Pricing Kernels and Stochastic Discount Factors*

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In this entry we characterize pricing kernels or stochastic discount factors that are used to represent valuation operators in dynamic stochastic economies. A kernel is commonly-used mathematical term used to represent an operator. The term stochastic discount factor extends concepts from economics and finance to include adjustments for risk. As we will see, there is a tight connection between the two terms. The terms pricing kernel and stochastic discount factor are often used interchangeably.

After deriving convenient representations for prices, we provide several examples of stochastic discount factors and discuss econometric methods for estimation and testing of asset pricing models that restrict the stochastic discount factors.

1 Representing Prices

We follow Ross (1976) and Harrison and Kreps (1979) by exploring the implications of no arbitrage in frictionless markets to obtain convenient representations of pricing operators. We build these operators as mappings that assign prices that trade in competitive markets to payoffs on portfolios of assets. The payoffs specify how much a numeraire good is provided in alternative states of the world. To write these operators we invoke the *Law of One Price*, which stipulates that any two assets with the same payoff must necessarily have the same price. This law is typically implied by but is weaker than the *Principle of No Arbitrage*. Formally, the Principle of No Arbitrage stipulates that nonnegative payoffs that are positive with positive (conditional) probability command a strictly positive price. To see why the Principle implies the Law of One Price, suppose that there is such a nonnegative portfolio payoff and call this portfolio a. If two portfolios, say b and c have the same payoff but the price of portfolio b is less than that of portfolio c, then an arbitrage opportunity exists. This can be seen by taking a long position on portfolio b, a short position on portfolio c and using the proceeds to purchase portfolio a. This newly constructed portfolio has zero price but a positive payoff violating the Principle of No-Arbitrage.

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Given that we can assign prices to payoffs, let $\pi_{t,t+1}$ be the date t valuation operator for payoffs (consumption claims) at date $t+1$. This operator assigns prices or values to portfolio payoffs p_{t+1} in a space P_{t+1} suitably restricted. We extend Debreu (1954)'s notion of a valuation functional to a valuation operator to allow for portfolio prices to depend on date t information. With this in mind let $\mathcal{F}_t$ be the sigma algebra used to depict the information available to all investors at date t. We use a representation of $\pi_{t,t+1}$ of the form:

$$\pi_{t,t+1}(p_{t+1}) = E\left(s_{t+1}p_{t+1}|\mathcal{F}_t\right) \tag{1}$$

for some positive random variable s_{t+1} with probability one and any admissible payoff $p_{t+1} \in P_{t+1}$ on a portfolio of assets.

Definition 1 *The positive random variable s_{t+1} is the* **pricing kernel** *of the valuation operator $\pi_{t,t+1}$ provided that representation* (1) *is valid.*

If a kernel representation exists, the valuation operator $\pi_{t,t+1}$ necessarily satisfies the Principal of No Arbitrage.

How does one construct a representation of this type? There are alternative ways to achieve this. First suppose that i) P_{t+1} consists of all random variables that are $\mathcal{F}_{t+1}$ measurable and have finite conditional (on $\mathcal{F}_t$) second moments, ii) $\pi_{t,t+1}$ is linear conditioned on $\mathcal{F}_t$, iii) $\pi_{t,t+1}$ also satisfies a conditional continuity restriction, and iv) the Principal of No-Arbitrage is satisfied. The existence of a kernel s_{t+1} follows from a conditional version of the Riesz Representation Theorem (Hansen and Richard (1987)). Second suppose that i) P_{t+1} consists of all bounded random variables that are $\mathcal{F}_{t+1}$ measurable, ii) $\pi_{t,t+1}$ is linear conditioned on $\mathcal{F}_t$, iii) $E(\pi_{t,t+1})$ induces an absolutely continuous measure on $\mathcal{F}_{t+1}$, and iv) the Principle of No Arbitrage is satisfied. We can apply the Radon Nikodym Theorem to justify the existence of a kernel. In both of these cases, security markets are *complete* in that payoffs of any indicator function for events in in $\mathcal{F}_{t+1}$ are included in the domain of the operator.[1] Kernels can be constructed for other mathematical restrictions on asset payoffs and prices as well.

