Pricing Growth-Rate Risk*

Lars Peter Hansen
University of Chicago

José A. Scheinkman
Princeton University

November 11, 2009

Abstract

We characterize the compensation demanded by investors in equilibrium for incremental exposure to growth-rate risk. Given an underlying Markov diffusion that governs the state variables in the economy, the economic model implies a stochastic discount factor process S and a reference stochastic growth process G for the macroeconomy. Both are modeled conveniently as multiplicative functionals of a multidimensional Brownian motion. To study pricing we consider the pricing implications of parameterized family of growth processes G^ϵ, with $G^0 = G$, as ϵ is made small. This parameterization defines a direction of growth-rate risk exposure that is priced using the stochastic discount factor S. By changing the investment horizon we trace a *term structure* of risk prices that shows how the valuation of risky cash flows depends on the investment horizon. Using methods of Hansen and Scheinkman (2009), we characterize the limiting behavior of the risk prices as the investment horizon is made arbitrarily long.

Keywords Markov process growth-rate risk pricing dynamics elasticities
Mathematics Subject Classification (2000) 60J70 · 91B28
JEL Classification C52 · G12

1 Introduction

A standard result from asset pricing theories is the characterization of the local risk-return tradeoff. This tradeoff is particularly simple in the case of Brownian information structures. In mathematical finance the risk prices are embedded in the transformation to a risk-neutral

*Forthcoming in *Finance and Stochastics*. We thank Rene Carmona for a helpful discussion and Juan Matias Ortner for research assistance. This material is based on work supported by the National Science Foundation under award number SES-07-18407 and SES-05-19372 .

measure. Applying Girsanov's Theorem, this change of measure adds a drift vector to the multivariate standard Brownian motion. The vector of local risk prices is the negative of the drift vector used in constructing the risk neutral transformation. This price vector reflects the local compensation in terms of the drift for exposure to alternative components of the Brownian motion. With these local prices, we price exposure to linear combinations of the Brownian risks by forming the corresponding linear combination of prices.

While derivative claims are often priced using the risk neutral measure, structural models of asset prices interpret these prices in terms of the fundamentals of the underlying economy. In this paper, as in Hansen and Scheinkman (2009) and Hansen (2008), we characterize the compensation demanded by investors for added risk at different time horizons, that is a *term-structure* of risk prices. This compensation will typically depend on how investors discount risky payoffs and the risk they already face. Our approach is as follows. There is an underlying Markov diffusion X that governs the state variables in the economy. The economic model implies a stochastic discount factor process S and a reference stochastic growth process G for the macroeconomy. Both are modeled conveniently as multiplicative functionals of a multi-dimensional Brownian motion. To feature the role of price dynamics, we normalize the reference growth functional to a be a martingale. More generaly this martingale can be the martingale component in a factorization of the growth functional (as in Hansen and Scheinkman (2009)). To study pricing we consider a parameterized family of growth processes G^ϵ, with $G^0 = G$ and study its pricing implications for payoffs at different horizons. We define the price of growth-rate risk as:

$$\rho_t = -\frac{d}{d\epsilon}\frac{1}{t}\log E\left[G_t^\epsilon S_t | X_0 = x\right]|_{\epsilon=0}.$$

It is the elasticity of the expected rate of return (per unit of time) with respect to the exposure to growth-rate risk. The expected return implicit in this calculation is the reciprocal of the price $E\left[G_t^\epsilon S_t | X_0 = x\right]$ since G_t^ϵ has expectation one by construction.

The resulting prices of growth-rate risk extend the local prices to arbitrary investment horizons. While we focus on scalar parameterizations, we can interpret our calculations as producing prices for an arbitrary linear combination of exposure to the Brownian motion risks. By changing the exposure weights, we feature alternative components of the Brownian increments and thus construct the counterpart to the local risk-price vector.

For a given investment horizon, we characterize our risk prices by applying tools that are used to compute sensitivities of option prices (the "Greeks"). The prices we compute reveal the local risk prices as the horizon t shrinks to zero:

$$\lim_{t \downarrow 0} \rho_t = \rho_0$$

We add to this a characterization of the limit prices as the investment horizon tends to ∞:

$$\lim_{t \uparrow \infty} \rho_t = \rho_\infty,$$

along with formulas for the intermediate investment horizons.

