Modeling the Long Run: Valuation in Dynamic Stochastic Economies

Lars Peter Hansen

2006 European Meetings of the Econometric Society

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Life Contains Endless Surprises

What if you’re facing a problem you haven’t even considered?

Zurich: We help you plan for the unexpected.
1 Introduction

I am very honored to give the Fisher-Schultz Lecture at these Meetings of the European Econometric Society. Portions of my talk today have a clear origin in Irving Fishers’ books *The Rate of Interest* and the *Theory of Interest*. Henry Schultz was an important empirical economist at my home institution, the University of Chicago.

In this talk I propose to augment the toolkit for economic dynamics and econometrics with methods that will reveal economic import of long run stochastic structure. These tools enable informative decompositions of a model’s dynamic implications for valuation. They are the outgrowth of my observation and participation in an empirical literature that aims to understand the low frequency links between financial market indicators and macroeconomic aggregates. Portions of this work are joint with John Heaton, Nan Li and Jose Scheinkman, and very much influenced by related work I have done with Xiaohong Chen and Tom Sargent.

A few months back I came across the following advertisement by Zurich North America in the New York Times. While I have little
ambition to sell you insurance or even to take up golf, to me, this picture says even a simple golf game can hold surprises! Landing a golf ball on the nose of an alligator may be a rare event but over time opportunities to experience such events certainly increase. When even rare events effect economic growth their consequences do not simply average out. It is best, however, that I resist any temptation to build an alligator theory of the stochastic components to growth.

A more direct source of motivation is the burgeoning empirical literature in macroeconomics and finance that features long run contributions to risk, including for instance the work by Alvarez and Jermann, Bansal and Yaron, Campbell and Vuolteenaho, Lettau and Wachter and Parker, in addition to some of my own work with Heaton and Li. My talk will not attempt to survey this literature, but instead I will put on my methodological hat and suggest new ways for understanding such models and the corresponding empirical evidence. I suspect these methods should have broader array of application.

Models of equilibrium valuation are the featured application in my talk because there is a natural link between macroeconomics and
the analysis of long-term risk. Why is macroeconomics crucial to asset pricing? The evolution of macroeconomic events is an essential component of risk, because these components are inherently undiversifiable. Being common to all investors, macroeconomic shocks cannot be smoothed over a cross section of agents. Therefore, security markets must price the macroeconomic risk components. Why is asset pricing informative to macroeconomics? Asset valuation are by their nature forward looking and encode information about investor beliefs, including their speculations about the long run stochastic growth.

Current dynamic models that relate macroeconomics and asset pricing are constructed from an amalgam of assumptions about preferences (such as risk aversion or habit persistence, etc) and technology (productivity of capital or adjustment costs to investment) and exposure to unforeseen shocks. Some of these components have more transitory effects while others have a lasting impact. In part my aim is to illuminate the roles of these model ingredients by presenting a structure that features long run implications.
These methods are designed to address three questions:

- What are the long run value implications of economic models?
- To which components of the *uncertainty* are long-run valuations most sensitive?
- What kind of hypothetical changes in preferences and technology have the most potent impact on the long run? What changes are transient?

Although aspects of these questions have been studied using log-linear models and log-linear approximations around a growth trajectory, the methods I describe offer a novel vantage point. These methods are designed for the study of valuation in the presence of stochastic inputs that have long run consequences. While the methods can exploit any linearity, by design they can accommodate nonlinearity as well. In this talk I will develop these tools, as well as describe their usefulness at addressing these three economic questions. I will draw upon some diverse results from stochastic process theory and time series analysis, although I will use these results in novel ways.
There are a variety of reasons to be interested in the first question. When we build dynamic economic models, we typically specify transitional dynamics over a unit of time for discrete time models or an instant of time for continuous time models. Long run implications are encoded in such specifications, but they can be hard to decipher, particularly in nonlinear stochastic models. I explore methods that describe long run limiting behavior, a concept which I will define formally. I see two reasons why this is important. First some economic inputs are more credible when they target low frequency behavior. Second these inputs may be essential for meaningful long-run extrapolation of value. Nonparametric statistical alternatives suffer because of limited empirical evidence on the long run behavior of macroeconomic aggregates and financial cash flows.