As featured by Harrison and Kreps (1979) and Hansen and Richard (1987) suppose that P_t is not so richly specified. Under the (conditional) Hilbert space formulation and appropriate restrictions on P_t we may still apply a (conditional) version of the Riesz Representation Theorem to represent:

$$\pi_{t,t+1}(p_{t+1}) = E\left(q_{t+1}p_{t+1}|\mathcal{F}_t\right)$$

for some $q_{t+1} \in P_{t+1}$ and all $p_{t+1} \in P_{t+1}$. Even if the Principal of No Arbitrage is satisfied, there is no guarantee that the resulting q_{t+1} is positive with probability one, however. Provided, however, that we can extend P_{t+1} to a larger space that includes indicator functions for events in $\mathcal{F}_{t+1}$ while preserving the Principal of No Arbitrage, then there exists a pricing kernel s_{t+1} for $\pi_{t,t+1}$. The pricing kernel may not be unique, though.

[1]See Rogers (1998) for a dynamic extension using this approach.

2 Stochastic discounting

The random variable s_{t+1} is also called the one-period *stochastic discount factor* process. Its multi-period counterpart is:

Definition 2 *The* **stochastic discount process** $\{S_{t+1} : t = 1, 2, ...\}$ *is*

$$S_{t+1} = \prod_{j=1}^{t+1} s_j$$

where s_j is the pricing kernel used to represent the valuation operator between dates $j-1$ and j.

Thus the $t+1$ period stochastic discount factor compounds the corresponding one-period stochastic discount factors. The compounding is justified by the Law of Iterated Values when trading is allowed at intermediate dates. As a consequence,

$$\pi_{0,t+1}(p_{t+1}) = E\left(S_{t+1} p_{t+1} | \mathcal{F}_0\right) \tag{2}$$

gives the date zero price of a security that pays p_{t+1} in the numeraire at date $t+1$. A stochastic discount factor discounts the future in a manner that depends on future outcomes. This outcome dependence is included in order to make adjustments for risk. Such adjustments are unnecessary for a discount bond, however; because such a bond has a payoff that is equal to one independent of the state realized at date $t+1$. The price of a date $t+1$ discount bond is obtained from formula (2) by letting $p_{t+1} = 1$ and hence is given by $E\left[S_{t+1} | \mathcal{F}_0\right]$.

More generally, the date τ price of p_{t+1} is given by:

$$\pi_{\tau,t+1}(p_{t+1}) = E\left[\left(\frac{S_{t+1}}{S_\tau}\right) p_{t+1} | \mathcal{F}_\tau\right].$$

for $\tau \leq t$. Thus the ratio $\frac{S_{t+1}}{S_\tau}$ is the pricing kernel for the valuation operator $\pi_{\tau,t+1}$.

3 Risk-neutral probabilities

Given a one-period pricing kernel, we build a so-called *risk neutral probability measure* recursively as follows. First rewrite the one-period pricing operator as;

$$\pi_{t,t+1}(p_{t+1}) = E\left(s_{t+1} | \mathcal{F}_t\right) \tilde{E}\left(p_{t+1} | \mathcal{F}_t\right)$$

where

$$\tilde{E}\left(p_{t+1} | \mathcal{F}_t\right) = E\left(\left[\frac{s_{t+1}}{E\left(s_{t+1} | \mathcal{F}_t\right)}\right] p_{t+1} | \mathcal{F}_t\right)$$

Thus one-period pricing is conveniently summarized by a one-period discount factor for a riskless payoff, $E\left(s_{t+1}|\mathcal{F}_t\right)$, and a change of the conditional probability measure implied by the conditional expectation operator $\tilde{E}\left(\cdot|\mathcal{F}_t\right)$. Risk adjustments are now absorbed into the change of probability measure.

Using this construction repeatedly, we decompose the date $t+1$ component of the stochastic discount factor process:

$$S_{t+1} = \bar{S}_{t+1} M_{t+1} \tag{3}$$

where

$$\bar{S}_{t+1} = \left[\prod_{j=1}^{t+1} E\left(s_j|\mathcal{F}_{j-1}\right)\right]$$

is a discount factor constructed from the sequence of one-period riskless discount factors, and

$$M_{t+1} = \prod_{j=1}^{t+1} \frac{s_j}{E\left(s_j|\mathcal{F}_{j-1}\right)} = \frac{S_{t+1}}{\bar{S}_{t+1}} \tag{4}$$

is a positive martingale adapted to $\{\mathcal{F}_t : t = 0, 1, ...\}$ with expectation unity. For each date t, M_t can be used to assign ˜ probabilities to events in $\mathcal{F}_t$. Since $E\left(M_{t+1}|\mathcal{F}_t\right) = M_t$, the date $t+1$ assignment of probabilities to events in $\mathcal{F}_{t+1}$ is compatible with the date t assignment to events in $\mathcal{F}_t \subset \mathcal{F}_{t+1}$. This follows from the Law of Iterated Expectations. In effect the Law of Iterated Expectations enforces the Law of Iterated Values.

