = -\rho u^M(x)\alpha^M(x, y) < 0$ whenever $y > a^M(x)$. Thus it is also true that $H_1(x, y) > 0$ for all $y > a^M(x)$. □

We can now proceed to our key observation: the men's acceptance threshold function a^M is *strictly* increasing whenever payoffs are *weakly* log-supermodular. The proof proceeds along the same lines as the proof of Proposition 3 in Smith [2006]. The key difference from Smith's case is that asymmetric divorce adds an additional benefit to higher-quality men: they have a lower probability of being divorced. This additional benefit (as captured by $H_1(x, y) > 0$) is sufficient to make higher-quality men strictly more selective then lower-quality men, even when payoffs are only *weakly* log-supermodular. The following lemma is proved in the Appendix.

Lemma 12. *When the payoff function $f(x, y)$ is* weakly *log-supermodular, the men's acceptance threshold function $a^M(x)$ is strictly increasing at all x such that $a^M(x) > 0$.*

We now have a complete characterization of all acceptance sets, and thus also of matching sets. In particular, we have obtained a sufficient condition for strict setwise PAM (recall Definition 4) in this model. That is, matching is *strictly* positively assortative setwise whenever payoffs are *weakly* log-supermodular. This result follows from the fact that matching sets are completely determined by men, whose thresholds are increasing by Lemma 12.

Proposition 3. *The asymmetric divorce model exhibits strict setwise PAM whenever payoffs are weakly log-supermodular.*

Proof. Let payoffs be weakly log-supermodular. By Lemma 9, $\alpha^W(y, x) = 1$ everywhere. By Lemma 6, $\alpha^M(x, y) = 1$ if and only if $y \geq a^M(x)$. Then, by Lemma 10, $\mathcal{M}^M(x) = [a^M(x), 1]$ for all x. Since $a^M(x)$ is strictly increasing whenever it is not zero (by Lemma 12), it follows that matching is strictly positively assortative (see the note after Definition 4). □

In particular, this result implies that the block segregation pattern from the baseline model is not robust to asymmetric divorce. With asymmetric divorce, *weak* supermodularity (and thus also multiplicative separability) of payoffs is sufficient for *strict* setwise PAM (which rules out block segregation). Note, however, that the asymmetric divorce result is considerably weaker than the symmetric divorce result. With symmetric divorce, the matching pattern converges to perfect PAM, whereas asymmetric divorce gives us only strict setwise PAM.

6 Conclusion

While marriage and matching have been studied extensively in economics, divorce and separation of matches have received comparatively little attention from our profession. In particular, a key question that has not been studied extensively is how the option to divorce might influence who ends up being married to whom. This paper contributes to filling this gap.

We show that the option to continue looking for a better match while already matched does in fact influence the long-term steady state pattern of matches. Furthermore, this influence can be welfare-improving in the aggregate, because the possibility of continued upgrades induces a strong form of sorting, unless there is a high turnover of agents (formally, a high external dissolution rate of matches).

By exhibiting the sorting potential of the option to divorce, we also contribute to the general literature of matching with nontransferable utility. A nagging problem in that body of research has been the prevalence in theory, but absence in practice, of the phenomenon of block-segregation, whereby agents freely intermarry within fixed bands of quality, but never across the boundaries of their bands. Smith [2006] showed that this peculiar artifact of the model can be ruled out by restricting payoff functions to be strictly log-supermodular. We show that the same result (and often one that is much stronger) can be achieved without strict log-supermodularity of payoffs if the agents are allowed to search for upgrades while matched.

Ultimately, however, the question of how divorce affects marriage patterns (and more generally, how the option to sever matches affects matching patterns in various markets) is one that can only be resolved empirically. This is an exciting avenue for future research.

