

# CARTAN ACTIONS OF HIGHER RANK ABELIAN GROUPS AND THEIR CLASSIFICATION

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ABSTRACT. We classify  $\mathbb{R}^k$  volume preserving Anosov actions with 1-dimensional coarse Lyapunov foliations, also called Cartan actions, when  $k \geq 2$  under mild ergodicity assumptions. This completes and improves known results when  $k \geq 3$ , and requires completely new methods and ideas.

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## 1. INTRODUCTION

Hyperbolic actions of higher rank abelian groups are markedly different from single hyperbolic diffeomorphisms and flows and display a multitude of rigidity properties such as measure rigidity and cocycle rigidity. We refer to [26, 5, 25] for details for the latter. In this paper, we concentrate on the third major rigidity property: classification and global differential rigidity of such actions.

Smale already conjectured in 1967 that generic diffeomorphisms only commute with its iterates [38]. In the hyperbolic case this was proved by Palis and Yoccoz in 1989 [28, 29]. Recently, Bonatti, Crovisier and Wilkinson proved this in full generality [1]. Much refined local rigidity properties were found in the works of Hurder and then Katok and Lewis on deformation and local rigidity of  $SL(n, \mathbb{Z})$  on the  $n$ -torus in which they used properties of the action of higher rank abelian subgroups [17, 22, 23]. This led to investigating such actions more systematically.

Let us first coin some terminology. Given a foliation  $\mathcal{F}$ , a diffeomorphism  $a$  acts *normally hyperbolically* w.r.t.  $\mathcal{F}$  if  $\mathcal{F}$  is invariant under  $a$  and if there exists a  $C^0$  splitting of the tangent space  $TM = E_a^s \oplus T\mathcal{F} \oplus E_a^u$  into stable and unstable subspaces for  $a$  and the tangent space of the leaves of  $\mathcal{F}$ . More precisely, we assume that  $E_a^s$  and  $E_a^u$  are contracted uniformly in either forward or backward time by  $a$  (see Definition 2.1 for precise definitions).

**Definition 1.1.** *Let  $A = \mathbb{Z}^k$  or  $A = \mathbb{R}^k$  with a faithful  $C^{1,\theta}$  action  $A \curvearrowright M$  on a compact manifold  $M$ . If  $a$  acts normally hyperbolic to the orbit foliation of  $A$  for some  $a \in A$ , we call  $a$  an Anosov element, and  $\alpha$  an Anosov action.*

*If  $k \geq 2$ , we call the action higher rank, and otherwise rank one. Finally call a higher rank Anosov action irreducible if no finite cover has a  $C^{1,\theta}$  rank one action on a compact manifold as a quotient, i.e. becomes rank one after factoring out by the kernel.*

Note that higher rank  $\mathbb{Z}^k$ -actions correspond to  $k$  commuting diffeomorphisms without nontrivial relations and at least one of them Anosov.

Irreducibility of actions could alternatively be defined by precluding different types of rank one factors, in particular measurable, topological and differentiable ( $C^{1,\theta}$  or  $C^\infty$ ). Asking for differentiable and especially  $C^\infty$  factors is clearly the most restrictive condition.

There are natural examples of higher rank actions coming from homogeneous actions, e.g. by the diagonal subgroup of  $SL(n, \mathbb{R})$  on a compact quotient  $SL(n, \mathbb{R})/\Gamma$  or actions by automorphisms of Lie groups, e.g. by commuting toral automorphisms. More generally, an *algebraic action* is the an action by  $\mathbb{R}^k \times \mathbb{Z}^l$  by compositions of translations and automorphisms on a compact homogeneous space  $G/\Gamma$  (see Section A.1 for examples of such Anosov actions). Katok and Spatzier proved local  $C^\infty$  rigidity of algebraic actions without rank one factors in [24], generalizing the earlier works by Hurder respectively Katok and Lewis. This gives evidence for the following conjecture of Katok and Spatzier (posed as a question in [3] and as a conjecture in [15, Conjecture 16.8]).

**Conjecture 1.2.** *(Katok-Spatzier) All irreducible higher rank Anosov actions on any compact manifold are smoothly conjugate to an algebraic action after passing to finite covers.*

This is reminiscent of the longstanding conjecture by Anosov and Smale that Anosov diffeomorphisms are topologically conjugate to an automorphism of an infra-nilmanifold [38].

The conclusion of the conjecture in this higher rank case however is significantly stronger: First of all, the conjugacy is claimed to be smooth. For single Anosov diffeomorphisms, smooth rigidity results are not possible as one can change derivatives at fixed points by local changes. Furthermore, Farrell and Jones constructed Anosov diffeomorphisms on exotic tori. Thus the differentiable structure of the underlying manifold is not determined. Secondly, the conjecture applies equally well to  $\mathbb{Z}^k$ - and  $\mathbb{R}^k$ -actions. For Anosov flows however, even a topological classification appears out of reach as there are many such flows cf. e.g. [14], even ones for which topological transitivity fails [9].

Significant progress has been made on this conjecture in the last decade. For higher rank Anosov  $\mathbb{Z}^k$  actions on tori and nilmanifolds, Rodriguez Hertz and Wang [35] have proved the ultimate result assuming that the linearization does not have rank one factors. This is also the only result known assuming existence of only one Anosov element in the action. In fact it followed earlier work by Fisher, Kalinin and Spatzier [8] for totally Anosov, i.e. Anosov actions with a dense set of Anosov elements. As knowledge of the underlying manifold is required, this is really more a global rigidity theorem. Finally, in the same vein, Spatzier and Yang classified nontrivially commuting expanding maps in [39]. This was possible as expanding maps were known to be  $C^0$  conjugate to endomorphisms of nilmanifolds (up to finite cover) thanks to work of Gromov and Shub [12, 37]. Of course, a positive resolution of the Anosov-Smale conjecture would automatically prove the Katok-Spatzier conjecture for higher rank  $\mathbb{Z}^k$  actions.

Much less is known for  $\mathbb{R}^k$  actions or when the underlying manifold is not a torus or nilmanifold. We will concentrate on the so-called totally Cartan actions. For a totally Anosov action  $\alpha$ , a *coarse Lyapunov space* is a maximal (nonempty) intersection of stable manifolds  $\cap \mathcal{W}_{a_i}^S$  for Anosov elements  $a_i$ . If  $\alpha$  preserves a measure  $\mu$  of full support, this can be phrased in terms of Lyapunov exponents (see Section 2.1.2), thus the name.