Applying factorization (3), we have an alternative way to represent $\pi_{\tau,t+1}$:

$$\pi_{\tau,t+1}(p_{t+1}) = \tilde{E}\left[\bar{S}_{t+1} p_{t+1}|\mathcal{F}_t\right]$$

Pricing is reduced to riskless discounting and a distorted (risk neutral) conditional expectation. The Ross (1976) and Harrison and Kreps (1979) insight is that dynamic asset pricing in the absence of arbitrage is captured by the existence of a positive (with probability one) martingale that can be used to represent prices in conjunction with a sequence of one period riskless discount factors. Moreover, it justifies an arguably mild modification of the "efficient market hypothesis" that states that discounted prices should behave as martingales with the appropriate cash-flow adjustment. While Rubinstein (1976) and Lucas (1978) had clearly shown that efficiency should not preclude risk-compensation, the notion of equivalent martingale measures reconciles the points of view under greater generality. The martingale property and associated "risk-neutral pricing" is recovered for some distortion of the historical probability measure that encapsulates risk compensation. This distortion preserves "equivalence" (the two probability measures agree about which events in $\mathcal{F}_t$ are assigned probability measure for each finite t) by ensuring the existence of a strictly positive stochastic discount factor.

The concept of equivalent martingale measure (for each t) has been tremendously influential in derivative asset pricing. The existence of such a measure allows risk-neutral pricing of all contingent claims that are attainable because their payoff can be perfectly duplicated by self-financing strategies. Basic results from probability theory are directly exploitable in characterizing asset pricing in an arbitrage free environment. More details can be found in the article "Econometrics of Option Pricing" in this Encyclopedia.

While we have developed this discussion for a discrete-time environment, there are well known continuous-time extensions. In the case of continuous-time Brownian motion information structures, the change of measure has a particularly simple structure. In accordance to the Girsanov Theorem, the Brownian motion under the original probability measure becomes a Brownian motion with drift under the risk-neutral measure. These drifts absorb the adjustments for exposure to Brownian motion risk.

4 Investors' preferences

Economic models show how exposure to risk is priced and what determines riskless rates of interest. As featured by Ross (1976), the risk exposure that is priced is the risk that cannot be diversified by averaging through the construction of portfolios of traded securities. Typical examples include macroeconomic shocks. Empirical macroeconomics seeks to identify macroeconomic shocks to quantify responses of macroeconomic aggregates to those shocks. Asset pricing models assign prices to exposure of cash flows to the identified shocks. The risk prices are encoded in stochastic discount factor processes and hence are implicit in the risk neutral probability measures used in financial engineering.

Stochastic discount factors implied by specific economic models often reflect investor preferences. Subjective rates of discount and intertemporal elasticities appear in formulas for risk-free interest rates, and investors' aversion to risk appears in formulas for the prices assigned to alternative risk exposures. Sometimes the reflection of investor preferences is direct and sometimes it is altered by the presence market frictions. In what follows we illustrate briefly some of the stochastic discount factors that have been derived in the literature.

4.1 Power utility

When there are no market frictions and markets are complete, investors' preferences can be subsumed into a utility function of a representative agent. In what follows suppose the representative investor has discounted time-separable utility with a constant elasticity of substitution $1/\rho$:

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}$$

This is the stochastic discount factor for the power utility specification of the model of Rubinstein (1976) and Lucas (1978).[2]

[2]See Breeden (1979) for a continuous-time counterpart.