2 Mathematical setup

The underlying state vector process X is n-dimensional and satisfies,

$$dX_t = \beta(X_t)dt + \alpha(X_t)dW_t, \tag{1}$$

where W is an n-dimensional Brownian motion in a probability space $\{\Omega, \mathcal{F}, \mathsf{Pr}\}$ and $\alpha(\cdot)$ is non-singular. We write $\{\mathcal{F}_t : t \geq 0\}$ for the (completed) Brownian filtration. Assuming that β and α are locally Lipschitz there exists a unique X_u that solves equation (1) when $X_0 = x$. In this section we think of $X_0 = x$ as fixed or known but construct assumptions and results that apply to all initializations. In section 6 we will introduce explicit randomness in X_0 and augment the filtration $\{\mathcal{F}_t : t \geq 0\}$. The "unconditional" expectations of this section will become expectations conditioned on $X_0 = x$ in section 5. Moreover, the resulting dependence on x will be of central interest in applications. We use multiplicative functionals M of the form

$$M_t = \exp\left[\int_0^t \delta(X_u)du + \int_0^t \gamma(X_u)dW_u\right] \tag{2}$$

where

$$\int_0^t |\delta(X_u)|du < \infty$$
$$\int_0^t |\gamma(X_u)|^2 du < \infty$$

for all t with probability one.[1] The multiplicative functional M given by equation (2) is refered to as parameterized by (δ, γ). Consider two multiplicative functionals: G parameterized by (δ_g, γ_g) and S parameterized by (δ_s, γ_s). The process G captures stochastic growth and the

[1] As in *e.g.* Ito and Watanabe (1965), we allow for multiplicative functionals that do not have bounded variation.

process S stochastic discounting. G and S depend on x, but since we only consider a fixed initial condition x, unless there is ambiguity, we will omit in the notation the dependence on x.

Asset valuation over a horizon t is represented as:

$$E\left(S_t G_t\right)$$

where G_t is the asset payoff that is priced. There are two channels that dictate the term structure of risk premia and the associated prices: stochastic discounting and stochastic growth. Our aim is to focus on the latter channel.

Hansen and Scheinkman (2009) (Corollary 6.1) establishes a multiplicative factorization of G:

$$G_t = \exp(\eta t) G_t^o \left[\frac{f(X_0)}{f(X_t)}\right]$$

where G_t^o is a multiplicative martingale.[2] The exponential growth term $\exp(\eta t)$ is of no consequence for risk prices and can be omitted. Since predictability in S and G alter the term structure of risk premia, one possibility is to feature the role of pricing dynamics by focusing exclusively on the martingale component G^o and constructing perturbations that preserve the martingale property. In what follows we will adopt this perspective where $G = G^o$ and hence is restricted to be a martingale. In this case

$$-\log E\left(S_t G_t\right)$$

is the logarithm of the expected return associated the martingale payoff G_t.

To construct risk prices for any given payoff horizon, we parameterize a family of growth functionals as G^ϵ with $G = G^0$ where G^ϵ is a martingale for each ϵ. The parameterized martingale is constructed to feature exposures to specific combination of shocks. By altering the parameterization, we explore sensitivity to alternative shocks thereby constructing counterparts to local risk prices.

As an alternative, we could work with a macro growth functional G that is not necessarily a martingale, but we explore parameterized perturbations that are martingales. Then the logarithm of the expected return associated with G_t is:

$$\log E\left(G_t\right) - \log E\left(S_t G_t\right).$$

For this formulation our baseline martingale G^0 is identically one and the counterpart to

[2] Strictly speaking, this corollary produces a local martingale rather than a martingale.

S for our analysis is either SG or G. With these changes, our forthcoming analysis will continue to be applicable.

Recall that the stochastic exponential of a semi-martingale N is a semi-martingale $\mathcal{E}(N)$ that solves $\mathcal{E}(N)_t = 1 + \int_0^t \mathcal{E}(N)_{s_-} dNs$. Since, in our case, sample paths are continuous,

$$\mathcal{E}(N) = \exp\left(N - \frac{1}{2}[N, N]\right). \tag{3}$$

We assume that the positive martingale G is the stochastic exponential $\mathcal{E}(Z^o)$ of a martingale $Z_t^o = \int_0^t \gamma_g(X_u)dW_u$. Consider a family of perturbabtions G^ϵ of the form:

$$G^\epsilon = \mathcal{E}(Z^o + \epsilon Z), \tag{4}$$

$\epsilon \in (-1, 1)$ where $Z_t = \int_0^t \gamma_d(X_u)dW_u$. For the stochastic integrals to be well behaved, $\int_0^t |\gamma_g(X_u)|^2 du < \infty$ and $\int_0^t |\gamma_d(X_u)|^2 du < \infty$ with probability one.

The process Z used to construct the perturbation can feature any of the individual components of the underlying Brownian motion. The resulting parameterized family expressed in logarithms is:

$$\log G_t^\epsilon = \int_0^t \gamma_g(X_u)dW_u + \epsilon \int_0^t \gamma_d(X_u)dW_u - \frac{1}{2}\int_0^t |\gamma_g(X_u) + \epsilon\gamma_d(X_u)|^2 du$$

In this specification $\epsilon \int_0^t \gamma_d(X_u)dW_u$ captures the (growth rate) risk exposure. By changing γ_d we alter which Brownian increments are featured in the pricing.