Appendix

Proof of Lemma 1. If some single x-agent of supertype i does not strictly prefer accepting a potential partner y to remaining single, it must be the case that $V^i(x) \geq V^i(x\,|\,y)$. Equation (7) then implies

$$V^i(x) \geq V^i(x\,|\,y) \geq f(x,y) + \frac{1}{r}\int_0^1 \max\left(V^i(x\,|\,y') - V^i(x), 0\right)\Omega^i(x,y')\,dy'.$$

The last term in the expression above is just $V^i(x)$ (by (6)). We thus obtain

$$V^i(x) \geq f(x,y) + V^i(x),$$

which is impossible, since $f(x,y) > 0$ everywhere. Contradiction. □

Proof of Lemma 2. By Lemma 1, $V^i(x\,|\,y) > V^i(x)$ for all x and y. This implies that the second summand in the right-hand-side of the definition of $V^i(x\,|\,y)$ (eq. (7)) is negative. Furthermore, the integrand in the third summand is no more than $\max(V^i(x\,|\,y) - V^i(x), 0)$, so that the integral is no more than $V^i(x)$ as defined in equation (6). Thus the entire right hand side is less than $f(x,y) + V^i(x)$. Thus (7)) implies $V^i(x\,|\,y) - V^i(x) < f(x,y)$. □

Proof of Lemma 3. Let $g(x,y) \equiv f(x,y) + \frac{1}{r}(V^i(x) - V^i(x\,|\,y))(\delta + D^{-i}(y,x))$.

First notice that $V^i(x \mid y)$ is increasing in y if and only if $g(x,y)$ is. For suppose $V^i(x \mid y_1) \geq V^i(x \mid y_2)$, while $g(x,y_1) < g(x,y_2)$. Then, using equation (7) for y_1, we obtain

$$\begin{aligned} V^i(x \mid y_1) &= g(x,y_1) + \frac{1}{r}\int_0^1 \max\big(V^i(x \mid y') - V^i(x \mid y_1), 0\big)\, \Omega^i(x,y')\, dy' \\ &< g(x,y_2) + \frac{1}{r}\int_0^1 \max\big(V^i(x \mid y') - V^i(x \mid y_2), 0\big)\, \Omega^i(x,y')\, dy' \\ &= V^i(x \mid y_2), \end{aligned}$$

RAA. It follows that $\operatorname{sgn}(V_2^i(x \mid y)) = \operatorname{sgn}(g_2(x,y))$.

Taking the derivative of g with respect to y, we obtain

$$g_2(x,y) = f_2(x,y) + \frac{1}{r}(V^i(x) - V^i(x \mid y))D_1^{-i}(y,x) - \frac{1}{r}V_2^i(x \mid y)(\delta + D^{-i}(y,x)).$$

Since $\operatorname{sgn}(V_2^i(x \mid y)) = \operatorname{sgn}(g_2(x,y))$ and $\delta + D^{-i}(y,x) > 0$, it follows that

$$\operatorname{sgn}(V_2^i(x \mid y)) = \operatorname{sgn}(g_2(x,y)) = \operatorname{sgn}\left(f_2(x,y) + \frac{1}{r}(V^i(x) - V^i(x \mid y))D_1^{-i}(y,x)\right).$$

$\square$

Proof of Theorem 1. Suppose everyone plays the conjectured "accept when single, upgrade when possible" strategy. Our first, and most difficult, task is to show that this strategy profile induces a well-defined steady-state match distribution. We begin by noting that the conjectured strategies transform the opportunity and divorce rate equations (4) and (5) into

$$\begin{aligned} \Omega^i(x,y) &= \rho\left(u^{-i}(y) + \int_0^x \mu^{-i}(y,x')\, dx'\right) = \rho\left(l(y) - \int_x^1 \mu^{-i}(y,x')\, dx'\right); \\ D^i(x,y) &= \rho\int_y^1 \left(u^{-i}(y') + \int_0^x \mu^{-i}(y',x')\, dx'\right) dy' \\ &= \rho\int_y^1 \left(l(y') - \int_x^1 \mu^{-i}(y',x')\, dx'\right) dy'. \end{aligned}$$