**Definition 1.3.** *A totally Anosov action of  $A = \mathbb{R}^k \times \mathbb{Z}^l$ ,  $A \curvearrowright M$ , is called totally Cartan if all coarse Lyapunov spaces are one dimensional.*

Kalinin and Spatzier classified totally Cartan actions of  $\mathbb{R}^k$  for  $k \geq 3$  in [21] on arbitrary manifolds under the additional hypothesis that every one-parameter subgroups of  $\mathbb{R}^k$  acts transitively and  $\alpha$  preserves a probability measure  $\mu$  of full support. Later, Kalinin and Sadovskaya proved several excellent results in this direction for totally nonsymplectic (TNS) actions, i.e. actions for which no two Lyapunov exponents (thought of as linear functionals on  $\mathbb{R}^k$ ) are negatively proportional. They also can treat higher dimensional coarse Lyapunov spaces but require additional conditions such as joint integrability of coarse Lyapunov foliations, non-resonance conditions and quasi-conformality of the action on the coarse Lyapunov foliations [20, 19]. Recently, Damjanovic and Xu [4] generalized their results relaxing the quasi-conformality conditions but still requiring joint integrability or non-resonance.

Remarkably, these results by Kalinin and Sadovskaya or Damjanovic and Xu hold also for  $k = 2$  when making additional conditions such as TNS. Naturally, one hopes to extend at least the classification of Cartan actions to  $k = 2$ . The techniques used in [21] in rank  $\geq 3$  break down completely. Indeed, the main idea there was to create isometries on integrable hulls of coarse Lyapunov spaces using the dynamics of elements in the intersection of kernels of Lyapunov exponents. This works well in rank at least 3 but obviously not for rank 2.

The main achievement of this paper is to classify Cartan actions for rank 2 acting groups without any of the integrability and TNS assumptions of prior work.

**Theorem 1.4.** *Let  $\alpha$  be a  $C^{1,\theta}$  totally Cartan action of  $\mathbb{R}^k$ ,  $k \geq 2$ , on a compact manifold preserving a volume  $\mu$  such that the action of every codimension one subgroup of  $\mathbb{R}^k$  is ergodic. Then, passing to a finite cover,  $\alpha$  is  $C^{1,\theta'}$  conjugate to an algebraic action. Moreover, if  $\alpha$  is  $C^\infty$  then  $\alpha$  is  $C^\infty$  conjugate to an algebraic action, up to passing to a finite cover.*

In particular, our new ideas allow us to drop the regularity requirements for such actions from  $C^\infty$  to  $C^{1,\theta}$ . This was not possible with the arguments in [21].

This theorem has an immediate application to the Zimmer program of classifying actions of higher rank semisimple Lie groups and their lattices on compact manifolds. We refer to Fisher's recent surveys [7, 6] for a more extensive discussion. This program has been one main reason for seeking rigidity results for hyperbolic actions of higher rank abelian groups.

**Corollary 1.5.** *Let  $G$  be a connected semisimple Lie group without compact factors, finite center and  $\mathbb{R}$ -rank at least 2. Suppose  $G$  acts on a compact manifold  $M$  by  $C^{1,\theta}$  diffeomorphisms such that the restriction to a split Cartan  $A \subset G$  is a Cartan action of  $A$ . Also assume that the action preserves a volume  $\mu$  on  $M$ , and that every simple factor of  $G$  acts ergodically w.r.t.  $\mu$ . Then the action is  $C^{1,\theta'}$  conjugate to an affine homogeneous action of  $G$  on a homogeneous space  $H/\Lambda$  via an embedding  $G \rightarrow H$ .*

*A similar result holds for  $\Gamma$  actions where  $\Gamma \subset G$  is a lattice and every factor of  $G$  has  $\mathbb{R}$  rank at least 2.*

A  $C^\infty$  version of the Corollary also holds. This had already been established for  $G$  of  $\mathbb{R}$ -rank at least 3 by work of Goetze and Spatzier [10]. For lattice actions as above by  $C^\infty$  diffeomorphisms even the  $\mathbb{R}$ -rank 2 case was proved in [10] thanks to earlier work by Qian [32].

As indicated above, the case of  $\mathbb{R}^2$  actions is significantly more difficult than that of actions of rank 3 and higher groups. Indeed, we need to develop several completely new tools. We expect these new ideas to be useful in other situations as well. Let us outline some of the steps and tools of our argument.

*Basic Higher Rank Tools:* Many basic technical tools, especially from Pesin theory, have been adapted from single diffeomorphisms to higher rank actions. We will use them freely, after summarizing them below in Section 2.1. In particular, we have decompositions into refined Oseledets spaces and Lyapunov exponents. The latter are also called *weights*, and we view them as linear maps from the acting group  $\mathbb{R}^k$  to  $\mathbb{R}$ . We refer to [2] for a detailed presentation.

*Equivariant Hölder Metrics:* The starting point of our argument is the main technical result about the Hölder cohomology of the derivative cocycle of [21, Theorem 1.2] which holds for Cartan actions of any higher rank abelian group (see Section 2.1.3). It implies immediately that we have a Hölder Riemannian metric  $g$  on  $M$  such that for any  $a \in \mathbb{R}^k$  and any Lyapunov exponent  $\chi$

$$\|a_*(v)\| = e^{\chi(a)} \|v\| \quad \text{for any } v \in E_\chi.$$

*Path and Cycle Groups and a Theorem of Gleason-Palais:* The Hölder metrics make each coarse Lyapunov manifold isometric to  $\mathbb{R}$ , so after passing to a cover, we may define Hölder flows along each such manifold which act by translations in each leaf. These  $\mathbb{R}$ -actions allow us to define an action of the free product  $\mathcal{P} := \mathbb{R}^{*d}$  on  $M$  where  $d = \dim M - k$ . We call this latter group the *path group*. With the free product topology,  $\mathcal{P}$  becomes a connected and path connected topological group which, when combined with the  $\mathbb{R}^k$  action in a precise way, gives a transitive topological group action on  $M$ . This group  $\mathcal{P}$  is enormous, infinite dimensional for sure, and not a Lie group (see Section 2.2). Our main achievement is to show that this action factors through the action of a Lie group. To this end, we will show that the *cycle subgroups*, the stabilizers of  $\mathcal{P}$  at  $x$ , all contain a normal subgroup  $\mathcal{C}$  for which  $\mathcal{P}/\mathcal{C}$  is locally compact and has no small subgroups. Therefore,  $G = \mathcal{P}/\mathcal{C}$  will be Lie, and since  $\mathcal{P}$  acts transitively on  $M$  by construction,  $M$  will be a homogeneous space. Moreover, the original action of  $\mathbb{R}^k$  naturally relates to  $\mathcal{P}$ , and the action becomes part of a homogeneous action.

The idea of using free products to build a homogeneous structures was first explored by the second author in [41] when proving local rigidity of certain algebraic actions. Basically it is a new tool to build global homogeneous structures from partial ones on complementary subfoliations. That this can be done away from algebraic actions, assuming just uniform hyperbolicity, is the main insight of this current paper.

*Cycle Relations:* Our other advances all concern how to prove constancy of the cycle relations. This falls into separate cases: in particular commutator cycles of both proportional (“symplectic”) weights and non-proportional weights. We call these types of cycles *pairwise cycle structures*. Then we consider cycles with legs in a stable set of weights  $E$  where we call  $E$  *stable* if for some  $a \in \mathbb{R}^k$ ,  $\lambda(a) < 0$  for all  $\lambda \in E$ . For more details see below.