Recent empirical research in macro-finance has highlighted economic modeling successes at low frequencies. After all, models are approximations, and applied economics necessarily employs models that are misspecified along some dimensions. Implications at higher frequencies are either skimmed over, or additional model compo-
nents, often *ad hoc*, are added to hopes of enlarging the empirical implications. In this context, then, I hope these methods for extracting long term implications from a dynamic stochastic model will be welcome additional research tools. Specifically, I will show how to deconstruct a dynamic stochastic equilibrium implied by a model, revealing what features dominate valuation over long time horizons. Conversely, I will formalize the notion of transient contributions to valuation. These tools will help to formalize long run approximation and to understand better what proposed model fixups do to long run implications.

This leads me to the second question. Many researchers study valuation under uncertainty by risk prices, and through them, the equilibrium risk-return tradeoff. In equilibrium, expected returns change in response to shifts in the exposure to various components of macroeconomic risk. The tradeoff is depicted over a single period in a discrete time model or over an instant of time in a continuous time model. I derive the long run counterpart to this familiar exercise by performing a sensitivity analysis that recovers prices of exposure
to the component parts of long run (growth rate) risk. These same methods facilitate long-run welfare comparisons in explicitly dynamic and stochastic environments.

Finally, consider the third question. Many components of a dynamic stochastic equilibrium model can contribute to value in the long run. Changing some of these components will have a more potent impact than others. To determine this, we could perform value calculations for an entire family of models indexed by the model ingredients. When this is not practical, an alternative is to explore local changes in the economic environment. We may assess, for example, how modification in the intertemporal preferences of investors alter long term risk prices and interest rates. The resulting derivatives can quantify these and other impacts and can inform statistical investigations.
Overview

**Aim:** Develop methods for long run analysis of value in dynamic stochastic equilibrium models with macroeconomic risk.

**Questions**

- What are the long run value implications of economic models?
- To which components of uncertainty are valuations (market or shadow prices) most sensitive?
- What hypothetical changes in preferences and technology have the most potent impact on the long run? What changes are transient?

An alternative to log-linear approximation around steady states.
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Game Plan

- Underlying Markov structure
- Stochastic growth: built by accumulating the impact of the Markov state and shock history
- Valuation with growth: families of operators indexed by horizon
- Representation of operators with processes
- Long run approximation
- Sensitivity and long run pricing: perturbation analysis
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Mathematical setup

- \( \{X_t : t \geq 0\} \) be a continuous time Markov process on a state space \( \mathcal{D} \). We will sometimes also assume that this process is stationary and ergodic.

- \( X = X^c + X^d \)

- \( X^c \) is the solution to \( dX_t^c = \mu(X_{t-})dt + \sigma(X_{t-})dW_t \) where \( W \) is an \( \{\mathcal{F}_t\} \) Brownian motion and \( X_{t-} = \lim_{\tau \downarrow 0} X_{t-\tau} \).

- \( X^d \) with a finite number of jumps in any finite interval and compensator \( \eta(dy|x)dt \). Jump intensity is \( \int \eta(dy|x) \) and \( \eta(dy|x) \) rescaled is the jump distribution.

Simple distinction between small shocks and big shocks.
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- \{X_t : t \geq 0\} be a continuous time Markov process on a state space \(D\). We will sometimes also assume that this process is stationary and ergodic.

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Simple distinction between small shocks and big shocks.
Construct real-valued process \( \{ A_t : t \geq 0 \} \) as a function of \( X_u \) for \( 0 \leq u \leq t \).