Inserting these values in the steady-state equation (8) and simplifying, we obtain

$$\begin{aligned} \mu(x,y)\left\{\delta + \rho\left[\int_y^1 \left(l(y') - \int_x^1 \mu(x',y')\, dx'\right) dy' + \int_x^1 \left(l(x') - \int_y^1 \mu(x',y')\, dy'\right) dx'\right]\right\} \\ = \rho\left(l(y) - \int_x^1 \mu(x',y)\, dx'\right)\left(l(x) - \int_y^1 \mu(x,y')\, dy'\right). \end{aligned}$$

Let $\mathcal{A}$ be the space of admissible match densities. That is, $\mathcal{A}$ consists of all integrable functions $\mu : [0,1]^2 \to \mathbb{R}_+$ that satisfy, for each $x \in [0,1]$, $\int_0^1 \mu(x,y)\, dy \leq l(x)$ and $\int_0^1 \mu(y,x)\, dy \leq l(x)$. Let $\phi \equiv \delta/\rho$. Define the operator T on $\mathcal{A}$ as mapping μ to the function $T\mu : [0,1]^2 \to \mathbb{R}_+$ such that, for any x and y

$$T\mu(x,y) = \frac{\left(l(y) - \int_x^1 \mu(x',y)\, dx'\right)\left(l(x) - \int_y^1 \mu(x,y')\, dy'\right)}{\phi + \int_y^1 \left(l(y') - \int_x^1 \mu(x',y')\, dx'\right) dy' + \int_x^1 \left(l(x') - \int_y^1 \mu(x',y')\, dy'\right) dx'}. \tag{24}$$

Note that $T\mu$ is defined everywhere, since $\phi > 0$ and all other terms are non-negative by admissibility of μ.

The match densities consistent with the conjectured strategies are precisely the fixed points of T. The standard approach would be to endow $\mathcal{A}$ with an appropriate topology and show that T is a contraction with respect to that topology, which would prove that T has a unique fixed point. Unfortunately, however, T does not even map $\mathcal{A}$ to itself (consider, for example, the case when $l(x) = l(y) = 1$ everywhere, $\mu(x, y) = 0$ everywhere, and $\phi = 0.1$). We therefore need to employ a more complicated solution technique.

Discretized problem and equilibrium match densities

We will find a fixed point of T by discretizing the problem, explicitly solving the discretized problem recursively, and then showing that the unique solutions of a sequence of discretized problems converge to the unique solution of the original, continuous, problem.

For a given $K \in \mathbb{N}$, consider 2^K discrete types, indexed by 0 through $2^K - 1$. Let $h_K = 1/2^K$. Consider an equally spaced grid of radius h_K on $[0, 1]$, and let the i^{th} type correspond to the grid point $1 - ih_K \in [0, 1]$. For any $x \in [0, 1]$, let $\iota_K(x)$ be the grid point closest to x:

$$\iota_K(x) = \arg\min_i |1 - ih_K - x| .$$

For any $0 \le i < 2^K$, define $l_i^K = l(1 - ih_K)$. Finally, for a given function μ and any $0 \le i, j < 2^K$, define

$$m_{ij,\mu}^K \equiv \mu(1 - ih_K, 1 - jh_K) .$$

We can now define an operator $\hat{T}_K$ on $\mathcal{A}$ by letting $\hat{T}_K\mu$ be the function that is defined as follows. Let

$$\hat{T}_K\mu(1 - ih_K, 1 - jh_K) = \frac{\left(l_j^K - \sum_{i' \le i} m_{i'j,\mu}^K h_K\right)\left(l_i^K - \sum_{j' \le j} m_{ij',\mu}^K h_K\right)}{\phi + \sum_{j' < j}\left(l_{j'}^K h_K - \sum_{i' \le i} m_{i'j',\mu}^K h_K^2\right) + \sum_{i' < i}\left(l_{i'}^K h_K - \sum_{j' \le j} m_{i'j',\mu}^K h_K^2\right)}$$

for all $0 \le i, j < 2^K$, and let $\hat{T}_K\mu(x, y)$ for all other (x, y) be determined by linear spline interpolation from the values of $\hat{T}_K\mu$ on the grid points $(1 - ih_K, 1 - jh_K)$.