*Endgame:* Once constancy of these three types of cycles is accomplished, we combine this information to prove constancy of all cycles in Section 6. In other related works on local rigidity problems, the latter was achieved via K-theoretic arguments. We found a new way to do this, avoiding the intricate K-theory arguments, by showing constancy of a dense set of relations using explicit relations between stable and unstable horocycle flows in  $PSL(2, \mathbb{R})$ . The K-theory argument was used in the past to treat the remaining potential relations. However, density of the good relations makes this unnecessary.

*Stable Cycles:* We first introduce a cyclic ordering of the Lyapunov hyperplanes (kernels of weights) to handle stable cycles. For  $\mathbb{R}^k$  actions for  $k \geq 3$ , restrict to a generic 2-plane in  $\mathbb{R}^k$  which is not contained in  $\ker \alpha$  for any weight  $\alpha$  to get an ordering. Then we simplify products  $\eta_{t_{i_1}}^{\lambda_{i_1}} * \dots * \eta_{t_{i_l}}^{\lambda_{i_l}}$  with all  $\lambda_{i_j}$  in a stable subset  $E$  by putting the  $\lambda_{i_j}$  into cyclic order by commuting them with other  $\lambda_{i_k}$ . Assuming commutator cycles are constant we can then easily show that stable cycles are constant using the dynamics of the action, cf. Section 5.1.

*Geometric Commutators:* To understand pairwise cycle structures, we introduce another crucial tool, the *geometric commutator*. The point is that we would like to consider brackets of the vectorfields generating the groups acting on different coarse Lyapunov spaces. A priori though, the coarse Lyapunov spaces and these group actions are only Hölder continuous

transversally. Thus we cannot use standard brackets from Lie theory. However, we can define geometric versions of the commutator of two  $\mathbb{R}$  actions  $\eta^\alpha$  and  $\eta^\beta$  as follows: create a path by following the  $\alpha$  and  $\beta$  coarse Lyapunov spaces to create a “rectangle” of the form  $\eta_{-t}^\alpha \eta_{-s}^\beta \eta_t^\alpha \eta_s^\beta$ . We produce a canonical way to close this path up using legs from other coarse Lyapunov spaces. Those combined paths define the *geometric commutator*. We will often call the individual segments in a coarse Lyapunov space *legs* of the geometric commutator.

*Symplectic Relations:* To prove constancy of pairwise cycle structures coming from negatively proportional weights  $\alpha, -c\alpha$  for  $c > 0$ , we follow the procedure in [10, 21]. Using ergodicity of  $\ker \alpha$  for  $\mu$ , we get a transitive action by isometries of our Hölder metric on the closure of the set of points accessible from a typical point w.r.t.  $\mu$  by moving by  $\alpha$  and  $-c\alpha$  legs. We use this homogeneous structure, to understand the cycles coming from  $\alpha, -c\alpha$ .

*Commutator Cycles:* Finally, we discuss the commutator cycles of two weights  $\alpha$  and  $\beta$  which are not proportional. This is by far the most difficult part of our argument, and requires yet again a set of diverse new tools.

*Higher Rank at least 3:* For  $\mathbb{R}^k$  actions with  $k \geq 3$ ,  $\ker \alpha \cap \ker \beta$  has dimension at least 1, and we immediately get constancy of the commutator of  $\alpha$  and  $\beta$  by ergodicity of one parameter subgroups, finishing the argument in this case.

$\mathbb{R}^2$  *actions:* From now on, consider  $\mathbb{R}^2$  actions. As the last argument fails for these, we have to proceed entirely differently. The simplest case is when for all weights  $\lambda$ , the coarse Lyapunov foliation  $\mathcal{W}^\lambda$  is transitive on  $M$ . For simplicity suppose that for some  $\alpha$  and  $\beta$ , we can find a  $\lambda$  which commutes with  $\alpha, \beta$  and anything appearing in its geometric commutator. Then the commutator relation of  $\alpha$  and  $\beta$  is constant along  $\mathcal{W}^\lambda$ , and hence constant. A more complicated argument always works, see Lemma 4.9.

As simple algebraic examples show, it is not always true that each coarse Lyapunov foliation  $\mathcal{W}^\lambda$  is transitive on  $M$ , cf. Section A.1. In such homogeneous examples,  $M$  is the total space of a smooth fiber bundle, with the corresponding tangent bundle splitting subordinate to the coarse Lyapunov splitting. We produce a topological version of this phenomenon by realizing of  $M$  as a bundle over a base space  $B$  where the latter has this transitivity property for all of its coarse Lyapunov foliations. This base space is constructed by factoring out the coarse Lyapunov spaces for some suitable set of weights  $E^{\max}$ .

*Commutators and Invariant Functions:* Crucial for constructing the base  $B$  is the following insight from Lemma 4.4: Consider a continuous function  $f : M \rightarrow \mathbb{R}$  invariant along the coarse Lyapunov foliation  $\mathcal{W}^\alpha$  for a weight  $\alpha$ . Then for  $\beta$  not proportional to  $\alpha$ , we show that  $f$  is also invariant under  $\mathcal{W}^\lambda$  for any leg  $\lambda$  in the commutator of  $\alpha$  and  $\beta$ . This is proved entirely using higher rank dynamics.

*Ideal Structure:* We call a set of weights  $E$  an ideal if for all  $\alpha \in E$  and  $\beta$  another weight, all legs in the geometric commutator  $[\alpha, \beta]$  belong to  $E$ . Then our lemma about invariant functions and commutators simply says that the set of weights for which  $f$  is invariant on their coarse Lyapunov spaces is an ideal. Ideals allow us to get factors. In particular, we can factor out by a maximal ideal, and obtain a base space  $B$  (not necessarily unique). It is important to observe that  $B$  does not naturally come with a differentiable structure, but that the actions and coarse Lyapunov flows descend to  $B$ . This forces us to define a notion

of a *topological Cartan action*, which carries the data of the flows along the coarse Lyapunov foliations and the intertwining behavior, as well as other technical assumptions implied by  $C^{1,\theta}$ -actions. We then develop the notions and ideas defined above in this setting. This allows us to conclude that on  $B$ , the remaining coarse Lyapunov spaces are all transitive, and that  $B$  will be a homogeneous space, by the arguments above and some classical results of Gleason and Yamabe (Theorem 2.11).

*Fiber Structure:* Finally, we need to understand the fiber structure. This will be done inductively. Roughly, we factor by a sequence of ideals of weights  $E_i$  such that each set  $E_i$  is a maximal ideal in  $E_{i-1}$ . This defines a sequence of factors  $M_i$ . We show that the fibers of  $M_i \rightarrow M_{i-1}$  are homogenous spaces, by the maximality of the ideals chosen.

Then we can explore various commutator relations between weights on the base and weights on the fiber. The arguments splits into base and fiber relations and base base relations, and involve various considerations of normal forms, cocycles and higher order polynomial relations.

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## 2. PRELIMINARIES

In this section, we recall some tools from smooth ergodic theory, topological groups, and classical Lie criteria.