3 Finite-Horizon Prices

In this section we apply an approach developed by Fournié et al. (1999, 2001) to show that

$$\rho_t = -\frac{E\left[S_t G_t \left(\int_0^t \gamma_d(X_u)dW_u - \int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u)du\right)\right]}{tE\left(S_t G_t\right)}. \tag{5}$$

We start by using the multiplicative martingale G to change measure. Then Girsanov's Theorem guarantees that $\frac{G^\epsilon}{G} = \mathcal{E}[\epsilon\tilde{Z}]$, and $\tilde{Z}_t = \int_0^t \gamma_d(X_u)d\tilde{W}_u$, where $\tilde{W}_u = -\int_0^u \gamma_g(X_v)dv + W_u$, and $\tilde{W}$ is a Brownian motion in $[0, t]$ under the changed measure $\tilde{\mathsf{Pr}}$. Notice that the functional form of G guarantees that X_t remains Markov under $\tilde{\mathsf{Pr}}$. We write $\tilde{E}$ for the associated expectations operator. Hence,

$$
\begin{aligned}
\frac{\frac{G_t^\epsilon}{G_t} - 1}{\epsilon} &= \int_0^t \frac{G_u^\epsilon}{G_u} d\tilde{Z}_u, \text{ or} \\
&= \int_0^t \left(\frac{G_u^\epsilon}{G_u}\right) \gamma_d(X_u) d\tilde{W}_u
\end{aligned}
\tag{6}
$$

If the right-hand side has a well-defined limit as $\epsilon \to 0$, then necessarily this limit is:

$$
\frac{\frac{G_t^\epsilon}{G_t} - 1}{\epsilon} \to \int_0^t \gamma_d(X_u) d\tilde{W}_u
$$

For each initial value x of X, we may write the price of an asset as a function of the perturbation on the growth factor as:

$$
\begin{aligned}
U(\epsilon) &= E\left(G_t^\epsilon S_t\right) \\
&= \tilde{E}\left(\frac{G_t^\epsilon}{G_t} S_t\right).
\end{aligned}
$$

Hence,

$$
U'(0) = \lim_{\epsilon \to 0} \frac{\tilde{E}\left[\left(\frac{G_t^\epsilon}{G_t} - 1\right) S_t\right]}{\epsilon}
$$

$$
= \lim_{\epsilon \to 0} \tilde{E}\left[S_t \int_0^t \left(\frac{G_u^\epsilon}{G_u}\right) \gamma_d(X_u) d\tilde{W}_u\right]
$$

Next we impose two assumptions that are sufficient for the main result in this section. After establishing this result, we provide sufficient conditions for the second of these assumptions.

Assumption 3.1. *For each x, $E\left[(S_t)^2 G_t\right] < \infty$.*

Imposing this restriction is equivalent to assuming that S_t has a finite conditional second moment (in the $\tilde{\mathsf{Pr}}$ measure).

Assumption 3.2. *For each x,*

$$
\lim_{\epsilon \to 0} \frac{\frac{G_t^\epsilon}{G_t} - 1}{\epsilon} \to \int_0^t \gamma_d(X_u) d\tilde{W}_u.
$$

in $L^2(\tilde{\mathsf{Pr}})$.

Proposition 3.3. *Suppose that Assumptions 3.1 and 3.2 are satisfied. Then for each x,*

$$U'(0) = \tilde{E}\left[S_t \int_0^t \gamma_d(X_u) d\tilde{W}_u\right]$$
$$= E\left(S_t G_t \left[\int_0^t \gamma_d(X_u) dW_u - \int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du\right]\right)$$

Proof. This follows directly from Holder's Inequality. □

The elasticity of interest is the ratio $-\frac{U'(0)}{tU(0)}$ and is given by (5).

We now provide sufficient conditions for Assumption 3.2. To insure that $\frac{G^\epsilon}{G} = \mathcal{E}(\epsilon \tilde{Z})$ is a martingale we assume Novikov's condition:

Assumption 3.4. *For each x*

$$\tilde{E}\left[\exp\left(\frac{1}{2}\int_0^t |\gamma_d(X_u)|^2 du\right)\right] < \infty.$$

For fixed $1 \leq m < \infty$ and $t > 0$, consider the space L^m of adapted stochastic processes $Y = \{Y_u\}_{0 \leq u \leq t}$, with norm $||Y|| = \left(\tilde{E}\left[\int_0^t |Y_u(\omega)|^m du\right]\right)^{1/m}$. Notice that $\frac{G^\epsilon}{G}$ converges to 1 almost surely as $\epsilon \to 0$. Another form of convergence is established in the following lemma.