An additive functional is parameterized by \((\beta, \gamma, \kappa)\) where:

i) \( \beta : D \rightarrow \mathbb{R} \) and \( \int_0^t \beta(X_u)du < \infty \) for every positive \( t \);

ii) \( \gamma : D \rightarrow \mathbb{R}^m \) and \( \int_0^t |\gamma(X_u)|^2 du < \infty \) for every positive \( t \);

iii) \( \kappa : D \times D \rightarrow \mathbb{R}, \kappa(x, x) = 0. \)

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A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u) \cdot dW_u + \sum_{0 \leq u \leq t} \kappa(X_u, X_u) 
\]

Process \( A \) is nonstationary and can grow linearly.

Sums of additive functionals are additive. Add the parameters.
Additive functional - definition

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Multiplicative functional - definition

- Exponential of additive functional.

\[ M_t = \exp(A_t) \]

- Parameterized by the additive functional \((\beta, \gamma, \kappa)\)
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Representation using multiplicative functionals

Form a family of operators using $M$:

$$\mathbb{M}_t f(x) = E [M_t f(X_t) | X_0 = x]$$

Harrison-Kreps, Hansen-Richard

Examples

- Stochastic discount factor functional $S$; use $M = S$ to price claim on $f(x)$ on the Markov state.

- Stochastic growth functional $G$; use $M = SG$ assign values to a cash flow or hypothetical consumption process $D_t = D_0 G_t f(X_t)$ where $D_0$ is an initial condition for the cash flow;

- Valuation functional $V$ such that $VS$ is a martingale; use $M = V$ to measure expected long run growth of investment.
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Form a family of operators using $M$:

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Harrison-Kreps, Hansen-Richard

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### Notation Summary

<table>
<thead>
<tr>
<th>object</th>
<th>multiplicative functional</th>
<th>operator family</th>
</tr>
</thead>
<tbody>
<tr>
<td>stochastic discount factor</td>
<td>$S$</td>
<td>${S_t}$</td>
</tr>
<tr>
<td>stochastic growth</td>
<td>$G$</td>
<td>${G_t}$</td>
</tr>
<tr>
<td>valuation with stochastic growth</td>
<td>$Q = GS$</td>
<td>${Q_t}$</td>
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<tr>
<td>cumulated return</td>
<td>$V$</td>
<td>${V_t}$</td>
</tr>
<tr>
<td>martingale restriction</td>
<td>$VS$</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Alternative Operator Families and Multiplicative Functionals

Move back and forth between operator families and multiplicative functionals.
Why is $M$ multiplicative?

\[ M_t f(x) = E[M_t f(X_t)|X_0 = x] \]

- Operator families that interest us obey the Law of Iterated Values:

\[ M_t M_\tau f = M_{t+\tau} f \]

for $t \geq 0$ and $\tau \geq 0$.

- Preserve the Markov structure.

A multiplicative functional does the trick!
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Model decomposition

Plan: Decompose multiplicative functionals

a) decompose consumption processes or cash flows into permanent and transient components as they contribute to value;

b) deconstruct model’s value implications - short run versus long run.
Additive decomposition

\[ A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u) \cdot dW_u + \sum_{0 \leq u \leq t} \kappa(X_u, X_{u-}). \]

\[ A_t = \rho t + \hat{A}_t + g(X_t) - g(X_0). \]

- \( \rho \) gives a linear growth rate
- \( \hat{A} \) is an additive martingale
- \( g \) is a transient component

Familiar from central limit theory for stochastic processes and from macro linear time series literature (Beveridge-Nelson decomposition).
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\[ A_t = \int_0^t \beta(X_u) \, du + \int_0^t \gamma(X_u) \cdot dW_u + \sum_{0 \leq u \leq t} \kappa(X_u, X_{u^-}). \]

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- Decomposition is unique and additive.
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Decomposition through exponentiation

Exploit log linearity by using $M = \exp(A)$ where:

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Limitations:

- Exponential of a martingale is not a martingale
- Log normal corrections - when volatility is state dependent will not work.
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Must use other methods to build multiplicative decomposition