### 2.1. Dynamical Preliminaries.

2.1.1. *Normal Hyperbolicity.* Let us first recall the definition of normally hyperbolic transformations.

**Definition 2.1.** *Let  $M$  be a compact manifold with some Riemannian metric  $\langle \cdot, \cdot \rangle$ . Let  $\mathcal{F}$  be a foliation of  $M$  with  $C^1$  leaves. We call a  $C^1$  diffeomorphism  $a : M \rightarrow M$  normally hyperbolic w.r.t.  $\mathcal{F}$  if  $\mathcal{F}$  is a equivariant and there exist constants  $C > 0$ , and  $\gamma > 0$  and a splitting of the tangent bundle  $TM = E_a^s \oplus T\mathcal{F}_x \oplus E_a^u$  such that for all  $n \geq 0$  and  $v^s \in E_a^s$ ,  $v^u \in E_a^u$  and  $v^0 \in T\mathcal{F}$ :*

$$\begin{aligned} \|da_x^n(v^s)\| &\leq Ce^{-\gamma n}\|v\| \\ \|da_x^{-n}(v^u)\| &\leq Ce^{-\gamma n}\|w\|. \\ \|da_x^{\pm n}(v^0)\| &\geq Ce^{-\gamma n}\|w\| \end{aligned}$$

It is clear that for the trivial foliation  $\mathcal{F}(x) = \{x\}$ , a normally hyperbolic diffeomorphism w.r.t.  $\mathcal{F}$  is just an Anosov diffeomorphism. We refer to [16, 34] for the standard facts about normally hyperbolic transformations.

2.1.2. *Lyapunov Functionals and Coarse Lyapunov Spaces.* Assume that  $A = \mathbb{R}^k$  or  $\mathbb{Z}^k$  acts on a manifold  $X$  preserving a probability measure  $\mu$ . Then there are linear functionals  $\lambda : A \rightarrow \mathbb{R}$  and a measurable  $A$ -invariant splitting of the tangent bundle  $TM = \oplus E^\lambda$  such that for all  $0 \neq v \in E^\lambda$  and  $a \in A$ , the Lyapunov exponent of  $v$  is  $\lambda(a)$ . We call this the *Oseledets splitting* of  $TM$  for  $A$ , and each  $\lambda$  a *Lyapunov functional* or simply *weight* of the action. We let  $\Delta$  denote the collection of Lyapunov functionals. The Lyapunov splitting is a refinement of the Oseledets splitting for any single  $a \in A$ . For each  $\lambda \in \Delta$ , we let  $\bar{E}^\lambda = \bigoplus_{t>0} E^{t\lambda}$  be the *coarse Lyapunov distributions*. We refer to [2] for an extensive discussion of all these topics.

We will call the kernels of the weights *Weyl chamber walls*. Removing all Weyl chamber walls from  $\mathbb{R}^k$ , we call the connected components of the remainder the *Weyl chambers* of the action for the measure  $\mu$ . It is easy to see that Weyl chambers and their walls are independent of the measure when the action is totally Anosov. In fact, having an Anosov element in every Weyl chamber is sufficient. Then the whole Weyl chamber will consist of Anosov elements.

When the action is totally Anosov, the coarse Lyapunov spaces become integrable, and are indeed tangent to maximal non-trivial intersections of stable manifolds of elements of  $A$ . We call the resulting foliations *coarse Lyapunov foliations*. They are always transversally Hölder with leaves the same regularity as the action.

2.1.3. *Hölder Metrics.* Kalinin and Spatzier proved the following cocycle rigidity theorem in [21, Theorem 1.2] which is fundamental to all further developments. We remark that this is not a general cocycle result but rather applies specifically to the derivative cocycle. The starting point of the argument is that the derivative cocycle of the Weyl chamber wall along a closed orbit has to be isometric since non-isometric subexponential growth along a periodic orbit is not possible (e.g. since the dynamics at closed orbits is  $C^0$ -conjugate to its derivative cocycle along the coarse Lyapunov foliation [30]). This requires that the Lyapunov functionals have the same kernels, independent of the invariant measure in question. This is one of the places in the argument where we really need many Anosov elements. It implies that the Lyapunov functionals for different measures are proportional though not necessarily equal.

**Theorem 2.2.** *Let  $\alpha$  be a totally Cartan action of  $\mathbb{R}^k$ ,  $k \geq 2$ , on a compact smooth manifold  $M$  preserving an ergodic probability measure  $\mu$  with full support. Suppose that every Lyapunov hyperplane contains a generic one-parameter subgroup with a dense orbit. Then there exists a Hölder continuous Riemannian metric  $g$  on  $M$  such that for any  $a \in \mathbb{R}^k$*

$$\|a_*(v)\| = e^{\chi(a)} \|v\| \quad \text{for any } v \in E_\chi.$$

2.1.4. *Parameterizations of coarse Lyapunov foliations.* Henceforth we assume that  $\mathbb{R}^k$  acts on a compact manifold  $M$  by a totally Cartan action  $\alpha$ . We can always pass to a finite cover of  $M$  on which the coarse Lyapunov foliations are oriented. We will assume that all Weyl chamber walls act ergodically.

**Definition 2.3.** *For each  $\chi \in \Delta$ , let  $\eta_t^\chi$  denote the positively oriented geodesic flow in  $W^\chi$ , which satisfies*



$$(2.1) \quad a \cdot \eta_t^\chi(x) = \eta_{e^{\chi(a)t}}^\chi(a \cdot x)$$

2.1.5. *The invariant volume and its disintegrations.*

**Lemma 2.4.** *If  $\mu$  is the invariant volume for a Cartan action, then  $\mu_x^\mathcal{W}$  is equivalent to Lebesgue measure for any coarse Lyapunov foliation  $\mathcal{W}$ .*

*Proof.* Fix a Lyapunov exponent  $\chi$  with foliation  $\mathcal{W}$ , and some  $a \in \mathbb{R}^k$  such that  $\chi(a) > 0$ . Then by the Pesin entropy formula,  $h_\mu(a) = \sum_{\beta: \beta(a) > 0} \beta(a)$ . Furthermore, by a result of

Ledrappier and Young, the Pesin formula holds if and only if the conditional measures  $\mu^{\mathcal{F}^u}$  along the full unstable manifolds are absolutely continuous (in particular, they have full dimension). From Corollary 13.4 of [2], the dimension of the unstable measure is equal to the sum of the dimensions along each coarse Lyapunov foliation. Therefore, the dimensions in almost every coarse Lyapunov leaf is 1, and thus is equivalent to Lebesgue measure.  $\square$