Lemma 3.5. *Suppose Assumption 3.4 is satisfied. Then for each x, $\lim_{\epsilon \to 0} \frac{G^\epsilon}{G} = 1$ in L^m for any $m \geq 1$.*

Proof. : Since $\frac{G^\epsilon}{G} \to 1$ a.s. and convergence a.s. plus convergence in norm implies the convergence in L^m, it suffices to show that for ϵ small, $\frac{G^\epsilon}{G} \in L^m$ and $||\frac{G^\epsilon}{G}|| \to 1$. Given $m > 1$, let $c_m = \frac{m}{2}(\sqrt{m} + \sqrt{m-1})^2$. If $\epsilon^2 < \frac{1}{2c_m}$ then for each $0 < u \leq t$,

$$\tilde{E}\left[\exp\left(c_m \int_0^u |\epsilon \gamma_d(X_\tau)|^2 d\tau\right)\right] < \tilde{E}\left[\exp\left(\frac{1}{2}\int_0^t |\gamma_d(X_\tau)|^2 d\tau\right)\right] < \infty. \tag{7}$$

Jensen's inequality and Theorem 1 of Grigelionis and Mackevicius (2003) guarantee that for $0 \leq u \leq t$,

$$1 \leq \tilde{E}\left[\left(\frac{G_u^\epsilon}{G_u}\right)^m\right] < \tilde{E}\left[\exp\left(c_m \int_0^t |\epsilon \gamma_d(X_\tau)|^2 d\tau\right)\right] < \infty.$$

Monotone convergence assures that

$$\lim_{\epsilon \downarrow 0} \tilde{E}\left[\exp\left(c_m \int_0^t |\epsilon \gamma_d(X_\tau)|^2 d\tau\right)\right] = 1,$$

and thus $\tilde{E}\left[\left(\frac{G^\epsilon_u}{G_u}\right)^m\right] \to 1$, uniformly in $u \le t$. Hence $||\frac{G^\epsilon}{G}|| = \left(\tilde{E}\left[\int_0^t \left(\frac{G^\epsilon_u}{G_u}\right)^m dv\right]\right)^{1/m} \to 1$. ☐

To control the term in $\gamma_d(X_t)$ we need to assume:

Assumption 3.6. *For each x, there exists a constant Γ (which may depend on x) such that*

$$\tilde{E}\left[|\gamma_d(X_u)|^4\right] \le \Gamma$$

for $0 < u \le t$.

Lemma 3.7. *Suppose Assumptions 3.4 and 3.6 are satisfied. Then Assumption 3.2 holds.*

Proof. Use (6) to represent

$$\frac{\frac{G^\epsilon_t}{G_t} - 1}{\epsilon} = \int_0^t \left(\frac{G^\epsilon_u}{G_u}\right) \gamma_d(X_u) d\tilde{W}_u.$$

Thus we must show that $\int_0^t \left(\frac{G^\epsilon_u}{G_u} - 1\right) \gamma_d(X_u) d\tilde{W}_u$ converges in mean-square to zero. Notice that the stochastic integral $\int_0^t \left(\frac{G^\epsilon_u}{G_u} - 1\right) \gamma_d(X_u) d\tilde{W}_u$ has second moment

$$\tilde{E}\left[\int_0^t \left(\frac{G^\epsilon_u}{G_u} - 1\right)^2 |\gamma_d(X_u)|^2 du\right]. \tag{8}$$

As $\epsilon \to 0$, expression (8) converges to zero from the assumptions, Lemma 3.5 for $m = 4$ and Holder's inequality. ☐

There are many alternative more primitive assumptions that are sufficient for Assumption 3.6. Here we present two conditions that together imply Assumption 3.6. The first is a slight strengthening of Novikov's condition for G.

Assumption 3.8. *For each x, there exists a $c > \frac{1}{2}$ such that*

$$E\left[\exp\left(c\int_0^t |\gamma_g(X_u)|^2 du\right)\right] < \infty.$$

It is a consequence of Assumption 3.8 that there exists a $p > 1$ such that for $u \le t$,[3]

$$E\left[(G_u)^p | X_0 = x\right] \le E\left[\exp\left(c\int_0^t |\gamma_g(X_u)|^2 du\right)\right].$$

[3] See Theorem 1 in Grigelionis and Mackevicius (2003)

The next assumption guarantees that for each initial value x there exists a $\Gamma'(x)$ such that for q satisfying $\frac{1}{q} + \frac{1}{p} = 1$, $u \leq t$,

$$E\left[|\gamma_d(X_u)|^q\right] \leq \Gamma'(x).$$

Holder's inequality then assures that Assumption 3.6 obtains.

Assumption 3.9. *(a) The functions $|\gamma_d(x)|$ are bounded by a polynomial in $|x|$ and*

(b) the coefficients β and α of equation (1) that defines the evolution of X satisfy a sublinear growth condition,

$$|\beta(x)|^2 + |\alpha(x)|^2 \leq K(1 + |x|^2),$$

for some constant K.