Note that μ is a fixed point of $\hat{T}_K$ if and only if the values of μ off the grid are obtained by linear spline interpolation from the values of μ on the grid, and the values on all grid points, $0 \le i, j < 2^K$, satisfy

$$m_{ij,\mu}^K = \frac{\left(l_j^K - \sum_{i' \le i} m_{i'j,\mu}^K h_K\right)\left(l_i^K - \sum_{j' \le j} m_{ij',\mu}^K h_K\right)}{\phi + \sum_{j' < j}\left(l_{j'}^K h_K - \sum_{i' \le i} m_{i'j',\mu}^K h_K^2\right) + \sum_{i' < i}\left(l_{i'}^K h_K - \sum_{j' \le j} m_{i'j',\mu}^K h_K^2\right)}. \tag{25}$$

Note that this equation represents the steady state in a model with 2^K discrete types, when all types follow the conjectured "accept anyone, upgrade when possible" strategy, the mass of the the i^{th} type is $L_i^K = l_i^K h_K$, and the mass of (i, j)-matches is $M_{ij}^K = m_{ij,\mu}^K h_K^2$.

The discrete steady-state equation (25) can be easily solved for $m_{i,j,\mu}^K$ recursively, starting with $i = j = 0$. It is trivial to check that there is exactly one admissible solution $\hat{m}_{ij}^K$ (satisfying $l_i^K \ge \sum_{j' \le j} \hat{m}_{ij'}^K h_K$ and $l_j^K \ge \sum_{j' \le j} \hat{m}_{ij}^K h_K$ for all i and j) and that this solution is symmetric, satisfying $\hat{m}_{ij}^K = \hat{m}_{ji}^K$ for all i and j.

Since the values of $\hat{T}_K\mu$ on the grid fully determine the values of $\hat{T}_K\mu$ everywhere, we have proven that $\hat{T}_K$ has a unique fixed point on $\mathcal{A}$, namely, the function $\hat{\mu}_k$ obtained from the unique solution to (25), $\{\hat{m}_{ij}^K\}$, by linear spline interpolation. Note that $\hat{\mu}_k$ is continuous by definition and symmetric due to the symmetry of the solution to (25). We thus have the following result:

Lemma 13. *For each K, $\hat{T}_K$ has a unique fixed point $\hat{\mu}_K$ on $\mathcal{A}$. The solution is symmetric ($\hat{\mu}_K(x,y) = \hat{\mu}_K(y,x)$ for all x and y) and everywhere continuous on $[0,1]^2$. For each $0 \leq i,j < 2^K$, $\hat{\mu}_K(1-ih_K, 1-jh_K) = \hat{m}^K_{ij}$, where $\{\hat{m}^K_{ij}\}$ is the unique admissible solution to* (25)*.*

Our next step is to observe that the sequence of fixed points $\hat{\mu}_K$ converges to a limit $\mu \in \mathcal{A}$ as K goes to infinity. Let $\|\bullet\|_\infty$ denote the L_∞ norm on $\mathcal{A}$. Then we require the following Lemma.