**2.2. Free Products of Lie Groups.** Let  $U_1, \dots, U_r$  be topological groups. The *topological free product* of the  $U_i$ , denoted  $\mathcal{P} = U_1 * \dots * U_r$  is a topological group whose underlying group structure is exactly the usual free product of groups. That is, elements of  $\mathcal{P}$  are given by

$$u_1^{(i_1)} * \dots * u_N^{(i_N)}$$

where each  $i_k \in \{1, \dots, r\}$  and each  $u_k \in U_{i_k}$ . We call the sequence  $(i_1, \dots, i_k)$  the *combinatorial pattern* of the word. Each term  $u_k^{(i_k)}$  is also called a *leg* and each word is also called a *path*. This is because in the case of a free product of connected Lie groups, the word can be represented by a path beginning at  $e$ , moving to  $u_1^{(i_1)}$ , then to  $u_1^{(i_1)} * u_2^{(i_2)}$ , and so on through the truncations of the word. The multiplication is given by concatenation of words, and the only group relations are given by

$$(2.2) \quad u^{(i)} * v^{(i)} = (uv)^{(i)}$$

$$(2.3) \quad e^{(i)} = e \in \mathcal{P}$$

Notice that the relations (2.2) and (2.3) give rise to canonical embeddings of each  $U_i$  into  $\mathcal{P}$ . We therefore identify each  $U_i$  with its embedded copy in  $\mathcal{P}$ . The usual free product is characterized by a universal property: given a group  $H$  and any collection of homomorphisms  $\varphi_i : U_i \rightarrow H$ , there exists a unique homomorphism  $\Phi : \mathcal{P} \rightarrow H$  such that  $\Phi|_{U_i} = \varphi_i$ . The group topology on  $\mathcal{P}$  may be similarly defined by a universal property, as first proved by Graev [11]:

**Proposition 2.5.** *There exists a unique topology  $\tau$  on  $\mathcal{P}$  (called the free product topology) such that*

- (1) *each inclusion  $U_i \hookrightarrow \mathcal{P}$  is a homeomorphism onto its image, and*
- (2) *if  $\varphi_i : U_i \rightarrow H$  are continuous group homomorphisms to a topological group  $H$ , then the unique extension  $\Phi$  is continuous with respect to  $\tau$ .*

In the case when each  $U_i$  is a Lie group (or more generally, a CW-complex), Ordman found a more constructive description of the topology [27]. Indeed, the free product of Lie groups is covered by a disjoint union of *combinatorial cells*. For each finite word  $\bar{i} = (i_1, \dots, i_N)$ , let  $C_{\bar{i}} = U_{i_1} \times \dots \times U_{i_N}$  be the combinatorial cell for  $\bar{i}$ , with its usual product topology. Notice that each  $C_{\bar{i}}$  has a map  $\pi_{\bar{i}} : C_{\bar{i}} \rightarrow \mathcal{P}$  given by  $(u_1, \dots, u_N) \mapsto u_1^{(i_1)} * \dots * u_N^{(i_N)}$ . Furthermore, if  $C = \bigsqcup_{\bar{i}} C_{\bar{i}}$  and  $\pi : C \rightarrow \mathcal{P}$  is defined by setting  $\pi(x) = \pi_{\bar{i}}(x)$  when  $x \in C_{\bar{i}}$ , then  $\pi$  is onto.

**Lemma 2.6** ([41] Proposition 4.2).  *$\tau$  is the quotient topology on  $\mathcal{P}$  induced by  $\pi$ . In particular,  $f : \mathcal{P} \rightarrow Z$  is a continuous function to a topological space  $Z$  if and only if its pullback  $f \circ \pi_{\bar{i}}$  to  $C_{\bar{i}}$  is continuous for every combinatorial pattern  $\bar{i}$ .*

**Corollary 2.7.**  *$\mathcal{P}$  is path-connected and locally path-connected.*

Let  $\mathcal{P} = \mathbb{R}^{*r}$  be the  $r$ -fold free product of  $\mathbb{R}$ . Given continuous flows  $\eta^1, \dots, \eta^r$  on a space  $X$ , we may induce a continuous action  $\tilde{\eta}$  of  $\mathcal{P}$  on  $X$  by setting:

$$\tilde{\eta}(t_1^{(i_1)} * \dots * t_n^{(i_n)})(x) = \eta_{t_1}^{i_1} \circ \dots \circ \eta_{t_n}^{i_n}(x)$$

This can be observed to be an action of  $\mathcal{P}$  immediately, and continuity can be checked with either the universal property (considering each  $\eta^i$  as a continuous function from  $\mathbb{R}$  to  $\text{Homeo}(X)$ ) or directly using the criterion of Lemma 2.6. Given a word  $t_1^{(i_1)} * \dots * t_N^{(i_N)}$  (which we often call a path as discussed above), we may associate a path in  $X$  defined by:

$$\gamma \left( \frac{s+k-1}{m} \right) = \eta_{st_k}^{i_k}(x_{k-1}), s \in [0, 1], k = 1, \dots, m$$

where  $x_0$  is a base point and  $x_k = \eta_{t_k}^{i_k}(x_{k-1})$ . This gives more justification for calling each term  $t_k^{(i_k)}$  a leg. The points  $x_k$  are called the *break points* of the path.

Given  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ , let  $\psi_\lambda$  denote the automorphism of  $\mathcal{P}$  defined by:

$$t_1^{(i_1)} * t_2^{(i_2)} * \dots * t_m^{(i_m)} \mapsto (\lambda_{i_1} t_1)^{(i_1)} * (\lambda_{i_2} t_2)^{(i_2)} * \dots * (\lambda_{i_m} t_m)^{(i_m)}$$

**Definition 2.8.** *Fix a finite collection  $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset (\mathbb{R}^k)^*$ , and let  $a \in \mathbb{R}^k$ . Define  $\psi_a = \psi_{(e^{\alpha_1(a)}, \dots, e^{\alpha_n(a)})}$ , and  $\hat{\mathcal{P}} = \mathbb{R}^k \ltimes \mathcal{P}$ , with the semidirect product structure given by*

$$(a_1, \rho_1) \cdot (a_2, \rho_2) = (a_1 a_2, \psi_{a_2}^{-1}(\rho_1) * \rho_2)$$

where the group operation of  $\mathbb{R}^k$  is written multiplicatively.

**Proposition 2.9.** *Let  $\psi_a$  denote the automorphism of  $\mathcal{P}$  as defined in Definition 2.8. Let  $\mathcal{C}$  be any closed normal subgroup of  $\mathcal{P}$ , and  $H = \mathcal{P}/\mathcal{C}$  be a corresponding topological group factor of  $\mathcal{P}$ . If  $\psi_a(\mathcal{C}) = \mathcal{C}$  for all  $a \in \mathbb{R}^k$ , then  $\psi_a$  descends to a homomorphism  $\bar{\psi}_a$  of  $H$ . Furthermore, if  $H$  is a Lie group with Lie algebra  $\mathfrak{h}$  each generating copy of  $\mathbb{R}$  in  $\mathcal{P}$  projects to an eigenspace for  $d\bar{\psi}_a$  on  $H$ , and if the  $i^{\text{th}}$  and  $j^{\text{th}}$  copies of  $\mathbb{R}$  do not commute, their Lie algebra commutator of their generators in  $H$  is an eigenspace of  $\bar{\psi}_a$  with eigenvalue  $e^{\alpha_i(a) + \alpha_j(a)}$ . Furthermore, any nonzero  $Y$  which can be written as  $[Z_1, [Z_2, \dots, [Z_N, Z_0] \dots]]$  with  $Z_k = X_i$  or  $X_j$  for every  $k$  is an eigenvector of  $\bar{\psi}_a$  with eigenvalue of the form  $e^{u\alpha_i(a) + v\alpha_j(a)}$  with  $u, v \in \mathbb{Z}_+$ .*