4.2 Recursive utility

Following Kreps and Porteus (1978), Epstein and Zin (1989) and Weil (1990), preferences are specified recursively using continuation values for hypothetical consumption processes. Using a double CES (constant elasticity of substitution) recursion, the resulting stochastic discount factor is

$$\frac{S_{t+1}}{S_t} = \exp(-\delta) \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left[\frac{U_{t+1}}{\mathsf{R}_t(U_{t+1})}\right]^{\rho-\gamma}$$

where $\gamma > 0$, $\rho > 0$, U_t is the continuation value associated with current and future consumption, and

$$\mathsf{R}_t(U_{t+1}) \doteq E\left[(U_{t+1})^{1-\gamma} | \mathcal{F}_t\right]$$

is the risk-adjusted continuation value. The parameter ρ continues to govern the elasticity of substitution while the parameter γ alters the risk preferences. See Hansen et al. (1999), Tallarini (2000), and Hansen et al. (2007) for a discussion of the use of continuation values in representing the stochastic discount factor and see Campbell (2003) and Hansen et al. (2007) for discussions of empirical implications. When $\rho = 1$, the recursive utility model coincides with a model in which investors have a concern about robustness as in Anderson et al. (2003).

The recursive utility model is just one of a variety of ways of altering investors' preferences. For instance, Constantinides (1990) and Heaton (1995) explore implications of introducing intertemporal complementarities in the form of "habit persistence".

4.3 Consumption externalities and reference utility

Following Abel (1990), Campbell and Cochrane (1999), and Menzly et al. (2004) develop asset pricing implications of models in which there are consumption externalities in models with stochastic consumption growth. These externalities can depend on a social stock of past consumptions. The implied one-period stochastic discount factor in these models has the form:

$$\frac{S_{t+1}}{S_t} = \exp(-\delta) \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \frac{\phi(H_t)}{\phi(H_0)}$$

The social stock of consumption is built as a possibly nonlinear function of current and past social consumption or innovations to social consumption. The construction of this stock differs across the various specifications. The process $\{H_t : t = 0, 1, ...\}$ is the ratio of the consumption to the social stock and ϕ is an appropriately specified function of this ratio. In a related approach Garcia et al. (2006) consider investors' preferences in consumption is evaluated relative to a reference level that is determined externally. The stochastic discount factor is:

$$\frac{S_{t+1}}{S_t} = \exp(-\delta) \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{H_t}{H_0}\right)^{\eta}$$

where the process $\{H_t : t = 0, 1, ...\}$ is the ratio of the consumption to the reference level. In effect, the social externality in these models induces a preference shock or wedge to the stochastic discount factor specification, which depends explicitly on the equilibrium aggregate consumption process. Finally, Bakshi and Chen (1996) develop implications for a model in which relative wealth enters the utility function of investors.

4.4 Incomplete markets

Suppose that individuals face private shocks that they cannot insure against but that they can write complete contracts over aggregate shocks. Let $\mathcal{F}_{t+1}$ denote the date $t+1$ sigma algebra generated by aggregate shocks available up until date $t+1$. This conditioning information set determines the type of risk exposure that can be traded as of date $t+1$. Under this partial risk sharing and power utility, the stochastic discount factor for pricing aggregate uncertainty is

$$\begin{aligned} \frac{S_{t+1}}{S_t} &= \exp(-\delta)\left(\frac{E\left[(C^j_{t+1})^{-\rho}|\mathcal{F}_{t+1}\right]}{E\left[(C^j_t)^{-\rho}|\mathcal{F}_t\right]}\right) \\ &= \exp(-\delta)\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho}\left(\frac{E\left[(C^j_{t+1}/C^a_{t+1})^{-\rho}|\mathcal{F}_{t+1}\right]}{E\left[(C^j_t/C^a_t)^{-\rho}|\mathcal{F}_t\right]}\right). \end{aligned} \tag{5}$$

In this example and the ones that follow, C^j_{t+1} is consumption for individual j and C^a_{t+1} is aggregate consumption. Characterization (5) of the stochastic discount factor displays the pricing implications of limited risk-sharing in security markets. It is satisfied, for instance, in the model of Constantinides and Duffie (1996).

4.5 Private information

Suppose that individuals have private information about labor productivity that is conditionally independent given aggregate information, leisure enters preferences in a manner that is additively separable and consumption allocations are Pareto optimal given the private information. As shown by Kocherlakota and Pistaferri (2009), the stochastic discount factor follows from the "inverse Euler equation" of Rogerson (1985) and Kocherlakota and Pistaferri (2009),

$$\begin{aligned} \frac{S_{t+1}}{S_t} &= \exp(-\delta)\left(\frac{E\left[(C^j_t)^{\rho}|\mathcal{F}_t\right]}{E\left[(C^j_{t+1})^{\rho}|\mathcal{F}_{t+1}\right]}\right) \\ &= \exp(-\delta)\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho}\left(\frac{E\left[(C^j_t/C^a_t)^{\rho}|\mathcal{F}_t\right]}{E\left[(C^j_{t+1}/C^a_{t+1})^{\rho}|\mathcal{F}_{t+1}\right]}\right) \end{aligned}$$