When Assumption 3.9 (b) holds, for each for each $m \geq 1$ there exists a $C = C(d, K, T, m)$ such that $E[\max_{u \leq t} |X_t|^{2m} | X_0 = x] \leq C(1 + |x|^{2m})e^{Ct}$, if $t \leq T$. (A more general result than this is problem 3.15 in Karatzas and Shreve (1991) page 306). Part (a) of Assumption 3.9 then implies that for each x, there exists a $\Gamma'(x)$ such that for $0 \leq u \leq t$, $E|\gamma_d(X_u)|^q \leq \Gamma'(x)$.

4 Short-term limits

We now use the formula:

$$\rho_t = \frac{E\left[S_t G_t \left(\int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du - \int_0^t \gamma_d(X_u) dW_u\right)\right]}{tE\left(S_t G_t\right)}$$

to study valuation over short time intervals. Formally we calculate short-horizon limits by computing the drift of an Ito process .

Recall that the SG has continuous sample paths that converge to one as t declines to zero. We add to this the assumption

Assumption 4.1. *For every x, $\lim_{t \downarrow 0} E\left(S_t G_t\right) = 1$.*

This assumption follows from the Dominated Convergence Theorem provided that we can dominate SG uniformly for small t.

With this restriction, we are lead to compute

$$\rho(x) = \lim_{t \downarrow 0} \frac{1}{t} E\left[S_t G_t \left(\int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du - \int_0^t \gamma_d(X_u) dW_u\right) | X_0 = x\right].$$

We calculate this limit as the drift of the Ito process

$$S_t G_t \left(\int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du - \int_0^t \gamma_d(X_u) dW_u \right)$$

at $t = 0$. Since

$$\rho_0 = \gamma_d(x) \cdot \gamma_g(x) - \gamma_d(x) \cdot [\gamma_g(x) + \gamma_s(x)],$$

the following proposition holds.

Proposition 4.2. *Suppose Assumption 4.1 is satisfied. Then*

$$\rho_0 = -\gamma_s(x) \cdot \gamma_d(x). \tag{9}$$

As we vary the risk exposure vector γ_d, we trace out the local risk prices. This results in the interpretation of $-\gamma_s$ a vector of local risk prices.[4] As is well known, the local risk price vector is the risk exposure of the stochastic discount factor S. The risk exposure of the stochastic growth process plays no role in this calculation.

5 An integral representation

In this section justify the integral representation:

$$\rho_t = -\frac{\hat{E}\left[\hat{e}(X_t) \int_0^t \gamma_d(X_u) \cdot [\kappa(X_u) + \phi(X_u, t-u)] du\right]}{t\hat{E}\left[\hat{e}(X_t)\right]} \tag{10}$$

under a particular change of measure. We will describe formally the construction of the distorted probability distribution and the the functions κ and ϕ.

Following Hansen and Scheinkman (2009), use the factorization:

$$S_t G_t = \exp(\delta t) \hat{M}_t \frac{e(x)}{e(X_t)}$$

where $\hat{M}$ is a multiplicative martingale and (δ, e), solve a principal eingenvalue problem: find $e >> 0$ such that

$$E\left[S_t G_t e(X_t)\right] = \exp(\delta t) e(x). \tag{11}$$

Since e solves a principal eigenvalue problem, it is smooth and Ito's Lemma can be used to show that $\hat{M}$ is a multiplicative process of the form defined in equation (2) above. Write

[4] In general this limit is computed as in Ito's Lemma by using stopping times. When the associated local martingale is in fact a square integrable martingale, stopping times can be dispensed with.

the volatility exposure (the coefficient on dW_t) of $\log(\hat{M})$ as $\kappa + \gamma_g$. Change measure using the martingale $\hat{M}$ and express the finite t derivative of interest as:

$$\begin{aligned}\rho_t &= \frac{E\left[S_t G_t \left(\int_0^t \gamma_d(X_u)\cdot\gamma_g(X_u)du - \int_0^t \gamma_d(X_u)dW_u\right)\right]}{tE\left(S_t G_t\right)} \\ &= -\frac{\hat{E}\left[\frac{1}{e(X_t)}\left(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du + \int_0^t \gamma_d(X_u)d\hat{W}_u\right)\right]}{t\hat{E}\left(\frac{1}{e(X_t)}\right)}.\end{aligned}$$

Under the $\hat{\cdot}$ change of measure,

$$dW_u = [\kappa(X_u) + \gamma_g(X_u)]du + d\hat{W}_u,$$

where $\hat{W}$ is a n dimensional Brownian motion (with respect to the filtration generated by the past values of W) , and X_u solves

$$dX_u = \hat{\beta}(X_u)du + \alpha(X_u)d\hat{W}_u,$$

with $X_0 = x$, and

$$\hat{\beta} = \beta + \alpha(\kappa + \gamma_g).$$

Write $\hat{e} = \frac{1}{e}$. Let $\hat{\mathcal{F}}_u,\ u \geq 0$ denote the (completed) Brownian filtration associated with $\hat{W}$ and note that $e(X_t)$ is measurable with respect to $\hat{\mathcal{F}}_t$. For each t, $u \leq t$, let $D_u\hat{e}(X_t)$ denote the Malliavin derivative of the random variable $\hat{e}(X_t)$. Sufficient conditions for the existence of the Malliavin derivatives of $\hat{e}(X_t)$ are as follows. If the functions $\hat{\beta}$ and α are smooth and with bounded derivatives then the random variable X_t is in the domain of the Malliavin derivative. In fact let Y be the first variation process associated to X, that is $Y_0 = I_n$ and

$$dY_u = \partial\hat{\beta}(X_u)Y_u du + \sum_i \partial\alpha_i(X_u)Y_u d\hat{W}_u^i. \tag{12}$$