*Proof.* If  $h = \rho\mathcal{C}$  is an element of the quotient group  $H$ , define  $\bar{\psi}_a(h) = \psi_a(\rho\mathcal{C}) = \psi_a(\rho)\mathcal{C}$ . Let  $\pi : \mathcal{P} \rightarrow H$  denote the projection from  $\mathcal{P}$  to  $H$ . For each  $i$ , let  $f_i : \mathbb{R} \rightarrow \mathcal{P}$  denote the inclusion of  $\mathbb{R}$  into the  $i^{\text{th}}$  copy of  $\mathbb{R}$  generating  $\mathcal{P}$ . Then  $\pi \circ f_i : \mathbb{R} \rightarrow H$  is a one-parameter subgroup of  $H$ , and we denote its corresponding Lie algebra element by  $X_i$ . Observe that since  $\bar{\psi}_a \circ \pi \circ f_i(t) = \pi \circ \psi_a \circ f_i = \pi \circ f_i(e^{\alpha_i(a)}t)$ ,  $d\bar{\psi}_a(X_i) = e^{\alpha_i(a)}X_i$ .

Finally, observe that

$$d\bar{\psi}_a[X_i, X_j] = [d\bar{\psi}_a X_i, d\bar{\psi}_a X_j] = [e^{\alpha_i(a)}X_i, e^{\alpha_j(a)}X_j] = e^{\alpha_i(a)+\alpha_j(a)}[X_i, X_j]$$

A similar argument shows it for arbitrary commutators.  $\square$

**Lemma 2.10.** *Suppose that a Lie group  $G$  is generated by subgroups  $U_1, \dots, U_n$ , and that  $\eta : G \curvearrowright X$  is an action of  $G$  on a compact metric space. If the restriction of  $\eta$  to each subgroup  $U_i$  is locally Lipschitz, then  $\eta$  is a locally Hölder action.*

**2.3. Lie Criteria.** In this subsection, we recall two deep results for Lie criteria of topological groups. The first was obtained by Gleason and Yamabe [40, Proposition 6.0.11]:

**Theorem 2.11** (Gleason-Yamabe). *Let  $G$  be a locally compact group. Then there exists an open subgroup  $G' \subset G$  such that, for any open neighborhood  $U$  of the identity in  $G'$  there exists a compact normal subgroup  $K \subset U \subset G'$  such that  $G'/K$  is isomorphic to a Lie group. Furthermore, if  $G$  is connected,  $G' = G$ .*

Recall that a locally compact group  $G$  has the *no small subgroups property* if for  $G'$  as in Theorem 2.11, a small enough neighborhood  $U \subset G'$  as above,  $U$  does not contain any compact normal subgroup besides  $\{1\}$ . Such a group  $G$  then is automatically a Lie group, by Theorem 2.11.

The second criterion is obtained by Gleason and Palais:

**Theorem 2.12** (Gleason-Palais). *If  $G$  is a locally path-connected topological group which admits an injective map from a neighborhood of  $G$  into a finite-dimensional topological space, then  $G$  is a Lie group.*

**Corollary 2.13.** *If  $\eta : \mathcal{P} \curvearrowright X$  is a group action on a topological space  $X$ , and  $\mathcal{C} \subset \text{Stab}_\eta(x)$  is a subgroup for every  $x$ , then  $\mathcal{C}$  is normal and the  $\eta$  action descends to  $\mathcal{P}/\mathcal{C}$ . Furthermore, if there is an injective continuous map from  $\mathcal{P}/\mathcal{C}$  to a finite-dimensional space  $Y$ , then  $\mathcal{P}/\mathcal{C}$  is a Lie group.*

### 3. BASIC STRUCTURES OF GEOMETRIC BRACKETS

First we define a fairly general notion, that of topological Cartan action. Our main result will be a classification of such actions when they also have local product structure. While interesting in its own right, this is absolutely needed for proving such a result even for smooth actions. Indeed, even for smooth actions, the coarse Lyapunov foliations are only Hölder transversely and it is not clear how to take Lie brackets of vector fields tangent to these foliations. Instead we will use “geometric brackets”. These are motivated by the usual geometric interpretation of the Lie bracket but explicitly use the rigid structure of the coarse Lyapunov foliations. Furthermore, one of the main steps is to define a factor as such an

action with much better transitivity properties for the coarse Lyapunov foliations. A priori, this factor action is not smooth, and in fact the quotient space is just a compact metric space. We then slowly show that all spaces and actions are indeed homogeneous.

The next definition imitates the structure one naturally gets from a smooth Cartan action. In particular, we get one dimensional laminations generalizing the coarse Lyapunov spaces and parametrizations rescaled by the Cartan actions from Theorem 2.2. Properties (4), (5) and (6) in the definition are stronger and stronger transversality and accessibility properties. (4) is crucial to define our geometric brackets below.

Fix  $a \in \mathbb{R}^k$ , and let  $\Phi \subset \Delta$  denote a subset of functionals  $\beta$  for which  $\beta(a)$  are negative and distinct for all  $\beta \in \Phi$ . We introduce an order on the set  $\Phi$ . Choose  $\mathbb{R}^2 \cong V \subset \mathbb{R}^k$  in which contains  $a$  and some nonzero  $\chi \in V^*$  such that  $\chi(a) = 0$ . Then  $\beta|_V \in V^* \cong \mathbb{R}^2$  for every  $\beta \in \Phi$ . Then  $\Phi|_V$  is contained completely on one side of the line, and we may order the weights in  $\Phi$  using the angle it makes with  $\chi$  (while the angles may depend on the a choice of metric in  $V^*$ , the ordering does not). Such an ordering is called a *circular ordering*.