where $\mathcal{F}_t$ is generated by the public information. As emphasized by Rogerson (1985) this is a model with a form of "savings constraints". Kocherlakota and Pistaferri (2009), While the stochastic discount factor for the incomplete information model is expressed in terms of the $(-\rho)^{th}$ moments of the cross-sectional distributions of consumption in adjacent time periods, show that in the private information model it is the ρ^{th} moments of these same distributions (see Kocherlakota and Pistaferri (2009)).

4.6 Solvency constraints

Luttmer (1996), He and Modest (1995) and Cochrane and Hansen (1992) study asset pricing implications in models with limits imposed on the state contingent debt that is allowed. Alvarez and Jermann (2000) motivate such constraints by appealing to limited commitment as in Kocherlakota (1996) and Kehoe and Levine (1993). When investors default they are punished by excluding participation in asset markets in the future. Chien and Lustig (2008) explore the consequences of alternative (out of equilibrium) punishments. Following Luttmer (1996), the stochastic discount factor in presence of solvency constraints and power utility is:

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\min_j \frac{C^j_{t+1}}{C^j_t}\right)^{-\rho},$$

and in particular,

$$\frac{S_{t+1}}{S_t} \geq \exp(-\delta)\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho}.$$

Thus the consumer with the smallest realized growth rate in consumption has a zero Lagrange multiplier on his or her solvency constraint, and hence the intertemporal marginal rate of substitution for this person is equal to the stochastic discount factor. For the other consumers the binding constraint prevents them shifting consumption from the future to the current period.

5 Long-term risk

The stochastic discount factor process assigns prices to risk exposures at alternative investment horizons. To study pricing over these horizons, Alvarez and Jermann (2005), Hansen and Scheinkman (2009) and Hansen (2008) use a Markov structure and apply factorizations of the form:

$$S_{t+1} = \exp(-\eta t) M_{t+1} \frac{\hat{f}(X_0)}{\hat{f}(X_{t+1})} \tag{6}$$

where $\{M_t : t = 0, 1, ...\}$ is a *multiplicative martingale*, η is a positive number, $\{X_t : t = 0, 1, ...\}$ is an underlying Markov process, and $\hat{f}$ is a positive function of the Markov state. The martingale is used as a convenient change of probability, one that is distinct from the "risk neutral" measure described previously. Using this change of measure, asset prices can be depicted as:

$$\pi_{0,t+1}(p_{t+1}) = \exp\left[-\eta(t+1)\right] \hat{E}\left[\frac{p_{t+1}}{\hat{f}(X_{t+1})} | X_0\right] \hat{f}(X_0).$$

where $\hat{\cdot}$ is the conditional expectation is built with the martingale $\{M_t : t \geq 0\}$ in (6). The additional discounting is now constant and simple expectations can now be computed by exploiting the Markov structure. Hansen and Scheinkman (2009) give necessary and sufficient conditions for this factorization for a process $\{M_t : t \geq 0\}$ that implies stable stochastic dynamics.[3] While Alvarez and Jermann (2005) use this factorization to investigate the long-term links between the bond market and the macro economy, Hansen and Scheinkman (2009) and Hansen (2008) extend this factorization to study the valuation of cash flows that grow stochastically over time. As argued by Hansen and Scheinkman (2009) and Hansen (2008), these more general factorizations are valuable for the study of risk-return tradeoffs for long investment horizons.

As argued by Bansal and Lehmann (1997), many alterations to the power utility model in section 4.1 can be represented as:

$$\frac{S^*_{t+1}}{S^*_t} = \left(\frac{S_{t+1}}{S_t}\right)\left[\frac{f(X_{t+1})}{f(X_t)}\right] \tag{7}$$

for a positive function f. Transient components in asset pricing models are included to produce short term alterations in asset prices and are expressed as the ratio of a function of the Markov state in adjacent dates. As shown by Bansal and Lehmann (1997), this representation arises in models with habit persistence; or as shown in Hansen (2008) the same is true for a limiting version of the recursive utility model. Combining (7) with (6) gives

$$S^*_{t+1} = \exp(-\eta) M_{t+1} \frac{f^*(X_0)}{f^*(X_{t+1})}$$

where $f^* = \hat{f}/f$.