Here, ∂F denotes the Jacobian matrix of an $\mathbb{R}^n$ valued function F and α_i is the i-th column of the matrix α. Then, for $u \leq t$, the $n \times n$ matrix

$$D_u X_t = Y_t Y_u^{-1}\alpha(X_u). \tag{13}$$

In addition, if $\hat{e}$ has bounded first derivatives, then

$$D_u\hat{e}(X_t) = \nabla\hat{e}(X_t)\cdot D_u X_t. \tag{14}$$

Then the Haussmann-Clark-Ocone formula guarantees[5] that:

$$\hat{e}(X_t) = \int_0^t \hat{E}\left[D_u \hat{e}(X_t)|\hat{\mathcal{F}}_u\right] \cdot d\hat{W}_u + \hat{E}\left[\hat{e}(X_t)\right].$$

Furthermore, it follows directly from equations (12) - (14) that $\hat{E}[D_u\hat{e}(X_t)|\hat{\mathcal{F}}_u] = \hat{E}[D_u\hat{e}(X_t)|X_u]$.

Define

$$\phi(y, t-u) = \frac{\hat{E}\left[D_u\hat{e}(X_t)|X_u = y\right]}{\hat{E}\left[\hat{e}(X_t)|X_u = y\right]} \tag{15}$$

Then, if we assume the necessary integrability conditions to apply Fubini's,

$$\begin{aligned}
\hat{E}\left[\hat{e}(X_t)\int_0^t \gamma_d(X_u)d\hat{W}_u\right] =& \hat{E}\left(\int_0^t \hat{E}\left[\hat{e}(X_t)|X_u\right]\phi(X_u, t-u)\cdot\gamma_d(X_u)du\right] \\
=& \hat{E}\left(\int_0^t \hat{E}\left[\hat{e}(X_t)\phi(X_u, t-u)\cdot\gamma_d(X_u)|X_u\right]du\right] \\
=& \int_0^t \hat{E}\left[\hat{e}(X_t)\phi(X_u, t-u)\cdot\gamma_d(X_u)\right]du \\
=& \hat{E}\left[\hat{e}(X_t)\int_0^t \phi(X_u, t-u)\cdot\gamma_d(X_u)du\right].
\end{aligned}$$

Therefore, formula (10) is justified.

Notice that we have an integral decomposition of ρ_t with key ingredient:

$$-[\kappa(X_u) + \phi(X_u, t-u)].$$

Now hold fixed $u = 0$ and depict as a function of t:

$$-\kappa(x) + \phi(x, t).$$

At $t = 0$ you obtain the local risk price. More generally you trace out the impact of the price elasticity of the shock exposure in the next instant to values over an interval of time t.

6 Long-term limits

In this section we establish the following limiting behavior:

$$\lim_{t\to\infty} \rho_t(x) \to -\int \gamma_d \cdot \kappa d\hat{Q}$$

[5]For a statement of results concerning the Malliavin derivative of functions of a Markov diffusion and the Hausmann-Clark-Occone formula see, for instance, Fournié et al. (1999), pages 395 and 396.

for some probability measure $\hat{Q}$. In what follows we show how to construct this measure and justify the limiting behavior.

As a precursor to studying the long horizon behavior of ρ_t it is convenient to alter the specification of the Markov process by choosing a probability distribution for the initial state X_0 other than the degenerate construction $X_0 = x$. Since our previous analysis applies for each x we have some flexibility as to how we do this.

For simplicity, we choose $\Omega = \mathbb{R}^n \times C_0([0,\infty), \mathbb{R}^n)$ with the first coordinate corresponding to the initial condition X_0 and the second coordinate to a realization of the Brownian motion. The random vector X_0 is independent of the Brownian motion. Let $\mathcal{F}_t^*$ be the augmented filtration generated by X_0 and W. Since α is non-singular, this coincides with the augmented filtration generated by X.