**Definition 3.1.** *An action of  $\mathbb{R}^k$  on a finite-dimensional compact metric space  $X$  is said to be a topological Cartan action if there is a set  $\Delta \subset (\mathbb{R}^k)^*$  (called the weights of the action) and collection of locally free, Hölder flows  $\{\eta^\chi : \chi \in \Delta\} \setminus \{0\}$  such that*

- (1) for any  $\beta \in \Delta$ ,  $c\beta \notin \Delta$  for any  $c > 0$ ,  $c \neq 1$
- (2) for every  $x \in X$ ,  $t \mapsto \eta_t^\chi(x)$  is locally Lipschitz
- (3)  $a \cdot \eta_t^\chi(x) = \eta_{\text{ex}(a)_t}^\chi(a \cdot x)$  for every  $a \in \mathbb{R}^k$ ,  $t \in \mathbb{R}$  and  $x \in X$
- (4) if  $a_1, \dots, a_m \in \mathbb{R}^k$ ,  $\Phi \subset \Delta$  is the subset of weights such that  $\chi(a_i) < 0$  for all  $i = 1, \dots, m$ , and  $\{\chi_1, \dots, \chi_r\}$  is a circular ordering of  $\Phi$ , then for any combinatorial pattern  $\bar{\beta}$  whose letters are all from  $\Phi$ ,  $\tilde{\eta}(C_{\bar{\beta}})x \subset \tilde{\eta}(C_{(\chi_1, \dots, \chi_r)})x =: W_{(a_i)}^s(x)$  for every  $x \in X$  (recall the definitions of  $C_{\bar{\beta}}$  in Section 2.2).
- (5) if  $\hat{\eta}$  is the corresponding action of  $\hat{\mathcal{P}}$  (see Definition 2.8), then  $\hat{\eta}$  is transitive in the sense that for every  $x, y \in X$ , there exists  $\rho \in \hat{\mathcal{P}}$  such that  $\hat{\eta}(\rho)x = y$ .

We say that the topological Cartan action has local product structures if it also satisfies

- (6) for any ordering  $\bar{\beta} = (\beta_1, \dots, \beta_n)$  of  $\Delta$  which lists every weight exactly once, the map from  $C_{\bar{\beta}} \times \mathbb{R}^k \rightarrow X$  is a local homeomorphism.
- (7) the action preserves a probability measure  $\mu$  of full support such that the conditional measures  $\mu_x^\eta$  of  $\mu$  for  $\mu$ -a.e. orbit  $\eta_t^\chi x$  have full support on this orbit.

First let us explain how smooth Cartan actions relate to these topological notions.

**Proposition 3.2.** *Let  $\mathbb{R}^k \curvearrowright X$  be a  $C^{1,0}$  totally Cartan action on a compact manifold  $M$  preserving a volume  $\mu$ . Further assume that every Lyapunov hyperplane contains a generic one-parameter subgroup with a dense orbit. Then after passing to a finite cover of  $M$ , this data defines a topological Cartan action with local product structure.*

*Proof.* First pass to a finite cover of  $M$  so that every coarse Lyapunov foliation has an orientation. Then the special Hölder metrics from Theorem 2.2 determine norm 1 vector fields, positively oriented, and thus flows which get scaled as in property (3) above. The remaining properties are immediate from transversality of coarse Lyapunov foliations. The

product structure of the measure follows from the disintegration properties of the measure in Section 2.1.5.  $\square$

Fix a topological Cartan action  $\mathbb{R}^k \curvearrowright X$ . Given a subset  $\Phi = \{\beta_1, \dots, \beta_n\} \subset \Delta$ , let  $\tilde{\eta}_\Phi$  denote the induced action of  $\mathcal{P}_\Phi = \mathbb{R}^{*|\Phi|}$  on  $X$ , and  $\mathcal{C}_\Phi(x)$  denote the stabilizer of  $x$  for  $\tilde{\eta}_\Phi$ .

**Lemma 3.3.** *Let  $\mathbb{R}^k \curvearrowright X$  be a Cartan action satisfying the assumptions of Theorem 1.4, and suppose that  $\alpha, -c\alpha \in \Delta$  are negatively proportional weights. Then for any  $x_0$  which has a dense  $\ker \alpha$ -orbit,  $\mathcal{C}_{\{\alpha, -c\alpha\}}(x_0) \subset \mathcal{C}_{\{\alpha, -c\alpha\}}(x)$  for all  $x \in x_0$ , and  $\mathcal{P}_{\{\alpha, -c\alpha\}}/\mathcal{C}_{\{\alpha, -c\alpha\}}(x_0)$  is a Lie group.*

*Proof.* Notice that in the group  $\hat{\mathcal{P}}$ , if  $a \in \mathbb{R}^k$ ,  $a\mathcal{C}_{\{\alpha, -c\alpha\}}(x)a^{-1} = \mathcal{C}_{\{\alpha, -c\alpha\}}(a \cdot x)$  by (2.1), and if  $a \in \ker \alpha$ ,  $a\mathcal{C}_{\{\alpha, -c\alpha\}}(x)a^{-1} = \mathcal{C}_{\{\alpha, -c\alpha\}}(x)$ . In particular,  $\mathcal{C}_{\{\alpha, -c\alpha\}}(\cdot)$  is constant on  $\ker \alpha$  orbits. Since  $\ker \alpha$  has a dense orbit by assumption,  $\mathcal{C}_{\{\alpha, -c\alpha\}}(x_0)$  is contained in any  $\mathcal{C}_{\{\alpha, -c\alpha\}}(x)$  for every  $x$ .

Hence, the quotient  $\mathcal{P}_{\{\alpha, -c\alpha\}}/\mathcal{C}_{\{\alpha, -c\alpha\}}(x_0)$  is a topological group, which is locally path-connected since  $\mathcal{P}_{\{\alpha, -c\alpha\}}$  is locally path-connected. Furthermore, the evaluation map  $\rho \mapsto \rho \cdot x_0$  is injective and continuous, so the quotient group is a Lie group by Lemma 2.12.  $\square$

Let  $\mathcal{G}_{\{\alpha, -c\alpha\}}$  be the group  $\mathcal{P}_{\{\alpha, -c\alpha\}}/\mathcal{C}_{\{\alpha, -c\alpha\}}$ . We now describe standard generators for the groups  $PSL(2, \mathbb{R})$  and the Heisenberg group  $\text{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$ , with

algebras  $\mathfrak{sl}(2, \mathbb{R})$  and  $\mathfrak{h}$ , respectively. For  $\mathfrak{sl}(2, \mathbb{R})$ , we call  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  the stan-

dard unipotent generators, and  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  the corresponding neutral element. For  $\mathfrak{h}$ , we call

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$  the standard unipotent generator and  $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  the corre-

sponding neutral element. Notice that in both cases, the map which multiplies one of the standard unipotent generators by  $\lambda$ , the other by  $\lambda^{-1}$  and fixes the corresponding neutral element is an automorphism of the Lie algebra.