Lemma 14. *There exists $\mu \in \mathcal{A}$ such that*

$$\lim_{K\to\infty} \|\hat{\mu}_K - \mu\|_\infty = 0.$$

μ is symmetric and everywhere continuous on $[0,1]$*.*

Proof of Lemma 14. We first show that the values of $\hat{\mu}_K$ and $\hat{\mu}_{K+1}$ evaluated on the grid corresponding to K get closer and closer to each other as K grows large. To see this, take any K and consider the grids K and $K+1$. Note that K is a subgrid of $K+1$, with the (i,j) node of K corresponding to the $(2i,2j)$ node of $K+1$. Also observe that, as $K \to \infty$,

$$\epsilon^K \equiv \max_{i,j<2^K} \left|\hat{m}^K_{i+1,j} - \hat{m}^K_{i,j}\right| \to 0$$

and

$$\hat{\epsilon}^K \equiv \max_{i<2^K} |l(1-ih_K) - l(1-(i+1)h_K)| \to 0.$$

Rewriting the discrete steady state equation (25) for K and $K+1$ and grouping neighboring terms in $K+1$ together, we can show by induction on i and j that

$$\left|\hat{m}^K_{ij} - \hat{m}^{K+1}_{2i,2j}\right| = \mathcal{O}\left(\max\{\epsilon^K, \hat{\epsilon}^K, h_K\}\right)$$

(details available on request). Since $\epsilon^K, \hat{\epsilon}^K$, and h_K all vanish as $K \to \infty$, this shows that

$$\lim_{K\to\infty} \max_{i,j\leq 2^K} \left|\hat{m}^K_{ij} - \hat{m}^{K+1}_{2i,2j}\right| = 0.$$

Next, since $\hat{\mu}_K$ is obtained by a linear spline from its values on the grid, we know that as K becomes infinite and the grid points get close to each other, the difference between $\hat{\mu}_K$ on any point and on its closest grid point becomes negligible:

$$\lim_{K\to\infty} \max_{(x,y)\in[0,1]^2} |\hat{\mu}_K(x,y) - \hat{\mu}_K(\iota_K(x), \iota_K(y))| = 0.$$

Since the values of $\hat{\mu}^K$ and $\hat{\mu}^{K+1}$ are close to each other on the grid, and values of these two functions elsewhere are close to their grid values, the triangle inequality implies that the values of the two functions are close to each other everywhere, i.e.,

$$\lim_{K\to\infty} \max_{(x,y)\in[0,1]^2} |\hat{\mu}_K(x,y) - \hat{\mu}_{K+1}(x,y)| = 0.$$

That is, the sequence of $\hat{\mu}_K$ is Cauchy in the norm L_∞. $\mathcal{A}$ is clearly complete with this norm, and the result of Lemma 14 follows. □

The key observation in the proof of Lemma 14 is that the solutions to (25) for successive values of K become increasingly close to each other, so that $\{\hat{\mu}_K\}$ is a Cauchy sequence and therefore converges in the complete space $(\mathcal{A}, L_\infty)$.

All that remains to be proven is that μ is a fixed point of T. The proof of this fact hinges on the observation that $\hat{T}_K$ is a quadrature approximation of T on the K-grid and thus converges to T on $\mathcal{A}$. Continuity properties of the functions involved, together with the facts that $\lim_{K\to\infty} \hat{\mu}_K = \mu$ and $\hat{\mu}_K$ is a fixed point of $\hat{T}_K$, yield the desired result:

Lemma 15. *$T\mu = \mu$.*

Proof of Lemma 15. We begin by observing that both T and $\hat{T}_K$ for any K are Lipschitz with respect to the L_∞ norm on $\mathcal{A}$.

Next, consider any bounded and uniformly continuous function $\phi : [0,1]^2 \to \mathbb{R}_+$. For any fixed (x, y), standard results on the convergence of quadrature estimates to the true values of integrals yield $\left|T\phi(x,y) - \hat{T}_K\phi(x,y)\right| \to 0$ as $K \to \infty$. Since ϕ is uniformly continuous and T and $\hat{T}_K$ are Lipschitz, hence also uniformly continuous, this also implies that $\left\|T\phi - \hat{T}_K\phi\right\|_\infty \to 0$ as $K \to \infty$.

Next, observe that μ is bounded (by 0 below and $\bar{l}^2\rho/\delta$ above) and continuous, hence also uniformly continuous on $[0,1]$. Thus $\left\|T\mu - \hat{T}_K\mu\right\|_\infty \to 0$ as $K \to \infty$.