Additive decompositions can still be put to good use.
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\[
\rho(M) = \lim_{t \to \infty} \frac{1}{t} \log M_t f(x) = \lim_{t \to \infty} \frac{1}{t} \log E [M_t f(X_t) | X_0 = x]
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- \(\rho(M)\) growth or decay rate.
- For what class of functions \(f\) do we obtain the same limit? \(f(X_t)\) gives a transient contribution to growth or value.
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\rho(M^1 M^2) \neq \rho(M^1) + \rho(M^2)
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**Table:** Alternative Operator Families and Multiplicative Functionals
Long run cash flow risk

\[ \rho(M) = \lim_{t \to \infty} \frac{1}{t} \log E [M_t f(X_t) | X_0 = x] \]

- **Cash flow return over horizon** $t$:
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  \frac{G_t f(X_t)}{Q_t f(X_0)}.\]

- **long run expected rate of return (risk adjusted)**:
  \[ \rho(G) - \rho(Q) = \rho(G) - \rho(GS). \]

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Long run cash flow risk-return tradeoff.

\[ \rho(M) = \lim_{t \to \infty} \frac{1}{t} \log E[M_t f(X_t) | X_0 = x] \]

▶ Excess expected long run rate of return (risk adjusted):

\[ \rho(G) + \rho(S) - \rho(GS) \]

▶ \( \rho(G) \) is a measure of cash flow growth and \(-\rho(GS)\) is a measure or cash flow duration as it contributes to value.

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▶ Change G but keep S fixed. Example: growth versus value.
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Expected rates of return by horizon

Expected Returns for $\theta = 20$

Horizon in Quarters

- Expected dividend growth increases with horizon.
- The expected returns for $\theta = 20$ are higher compared to $\theta = 5$.

Graphical representation shows the trend of expected returns over different horizons.
Multiplicative decomposition

\[ M_t = \exp(\rho t) \hat{M}_t \left[ \frac{\hat{e}(X_t)}{\hat{e}(X_0)} \right] \]

- \( \rho \) is a deterministic growth rate;
- \( \hat{M}_t \) is a multiplicative martingale;
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Observations

- Reminiscent of a permanent-transitory decomposition from time series.
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- raise a matrix to a power - raise eigenvalues to same power and preserve eigenvectors; one eigenvalue dominates in the long run as we increase the powers.

- exponentiate a matrix - exponentiate eigenvalues and preserve eigenvectors; one eigenvalue dominates in the long run as we increase the scale of the matrix.

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\[ \mathbb{M}_t e(x) = E[M_t e(X_t)|X_0 = x] = \exp(\rho t)e(x) \]

where \( e \) is strictly positive. Eigenvalue problem.

- Construct martingale

\[ \hat{M}_t = \exp(-\rho t)M_t \left[ \frac{e(X_t)}{e(X_0)} \right] . \]

- For \( \hat{e} = 1/e \)

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### Long Run Approximation

- Use the multiplicative martingale \( \hat{M} \) to produce a new probability measure:

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\hat{E}[f(X_t)] = E[\hat{M}_t f(X_t)].
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The process \( X \) remains Markovian under this change in measure.

- Suppose that in addition it is stationary and:

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Important Tool

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- Stationary distribution under the \( \hat{\cdot} \) determines range of approximation. Determines which choice of \( f \) for which the value contribution of \( f(X_t) \) is transitory.
- Use tools developed for continuous time Markov processes - for instance Meyn-Tweedie - strong dependence tolerated.
- Only one eigenfunction/martingale leaves \( X \) stochastically stable. - Hansen-Scheinkman
Important Tool

\[
\lim_{t \to \infty} \exp(-\rho t) E[M_t f(X_t) \mid X_0 = x] = \hat{E} \left[ \frac{f(X_t)}{e(X_t)} \right] e(x)
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Illustration: Long term bond prices

Backus-Zin and Alvarez-Jermann - Term structure encodes information about macroeconomic risk.