6 Inferring stochastic discount factors from data

Typically a finite number of asset payoffs are used in econometric practice. Also the information used in an econometric investigation may be less than that used by investors. With this in mind, let Y_{t+1} denote an n-dimensional vector of asset payoffs observed by the econometrician such that

$$E\left(|Y_{t+1}|^2 | \mathcal{G}_t\right) < \infty$$

with $E\left(Y_{t+1} {Y_{t+1}}' | \mathcal{G}_t\right)$ nonsingular with probability one and $\mathcal{G}_t \subset \mathcal{F}_t$. Let Q_t denote the corresponding price vector that is measurable with respect to $\mathcal{G}_t$ implying that

$$E\left(s_{t+1} Y_{t+1} | \mathcal{G}_t\right) = Q_t \tag{8}$$

where $s_{t+1} = S_{t+1}/S_t$. We may construct a counterpart to a (one-period) stochastic discount factor by forming:

$$p^*_{t+1} = {Y_{t+1}}' \left[E\left(Y_{t+1} {Y_{t+1}}' | \mathcal{G}_t\right)\right]^{-1} Q_t.$$

[3] While Hansen and Scheinkman (2009) use a continuous-time formulation, discrete-time counterparts to their analysis are straightforward to obtain.

Notice that

$$E\left(p^*_{t+1} Y_{t+1} \middle| \mathcal{G}_t\right) = Q_t,$$

suggesting that we could just replace s_{t+1} in (8) with p^*_{t+1}. We refer to p^*_{t+1} as a "counterpart" to a stochastic discount factor because we have not restricted p^*_{t+1} to be positive.

This construction is a special case of the representation of the prices implied by a conditional version of the Riesz-Representation Theorem (see Hansen and Richard (1987)). Since p^*_{t+1} is not guaranteed to be positive, if we used it to assign prices to derivative claims (nonlinear functions of Y_{t+1}), we might induce arbitrage opportunities. Nevertheless, provided that s_{t+1} has a finite conditional second moment,

$$E\left[\left(s_{t+1} - p^*_{t+1}\right) Y_{t+1} | \mathcal{G}_t\right] = 0.$$

This orthogonality informs us that p^*_{t+1} is the conditional least squares projection of s_{t+1} onto Y_{t+1}. While a limited set of asset price data will not reveal s_{t+1}, the data can provide information about the date $t+1$ kernel for pricing over a unit time interval.

Suppose that Y_{t+1} contains a conditionally riskless payoff. Then

$$E\left(s_{t+1} | \mathcal{G}_t\right) = E\left(p^*_{t+1} | \mathcal{G}_t\right)$$

By a standard least squares argument, the conditional volatility of s_{t+1} must be at least as large as the conditional volatility of p^*_{t+1}. There are a variety of other restrictions that can be derived. For instance, see Hansen and Jagannathan (1991), Snow (1991), and Bansal and Lehmann (1997).[4]

This construction has a direct extension to the case in which a complete set of contracts can be written over the derivative claims. Let H_{t+1} be the set of all payoffs that have finite second moments conditioned on $\mathcal{G}_t$ and are of the form $h_{t+1} = \phi(Y_{t+1})$ for some Borel measurable function ϕ. Then we may obtain a kernel representation for pricing claims with payoffs in H_{t+1} by applying the Riesz Representation Theorem:

$$\pi_t(h_{t+1}) = E\left(h^*_{t+1} h_{t+1} | \mathcal{G}_t\right)$$

for some h^*_{t+1} in H_{t+1}. Then

$$E\left[\left(s_{t+1} - h^*_{t+1}\right) h_{t+1} | \mathcal{G}_t\right) = 0,$$

which shows that the conditional least-squares projection of s_{t+1} onto H_{t+1} is h^*_{t+1}. In this case h^*_{t+1} will be positive. Thus this richer collection of observed tradable assets implies a more refined characterization of s_{t+1} including the possibility that $s_{t+1} = h^*_{t+1}$ provided that H_{t+1} coincides with the full collection of one-period payoffs that could be traded by investors. The estimation procedures of Ait-Sahalia and Lo (1998) and Rosenberg and Engle (2002) can be interpreted as estimating h^*_{t+1} projected on the return information.

[4] These authors do not explicitly use conditioning information. In contrast Gallant et al. (1990) estimate conditional moment restrictions using a flexible parameterization for the dynamic evolution of the data.