Furthermore, $\left\|\hat{T}_K\mu - \hat{\mu}_K\right\|_\infty = \left\|\hat{T}_K\mu - \hat{T}_K\hat{\mu}_K\right\|_\infty$, since $\hat{\mu}_K$ is a fixed point of $\hat{T}_K$. Since $\hat{T}_K$ is Lipschitz, there exists z such that the left hand side is no more than $z\left\|\mu - \hat{\mu}_K\right\|_\infty$, which goes to zero as $K \to \infty$ by Lemma 14. Thus $\left\|\hat{T}_K\mu - \hat{\mu}_K\right\|_\infty \to 0$ as $K \to \infty$.

Finally, $\left\|\hat{\mu}_K - \mu\right\|_\infty \to 0$ as $K \to \infty$ by Lemma 14.

By the triangle inequality, $\left\|T\mu - \mu\right\|_\infty \le \left\|T\mu - \hat{T}_K\mu\right\|_\infty + \left\|\hat{T}_K\mu - \hat{\mu}_K\right\|_\infty + \left\|\hat{\mu}_K - \mu\right\|_\infty$. Since each of the terms on the right goes to zero as K goes to infinity, we conclude that the left-hand side must be zero, i.e., $T\mu = \mu$. □

Value functions in conjectured equilibrium

The result above shows that the conjectured equilibrium strategies give rise to a well-defined, symmetric match density function. Since there is symmetry in both strategies and match distributions, the value function equations (6) and (7) are the same for men and women. Thus, if value functions are well defined, they are symmetric, and we can drop the M and W superscripts. Rewriting and merging the value equations, we see that values are fully defined by solutions $V(x\,|\,y)$ to the following equation:

$$V(x\,|\,y) = \varphi_1(x,y) + \int_0^1 V(x\,|\,y')\varphi_2(x,y,y')\,dy' + \int_y^1 V(x\,|\,y')\varphi_3(x,y,y'))\,dy',$$

where the φ_i are bounded and continuous functions that are differentiable in y and are derived from the conjectured strategies via the steady state matching densities. Thus, for each x, $V(x\,|\,y)$ is given by a linear Volterra-Fredholm integral equation on a compact domain, where all kernels are bounded, continuous, and differentiable in y. Standard results in the theory of integral equations imply that a unique solution $V(x\,|\,y)$ exists, and that it is bounded, continuous, and differentiable with respect to y.

□

Proof of Theorem 2. We begin by looking at the discretized model for some K.

First, let us show that $L_i^K - \hat{M}_{ii}^K = \mathcal{O}(\delta^{2^{-(K+1)}})$ for all $i \leq K$. We do this by using induction on n to prove the statement that $L_i^K - \hat{M}_{ii}^K = \mathcal{O}(\delta^{2^{-(n+1)}})$ for all $i \leq n$.

The basis is straightforward: $\hat{M}_{00}^K \times \delta = \rho(L_0^K - \hat{M}_{00}^K)^2$. Since $\hat{M}_{00}^K$ is bounded above by L_0^K, we see immediately that the LHS is $\mathcal{O}(\delta)$. It follows that $L_0^K - \hat{M}_{00}^K = \mathcal{O}(\delta^{\frac{1}{2}})$.

Now suppose $L_i^K - \hat{M}_{ii}^K = \mathcal{O}(\delta^{2^{-n}})$ for all $i \leq n-1$. Then, in particular, $\hat{M}_{ij}^K = \mathcal{O}(\delta^{2^{-n}})$ for all $i \leq n-1$ and $j \neq i$. By symmetry of $\hat{M}$, $\hat{M}_{ji}^K = \mathcal{O}(\delta^{2^{-n}})$ for all $i \leq n-1$ and $j \neq i$. Then also $\sum_{i'<n} \hat{M}_{i'n}^K = \mathcal{O}(\delta^{2^{-n}})$ and $\sum_{i'<n} \hat{M}_{ni'}^K = \mathcal{O}(\delta^{2^{-n}})$. Finally,