We will again use the decomposition of Hansen and Scheinkman (2009),

$$S_t G_t = \exp(\delta t) \hat{M}_t \frac{e(X_0)}{e(X_t)} \tag{16}$$

where $\hat{M}$ is a multiplicative martingale and (δ, e), solve a principal eingenvalue problem: find $e >> 0$ such that

$$E\left[S_t G_t e(X_t) | \, X_0 = x\right] = \exp(\delta t) e(x). \tag{17}$$

which is the same as (11) except that we noted explicitly the conditioning. Given a decomposition in this form we use $\hat{M}$ to change the transition probabilities from date zero to all other dates. Since $\hat{M}$ is a multiplicative martingale with unitary expectation (conditioned on X_0), this change of measure preserves the Markov structure and it depends on $X_0 = x$. We still have freedom to assign an initial probability to X_0, and we will do so in a convenient manner so as to make the process X stationary under the change of measure.

Associated with the multiplicative functional is a generator $\hat{\mathcal{A}}$ that is an extension of the second-order differential operator:

$$\hat{\beta}(x) \cdot \frac{\partial f(x)}{\partial x} + \frac{1}{2} \text{trace}\left[\alpha(x)\alpha(x)' \frac{\partial^2 f(x)}{\partial x \partial x'}\right] \tag{18}$$

for functions f that are twice continuously differentiable and have compact support on the interior of the state space. As remarked in the previous section, our use of $\hat{\beta}$ instead of β reflects the addition of a drift term in our representation of W under the change of measure associated with $\hat{M}$.

Assumption 6.1. *There exists a probability measure $\hat{Q}$ on $\mathbb{R}^n$ such that*

$$\int \hat{\mathcal{A}} f(x) d\hat{Q}(x) = 0$$

for all f that are twice continuously differentiable and have compact support.

By using $\hat{Q}$ to initialize the state, the process X is stationary under this change of measure. There are many well known results for the existence of stationary distributions. See for example Meyn and Tweedie (1993). While there may be multiple solutions to the principal eigenvalue problem, Hansen and Scheinkman (2009) show that there is at most one solution for which the resulting probability measure makes X stationary and Harris recurrent.

Associated with the Markov process X there is a semigroup of conditional expectation operators, which may be extended to the space $\hat{L}^p$ of Borel measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ with $\int |f(x)|^p d\hat{Q}(x) < \infty$.[6] Let $Z = \left\{ f \in \hat{L}^2 : \int f(x) d\hat{Q}(x) = 0 \right\}$.

Assumption 6.2. *The semigroup of conditional expectation operators associated with X under the change of measure implied by $\hat{M}$ and $\hat{Q}$ is a strong contraction semigroup on Z.*

As discussed by Rosenblatt (1971) and Hansen and Scheinkman (1995), Assumption 6.2 is ρ-mixing with mixing coefficients that necessarily decay exponentially to zero.

Proposition 6.3. *Suppose that $\gamma_d \cdot \kappa$, γ_d and $\frac{1}{e}$ are in $\hat{L}^2$. Then*

$$\lim_{t \to \infty} \rho_t(x) \to - \int \gamma_d \cdot \kappa d\hat{Q}$$

in probability under the $\hat{Q}$ measure.

Thus long-term risk prices are obtained by changing the state-dependent risk exposure γ_d in the representation given by Proposition 6.3. As in local counterpart given in Proposition 4.2, we think of γ_d as parameterizing the exposure to (growth-rate) risk, which we allow to be state dependent. The vector $(\kappa + \gamma_g)$ is the risk exposure of the martingale component of SG and γ_g is the risk exposure of the multiplicative martingale growth functional. In effect, the state dependent vector κ in conjunction with the probability distribution $\hat{Q}$ determine the long-term counterpart to the local risk price vector $-\gamma_s$ given in Proposition 4.2.

[6] See for example Hansen and Scheinkman (1995) for a discussion of the construction of the semigroup of conditional expectation operators in $\hat{L}^2$ and the construction of its associated generator.

Proof. Recall that if $\hat{e} = \frac{1}{e}$ then,

$$\rho_t(x) = \frac{\frac{1}{t}\hat{E}\left(\hat{e}(X_t)\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du + \int_0^t \gamma_d(X_u)d\hat{W}_u\right]|X_0 = x\right)}{\hat{E}\left(\hat{e}(X_t)|X_0 = x\right)}.$$

First notice that

$$\begin{aligned}\frac{1}{t}\hat{E}\left(\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du + \int_0^t \gamma_d(X_u)d\hat{W}_u\right]|X_0 = x\right) &= \frac{1}{t}\hat{E}\left(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du|X_0 = x\right) \\ &\to \int \gamma_d\cdot\kappa d\hat{Q}\end{aligned} \tag{19}$$

in $\hat{L}^2$ under Assumption 6.2. $\hat{L}^2$ convergence implies convergence in $\hat{Q}$ probability.

Next we show that

$$\frac{1}{t}\hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t)|X_0 = x\right]\right)\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du + \int_0^t \gamma_d(X_u)d\hat{W}_u\right]|X_0 = x\right] \to 0. \tag{20}$$

in $\hat{L}^1$.