7 Linear Asset Pricing Models

The long history of linear beta pricing can be fruitfully revisited within the stochastic discount factor (SDF) framework. Suppose that

$$Q_t = E\left[(\lambda_t \cdot z_{t+1} + \alpha_t)\, Y_{t+1} | \mathcal{G}_t\right] \tag{9}$$

for some vector λ_t and some scalar α_t that are $\mathcal{G}_t$ measurable. Then

$$E\left[Y_{t+1}|\mathcal{G}_t\right] = \alpha_t^* Q_t + \lambda_t^* \cdot \text{cov}(Y_{t+1}, z_{t+1}|\mathcal{G}_t)$$

where

$$\begin{aligned} \alpha_t^* &= \frac{1}{\alpha_t + \lambda_t \cdot E(z_{t+1}|\mathcal{G}_t)} \\ \lambda_t^* &= -\lambda_t \alpha_t^*, \end{aligned}$$

which produces the familiar result that risk compensation expressed in terms of the conditional mean discrepancy: $E\left[Y_{t+1}|\mathcal{G}_t\right] - \alpha_t^* Q_t$ depends on the conditional covariances with the factors. When the entries of z_{t+1} are standardized to a have conditional variances equal to unity, the entries of λ_t^* become the conditional regression coefficients the "beta's" of the asset payoffs onto the alternative observable factors. When z_{t+1} is the scalar market return, this specification gives the familiar CAPM from empirical finance. When z_{t+1} is augmented to include the payoff on a portfolio designed to capture risk associated with size (market capitalization) and the payoff on a portfolio designed to capture risk associated with book to market equity, this linear specification gives a conditional version of the Fama and French (1992) three factor model designed to explain cross-sectional differences in expected returns. See Connor and Korajczyk's chapter in this Encyclopedia for more on factor models.

If the factors are among the asset payoffs and a conditional (on $\mathcal{G}_t$) riskless payoff is included in Y_{t+1}, then

$$p_{t+1}^* = \lambda_t \cdot z_{t+1} + \alpha_t$$

and thus $\lambda_t \cdot z_{t+1} + \alpha_t$ is the conditional least-squares regression of s_{t+1} onto Y_{t+1}.

For estimation and inference, consider the special case in which $\mathcal{G}_t$ is degenerate and the vector Y_{t+1} consists of excess returns (Q_t is a vector of zeros). Suppose that the data generation process for $\{(z_{t+1}, Y_{t+1})\}$ is i.i.d. and multivariate normal. Then α_t and λ_t are time invariant. The coefficient vectors can be efficiently estimated by maximum likelihood and the pricing restrictions tested by likelihood ratio statistics. See Gibbons et al. (1989). As MacKinlay and Richardson (1991) point out, it is important to relax the i.i.d normal assumption in many applications. In contrast with the parametric maximum likelihood approach, generalized method of moments (GMM) provides an econometric framework that allows conditional heteroskedasticity and temporal dependence. See Hansen (1982). Moreover, the GMM approach offers the important advantage to provide an unified framework to testing of conditional linear beta pricing models. See Jagannathan et al. (2009) for an extensive discussion of the application of GMM to linear factor models. We will have more to say about estimation when $\mathcal{G}_t$ is not degenerate in our subsequent discussion.

Suppose that the conditional linear pricing model is misspecified. Hansen and Jagannathan (1997) show that choosing (λ_t, α_t) to minimize the maximum pricing error of payoffs with $E(|p_{t+1}|^2|\mathcal{G}_t) = 1$ is equivalent to solving the least squares problem:[5]

$$\min_{\lambda_t, \alpha_t, v_{t+1}, E((v_{t+1})^2|\mathcal{G}_t)<\infty} E\left[(\lambda_t \cdot z_{t+1} + \alpha_t - v_{t+1})^2 |\mathcal{G}_t\right]$$
$$\text{subject to} \;\; Q_t = E\left(v_{t+1} Y_{t+1}|\mathcal{G}_t\right).$$

This latter problem finds a random variable v_{t+1} that is close to $\lambda_t \cdot z_{t+1} + \alpha_t$ allowing for departures from pricing formula (9) where v_{t+1} is required to represent prices correctly. For a fixed (λ_t, α_t), the "concentrated" objective is:

$$[E[\lambda_t \cdot z_{t+1} + \alpha_t)Y_{t+1}|\mathcal{G}_t] - Q_t]' \left[E\left(Y_{t+1}Y_{t+1}'|\mathcal{G}_t\right)\right]^{-1} [E[\lambda_t \cdot z_{t+1} + \alpha_t)Y_{t+1}|\mathcal{G}_t] - Q_t]$$

which is a quadratic form the vector of pricing error. The random vector (λ_t, α_t) is chosen to minimize this pricing error. In the case of a correct specification, this minimization results in a zero objective; but otherwise it provides a measure of model misspecification.