$$\sum_{j'<n} \left(L_{j'}^K - \sum_{i' \leq n} \hat{M}_{i'j'}^K \right) = \sum_{j'<n} \left(L_{j'}^K - \hat{M}_{j'j'}^K \right) + \mathcal{O}(\delta^{2^{-n}}) = \mathcal{O}(\delta^{2^{-n}})$$

by the induction assumption, and

$$L_n^K - \sum_{i'<n} \hat{M}_{i'n}^K - \hat{M}_{nn}^K = L_n^K - \hat{M}_{nn}^K + \mathcal{O}(\delta^{2^{-n}}).$$

The SS equation thus implies that

$$\rho(L_n^K - \hat{M}_{nn}^K + \mathcal{O}(\delta^{2^{-n}}))^2 = \mathcal{O}(\delta^{2^{-n}}),$$

which in turn shows that $L_n^K - \hat{M}_{nn}^K = \mathcal{O}(\delta^{2^{-(n+1)}})$, or $l_n^K - \hat{m}_{nn}^K h_K = \mathcal{O}(\delta^{2^{-(n+1)}})$, which completes the induction.

Now, pick any $R, \epsilon > 0$. By the triangle inequality,

$$\left\| l(x) - \int_{x-R}^{x+R} \mu(x,y)\, dy \right\|_\infty \leq \left\| l(x) - l_{\iota_K(x)}^K \right\|_\infty + \left\| l_{\iota_K(x)}^K - \sum_{1-ih_K \in (x-R, x+R)} \hat{m}_{\iota_K(x),i}^K h_K \right\|_\infty + \left\| \sum_{1-ih_K \in (x-R, x+R)} \hat{m}_{\iota_K(x),i}^K h_K - \int_{x-R}^{x+R} \mu(x,y) \right\|_\infty . \tag{26}$$

By our previous results on the convergence of the discrete to the continuous model, there exists $K_0 > 0$ such that the first and last terms on the right-hand side of (26) are less than $\epsilon/4$ for each $K \geq K_0$. Note also that $1 - \iota_K(x)h_K \to x$ as $K \to \infty$, so that there exists K_1 such that $1 - \iota_K(x)h_K \in (x-R, x+R)$ for all $K \geq K_1$. Let $\bar{K} = \max\{K_0, K_1\}$.

By the result we just proved inductively, $l_{\iota_{\bar{K}}(x)}^{\bar{K}} - m_{\iota_{\bar{K}}(x),\iota_{\bar{K}}(x)}^{\bar{K}} h_K = \mathcal{O}(\delta^{2^{-(\bar{K}+1)}})$. Since $1 - \iota_{\bar{K}}(x)h_{\bar{K}} \in (x-R, x+R)$, this implies that the second term on the right-hand side of (26) is $\mathcal{O}(\delta^{2^{-(\bar{K}+1)}})$. Since the first and the lest terms are each less than $\epsilon/4$, this implies that there exists $\delta_0 > 0$ such that the entire expression is less than ϵ for all $\delta < \delta_0$. That is,

$$\lim_{\delta \to 0_+} \left\| l(x) - \int_{x-R}^{x+R} \mu(x,y)\, dy \right\|_\infty = 0.$$

□

Proof of Lemma 6. First observe that if $f(x,0) \geq V^M(x)$, then, by Lemma 5, $A^M(x) = [0,1]$. Now, let $f(x,0) < V^M(x)$.

Suppose $f(x,y) < V^M(x)$ for all $y \in [0,1)$. Then, by Lemma 5, $A^M(x) \subset \{1\}$, so that, by equation (16), an x-man's value of being single would be zero. But then, since f is everywhere positive, $f(x,y) > 0 = V^M(x)$ for all $y \in [0,1]$. Then Lemma 5 implies that $A^M(x) = [0,1]$, which contradicts $A^M(x) \subset \{1\}$. RAA.