We consider this in two parts.

i)

$$\begin{aligned}&\hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)|X_0 = x\right)\right]\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du\right]|X_0 = x\right) \\ &= \hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)|X_0 = x\right)\right]\int_0^t \left[\gamma_d(X_u)\cdot\kappa(X_u) - \hat{E}\left(\gamma_d(X_u)\cdot\kappa(X_u)\right)\right]du|X_0 = x\right)\end{aligned}$$

Notice that

$$\hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)|X_0 = x\right)\right]^2|X_0 = x\right) \le \hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)\right)\right]^2|X_0 = x\right),$$

and

$$\hat{E}\left[\hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)\right)\right]^2|X_0 = x\right)\right] \le \hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)\right)\right]^2\right) < \infty$$

since $\hat{e}(X_t)$ has finite second moment under the $\hat{Q}$ stationary distribution. The bound can be chosen to be independent of t. Moreover,

$$\hat{E}\left(\int_0^t \left[\gamma_d(X_u)\cdot\kappa(X_u) - \hat{E}\gamma_d(X_u)\cdot\kappa(X_u)\right]|X_0 = x\right]$$

converges in $\hat{L}^2$ to a function of x with a finite ($\hat{Q}$) second moment under Assumption 6.2. It follows from the Cauchy-Schwarz Inequality that

$$\frac{1}{t}\hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)|X_0 = x\right)\right]\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du\right]|X_0 = x\right) \to 0$$

in $\hat{L}^1$.

ii) Consider next

$$\begin{aligned}\frac{1}{t}\hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)|X_0 = x\right)\right]\left[\int_0^t \gamma_d(X_u)d\hat{W}_u\right]|X_0 = x\right)\\ \leq \frac{1}{\sqrt{t}}\sqrt{\hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\hat{e}(X_t)\right]^2|X_0 = x\right)}\\ \times\sqrt{\left(\hat{E}\left[\frac{1}{t}\int_0^t |\gamma_d(X_u)|^2 du|X_0 = x\right]\right)}\end{aligned}$$

where the inequality is application of the conditional Cauchy-Schwarz Inequality. Provided that $\gamma_d(X_u)$ has a finite second moment under the $\hat{}$ distribution, the right-hand side converges to zero in $\hat{L}^1$ since the unconditional second moments of

$$\sqrt{\hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\hat{e}(X_t)\right]^2|X_0 = x\right)}$$

and

$$\sqrt{\left(\hat{E}\left[\frac{1}{t}\int_0^t |\gamma_d(X_u)|^2 du|X_0 = x\right]\right)}$$

are finite and independent of t.

Given these two intermediate results, (20) follows. Finally,

$$\hat{E}\left[\hat{e}(X_t)|X_0 = x\right] \to \int \hat{e}d\hat{Q} > 0.$$

in $\hat{L}^2$. Thus

$$\frac{1}{t}\frac{\hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t)|X_0 = x\right]\right)\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du + \int_0^t \gamma_d(X_u)d\hat{W}_u\right]|X_0 = x\right]}{\hat{E}\left[\hat{e}(X_t)|X_0 = x\right]} \to 0$$

in $\hat{Q}$ probability. The conclusion follows from this result combined with (19).

□

References

Fournié, Eric, Jean-Michel Lasry, Jérôme Lebuchoux, and Pierre-Louis Lions. 1999. Applications of Malliavin Calculus to Monte-Carlo Methods in Finance. *Finance and Stochastics* 3 (4):391–412.

———. 2001. Applications of Malliavin Calculus to Monte-Carlo Methods in Finance II. *Finance and Stochastics* 5 (2):201–236.

Grigelionis, Bronius and Vigirdas Mackevicius. 2003. The finiteness of moments of a stochastic exponential. *Statistics & Probability Letters* 64 (3):243 – 248.

Hansen, Lars Peter. 2008. Modeling the Long Run: Valuation in Dynamic Stochastic Economies. Fisher-Schultz Lecture at the European Meetings of the Econometric Society.

Hansen, Lars Peter and Jose Alexandre Scheinkman. 1995. Back to the Future: Generating Moment Implications for Continuous-Time Markov Processes. *Econometrica* 63 (4):767–804.

———. 2009. Long-term Risk: A Operator Approach. *Econometrica* 77:177–234.

Ito, K. and T. Watanabe. 1965. Transformation of Markov processes by multiplicative functionals. *Annales de L'Institut Fourier* 15 (1):13–30.

Karatzas, Ioannis and Stephen E. Shreve. 1991. *Bownian Motion and Stochastic Calculus*. New York: Springer-Verlag, second ed.

Meyn, S. P. and R. L. Tweedie. 1993. *Markov Chains and Stochastic Stability*. London: Springer-Verlag.

Rosenblatt, Murray. 1971. *Markov Processes: Structure and Asymptotic Behavior*. New York: Springer-Verlag.