- $S$ the stochastic discount factor.
- The price of a claim $f(X_s)$ to the Markov state

$E [S_t f(X_t)|X_0 = x].$

- Decomposition

$S_t = \exp(\rho t) \hat{M}_t \hat{e}(X_t) \hat{e}(X_0)$

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- Prices of long term discount bonds:

$\exp(-\rho t) E (S_t|X_0 = x) \approx \hat{E} [\hat{e}(X_t)] e(x).$
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Want to solve,

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for all \( t > 0 \).

Take the derivative with respect to time:

\[ \mathbb{B} f = \lim_{t \downarrow 0} \frac{\mathbb{M}_t f - f}{t} \]

Principal eigenvalue problem:

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\[ M_t^* = M_t \frac{\hat{f}(X_t)}{\hat{f}(X_0)} \]

for some \( \hat{f} \) where \( M \) is used to represent a benchmark model and \( M^* \) an alternative model.

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Transient Components to Stochastic Discount Factors

Decomposition: \( S_t^* = S_t \frac{\hat{f}(X_t)}{\hat{f}(X_0)} \)

Moment restriction: \( \hat{E} \left[ f(X_t)\hat{e}(X_t)\hat{f}(X_t) \right] < \infty. \)

Examples

- models of habit persistence Constantinides, Heaton and others.
- Solvency constraint models of Luttmer, Lustig and others
- preference shock models and social externalities

i) Santos-Veronesi - rich array of \( f \)'s satisfy the moment restriction
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Sensitivity Analysis

- Parameterized family $M(\alpha)$ of multiplicative functionals or of the associated generators $B(\alpha)$. Recall

$$\rho[M(\alpha)] = \lim_{t \to \infty} \frac{1}{t} \log E[M_t(\alpha)f(X_t)|X_0 = x]$$

- Derivative: For any $t > 0$

$$\frac{d}{d\alpha} \rho[M(\alpha)]|_{\alpha=0} = \frac{1}{t} \hat{E} \left( \frac{\partial \log M_t(\alpha)}{\partial \alpha} \right)_{\alpha=0}$$

- $\hat{E}$ computed under $\alpha = 0$ model;
- Take limits as $t \to 0$.

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  - Change model ingredients or statistical specification.
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Recursive utility models with CES aggregators as in Kreps-Porteus, Epstein-Zin and Weil

- sensitivity to changes in intertemporal substitution or risk aversion;
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Long run cash flow risk

- Let $S$ be the stochastic discount factor and $G$ be a stochastic growth functional.
- Cash flow $D_t = D_0 G_t f(X_t)$
- Return to equity is a portfolio of holding period returns. Limiting return:

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as $t$ gets large.

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Alternative to log-linear decomposition of Campbell and Shiller.
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- **Approximating model solutions**: to accommodate stochastic growth use a change of measure to approximate transient components.

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**Summary**

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1 Conclusion

To conclude I want to be clear on two matters that I consider to be of particular importance.

First, while a concern about the role in economics in model specification is a prime motivator for this analysis, I do not mean to focus exclusively on the limiting characterizations. Specifically, my analysis of long run approximation in this talk is not meant to pull discussions of transient implications off the table. Instead I mean to add some clarity into our understanding of how valuation models work by understanding better which model levers move which parts of the complex machinery. Moreover, I find the outcome of this analysis to be informative even if it reveals that some models blur the distinction between permanent and transitory components.

Second, while my discussion of statistical approximation has been notably brief, I do not have to remind time series econometricians of the particular measurement challenges associated with the long run. Indeed there is a substantial literature on such issues including contributions presented at this conference. In part my aim is to suggest
an econometric framework for the use of such measurements. But some of the measurement challenges remain. My own view is that many of these same statistical challenges that we as econometricians struggle with should be passed along to the hypothetical investors that populate our economic models. Difficulties in selecting a statistical model to use in extrapolation and associated ambiguities in inferences may well be an important component to the behavior of asset prices.
Some fun reading for remainder of the summer

► L. P. Hansen, J. C. Heaton and N. Li, “Consumption Strikes Back?: Measuring Long Run Risk.”