Hence $(\exists y \in [0,1))(f(x,y) \geq V^M(x))$. Since f is continuous and $f(x,0) < V^M(x)$, the Intermediate Value Theorem implies that

$$(\exists a^M(x) \in (0,1))(f(x,a^M(x)) = V^M(x)).$$

By Lemma 5, $V^M(x \,|\, a^M(x)) = V^M(x)$ and $A^M(x) = [a^M(x),1]$. □

Proof of Lemma 9. Since $V^W(x \,|\, y)$ is increasing in y by Lemma 7, we only need to show that $V^W(x \,|\, 0) \geq V^W(x)$. Suppose not, i.e., $V^W(x \,|\, 0) < V^W(x)$. Then we have

$$\int_0^1 \big(V^W(x \,|\, y') - V^W(x \,|\, 0), 0\big)\, u^M(y')\alpha^M(y',x)\, dy' \geq$$
$$\int_0^1 \big(V^W(x \,|\, y') - V^W(x), 0\big)\, u^M(y')\alpha^M(y',x)\, dy' = V^W(x),$$

where the equality follows by the definition of $V^W(x)$ (equation (12)).

Now, look at the right-hand side of the definition of $V^W(x \,|\, 0)$ (equation (13)). We have just shown that the last term there is no less than $V^W(x)$. Furthermore, the first term, $f(x,0)$ is positive by definition, and the second term, $(\delta/r)(V^W(x) - V^W(x \,|\, y))$ is positive by assumption. But then the entire right-hand side of (13) is greater than $V^W(x)$. Therefore, the same must be true of the left-hand side: $V^W(x \,|\, 0) > V^W(x)$. RAA. □

Proof of Lemma 12. Take any x such that $a^M(x) > 0$. By the Envelope Theorem and (21),

$$V^{M\prime}(x) = \frac{\left(\int H_1 f + \int H f_1\right)\left(1 + \int H\right) - \left(\int H f\right)\left(\int H_1\right)}{\left(1 + \int H\right)^2},$$

where all integrals are taken with respect to dy over the interval $[a^M(x),1]$; we have omitted the arguments (x,y) of all functions for brevity.

By Lemmas 5 and 6, $f(x,y) > V^M(x)$ for all $y > a^M(x)$. Furthermore, by Lemma 11, $H_1(x,y) > 0$ for $y > a^M(x)$ where the derivative exists. Therefore,

$$\int H_1 f > V^M(x) \int H_1 = \frac{\int H f}{1 + \int H} \int H_1,$$

so that $V^{M\prime}(x) > (\int H f_1)/(1 + \int H)$ and

$$\frac{V^{M\prime}(x)}{V^M(x)} > \frac{\int_{a^M(x)}^1 H(x,y) f_1(x,y)\, dy}{\int_{a^M(x)}^1 H(x,y) f(x,y)\, dy} \geq \frac{f_1(x,a^M(x))}{f(x,a^M(x))}, \tag{27}$$

where the weak inequality follows by weak log-supermodularity of f.

Finally, note that, by Lemmas 6 and 5, $f(x,a^M(x)) = V^M(x)$ for all x where $a^M(x) > 0$. Differentiating this identity yields

$$f_1(x,a^M(x)) + f_2(x,a^M(x)) a^{M\prime}(x) = V^{M\prime}(x),$$

so that

$$\frac{V^{M\prime}(x)}{V^{M}(x)} = \frac{f_1(x, a^M(x)) + f_2(x, a^M(x))a^{M\prime}(x)}{f(x, a^M(x))}.$$

Substituting this identity into (27) yields $(f_2(x, a^M(x))/f(x, a^M(x)))a^{M\prime}(x) > 0$. Since $f_2, f > 0$ everywhere, we must conclude that $a^{M\prime}(x) > 0$. $\square$

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