8 Estimating parametric models

In examples of stochastic discount factors like those given in section 4, there are typically unknown parameters to estimate. This parametric structure often permits the identification of the stochastic discount factor even with limited data on security market payoffs and prices. To explore this approach, we introduce a parameter θ that governs say investors' preferences. It is worth noting that inference about θ is a semi-parametric statistical problem since we avoid the specification of the law of motion of asset payoffs and prices along with the determinants of stochastic discount factors. In what follows we sketch the main statistical issues in the following simplified context.

Suppose that $s_{t+1} = \phi_{t+1}(\theta_o)$ is a parameterized model of a stochastic discount factor that satisfies:

$$E\left[\phi_{t+1}(\theta_o) Y_{t+1}|\mathcal{G}_t\right] - Q_t = 0$$

where ϕ_{t+1} can depend on observed data but the parameter vector θ_o is unknown. This conditional moment restriction implies a corresponding unconditional moment condition:

$$E\left[\phi_{t+1}(\theta_o) Y_{t+1} - Q_t\right] = 0. \tag{10}$$

As emphasized by Hansen and Richard (1987), conditioning information can be brought in through the back door by scaling payoffs and their corresponding prices by random variables

[5]Hansen and Jagannathan (1997) abstract from conditioning information, but what follows is a straightforward extension.

that are $\mathcal{G}_t$ measurable. Hansen and Singleton (1982) show how to use the unconditional moment condition to construct a generalized method of moments (GMM) estimator of the parameter vector θ_o with properties characterized by Hansen (1982). See also the GMM entry by A. Hall in this Encyclopedia.

As an alternative, the parameterized family of models might be misspecified. Then the approach of Hansen and Jagannathan (1997) described previously could be used whereby an estimator of θ_o is obtained by minimizing:

$$\left[\frac{1}{N}\sum_{t=1}^{N}\phi_{t+1}(\theta)Y_{t+1} - Q_t\right]\left[\frac{1}{N}\sum_{t=1}^{N}Y_{t+1}{Y_{t+1}}'\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N}\phi_{t+1}(\theta)Y_{t+1} - Q_t\right].$$

with respect to θ. This differs from a GMM formulation because the weighting matrix in the quadratic form does not depend on the stochastic discount factor, but only on the second moment matrix of the payoff vector Y_{t+1}. This approach suffers from a loss of statistical efficiency in estimation when the model is correctly specified, but it facilitates comparisons across models because the choice of stochastic discount factor does not alter how the overall magnitude of the pricing errors is measured.

As in Hansen and Jagannathan (1997) and Hansen et al. (1995), the analysis can be modified to incorporate the restriction that pricing errors should also be small for payoffs on derivative claims. This can be formalized by computing the time series approximation to the least squares distance between a candidate, but perhaps misspecified, stochastic discount factor from the family of strictly positive random variables z_{t+1} that solve the pricing restriction:

$$E(z_{t+1}Y_{t+1} - Q_t) = 0.$$

See Hansen et al. (2007) for more discussion of these alternative approaches to estimation.

So far we have described econometric methods that reduce conditional moment restrictions (9) and (10) to their unconditional counterparts. From a statistical perspective, it is more challenging to work with the original conditional moment restrictsion. Along this vein, Ai and Chen (2003) and Antoine et al. (2007) develop nonparametric methods to estimate conditional moment restrictions used to represent pricing restrictions. In particular, Antoine et al. (2007) develop a conditional counterpart to the continuously-updated GMM estimator introduced by Hansen et al. (1996) in which the weighting matrix in GMM objective depends explicitly on the unknown parameter to be estimated.[6] These same methods can be adapted to take account explicitly of conditioning information while allowing for misspecification as in Hansen and Jagannathan (1997).

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[6]The continuously updated estimator is also closely related to the "mimimum entropy" and "empirical likelihood" approaches (see Kitamura and Stutzer's chapter in this Encyclopedia).

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