**Proposition 3.4.** *If  $\chi, -c\chi \in \Delta$ , then  $\mathcal{P}_{\{\alpha, -c\alpha\}}/\mathcal{C}_{\{\alpha, -c\alpha\}}$  is either isomorphic to  $\mathbb{R}^2$ , (some cover of)  $PSL(2, \mathbb{R})$  or the Heisenberg group. Furthermore, the flows  $\eta^\chi$  and  $\eta^{-c\chi}$  are the one-parameter subgroups of the standard unipotent generators and the corresponding neutral element generates a one-parameter subgroup of the topological Cartan action.*

*Proof.*  $\mathcal{G}_{\{\chi, -c\chi\}}$  is a Lie group by Lemma 3.3, which is generated by two one-parameter subgroups corresponding to the flows  $\eta^\chi$  and  $\eta^{-c\chi}$ . Let  $v_\pm$  denote the elements of  $\text{Lie}(\mathcal{G}_{\{\chi, -c\chi\}})$  which generate  $\eta^{\pm\chi}$ , and assume they don't commute (if they commute we are done). Notice that  $v_0 = [v_+, v_-]$  will generate a one-parameter subgroup which expands or contracts at the rate  $(1 - c)\chi(a)$  for  $a \in \mathbb{R}^k$  in the distance given by the Lie group. By Lemma 2.10, the distance in the Lie group is distorted in only a Hölder way. So for each element  $a \in \mathbb{R}^k$ , if  $c > 1$  and  $\chi(a) > 0$ , each point in the orbit of the one parameter subgroup generated by  $v_0$  will diverge at an exponential rate. Similarly, one may see exponential expansion or decay

properties for different choices of  $a$  and different possibilities for  $c$ . Therefore, if  $c \neq 1$ , the orbit of this one-parameter subgroup is contained in the coarse Lyapunov submanifold for either  $\chi$  or  $-\chi$ . Since Cartan implies that each coarse Lyapunov foliation is one-dimensional,  $c = 1$  and  $v_0$  is neither expanded nor contracted. By a similar argument,  $[v_0, v_{\pm}] \subset \mathbb{R}v_{\pm}$ . In particular, as a vector space  $\text{Lie}(\mathcal{G}_{\{\chi, -\chi\}})$  is generated by  $\{v_-, v_0, v_+\}$ . One quickly sees that the only possibilities for the 3-dimensional Lie algebras are the ones listed.

Assume that  $[v_-, v_+] \neq 0$ , and let  $f^X$  denote the flow generated by  $v_0$ . Then notice that  $f_t^X(a \cdot x) = a \cdot f_t^X(x)$ , since by Proposition 2.9,  $v_0$  is an eigenvector of eigenvalue 1 for the resulting automorphism of  $\mathcal{G}_{\{\chi, -\chi\}}$ . Fix a regular element  $a_1 \in \mathbb{R}^k$ . We show this for sufficiently small  $t$ , so we may assume that if  $f = f_t^X$  and  $d(f(x), x) < \delta$ . For each  $x$ , there exists a path of the form  $\rho_x = t_1^{(\alpha_1)} * \dots * t_m^{(\alpha_m)} * s_1^{(\beta_1)} * \dots * s_l^{(\beta_l)} * b$  such that  $f(x) = \rho_x * x$ , and  $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l$  is a circular ordering on the weights, with  $\alpha_i(a) < 0$  and  $\beta_j(a) > 0$ . We claim that  $t_i = 0$  for every  $i$ . Indeed, by applying  $a_1$  to  $\rho$ , we know that the  $\alpha_i$  components will all expand, and the  $b$  and  $s_j$  components remain small. Thus, if  $\delta$  is sufficiently small (relative to the injectivity radius of  $X$ ), we may apply conclude that  $d(f(x), x) > \delta$ , a contradiction. A similar argument shows that  $s_j = 0$  for every  $j$ .

Therefore,  $f(x) = b_x \cdot x$  for some  $b_x \in \mathbb{R}^k$ . Notice that on a free orbit,  $b_x$  is unique and continuously varying. Since  $f$  commutes with the  $\mathbb{R}^k$  action,  $f(a \cdot x) = b_{a \cdot x} a \cdot x = ab_x \cdot x$ , thus  $b_x$  is constant on free orbits.  $\square$

Fix regular elements  $a_1, \dots, a_m \in \mathbb{R}^k$ , and let  $W_{(a_j)}^s$  be the common stable manifold through  $x$  as defined in Definition 3.1(4). Let  $\Phi = \{\beta_1, \dots, \beta_n\} \subset \Delta$  denote the set of weights  $\beta$  for which  $\beta(a_j) < 0$  for  $j = 1, \dots, m$  listed with a circular ordering.

**Proposition 3.5.** *The map  $f_{(a_j),x} : \mathbb{R}^n \rightarrow W_{(a_j)}^s(x)$  defined by*

$$f_{(a_j),x}(u_1, \dots, u_n) = \eta_{\beta_n}(u_n) \eta_{\beta_{n-1}}(u_{n-1}) \dots \eta_{\beta_1}(u_1)x$$

*is a local homeomorphism.*

*Proof.* It is clear that the image of  $f_{(a_j),x}$  is contained in  $W_{(a_j)}^s(x)$ . We first show injectivity. Suppose  $f_{(a_j),x}(u_1, \dots, u_n) = f_{(a_j),x}(v_1, \dots, v_n)$ . Then  $\eta_{\beta_n}(u_n) \eta_{\beta_{n-1}}(u_{n-1}) \dots \eta_{\beta_1}(u_1)x = \eta_{\beta_n}(v_n) \eta_{\beta_{n-1}}(v_{n-1}) \dots \eta_{\beta_1}(v_1)x$  so

$$(3.1) \quad \eta_{\beta_1}(u_1)^{-1} \dots \eta_{\beta_n}(v_n^{-1}) \eta_{\beta_n}(u_n) \dots \eta_{\beta_1}(u_1)x = x.$$

Let  $C_0 \subset \mathbb{R}^k$  be the cone of nonnegative linear combinations of elements of  $\Phi$ . By construction, if  $\chi \in C_0 \cap \Delta$ ,  $\chi \in \Phi$ . Let  $C$  be the intersection of  $C_0$  with the 2-dimensional subspace determining the circular ordering. Since  $\beta_n$  is the bounding weight for the cone  $C$ , we may choose  $b_n$  such that  $\beta_n(b_n) = 0$  and  $\beta_i(b_n) < 0$  for all  $i < n$ . Applying  $b_n$  to both sides of (3.1) and using the relation (2.1) shows that in fact  $v_n = u_n$ , and we may cancel the innermost term. Now we proceed by induction. Suppose that we have concluded that  $v_k = u_k$  for  $k = \ell + 1, \dots, n$ . Then choose  $b_\ell$  such that  $\beta_\ell(b_\ell) = 0$  and  $\beta_i(b_\ell) < 0$  for  $i < \ell$ . Then since we have already canceled all terms greater than  $\ell$  in (3.1), applying  $b_\ell$  to both sides and applying (2.1) again implies that  $v_\ell = u_\ell$ .

So  $f_{(a_j),x}$  is injective.  $C$  is contained in a half-space  $H$  for a root  $\beta$  if and only if  $\beta(a_j) < 0$  for every  $j$ . Since the  $\beta_j$  are *all* such weights satisfying this property by definition,  $\dim(W_{(a_j)}^s) = n$ , and  $f_{(a_j),x}$  is a local homeomorphism by invariance of domain.  $\square$

Given  $\chi_1, \chi_2 \in \Delta$  which are not proportional, let  $D = D(\chi_1, \chi_2) = \{t\chi_1 + s\chi_2 : t, s \geq 0\} \cap \Delta$ . We pick some  $a \in \mathbb{R}^k$  such that  $\chi_1(a), \chi_2(a) < 0$  and every weight  $\chi$  evaluates to a different number on  $a$ . We may then relabel and reorder the elements of  $D$  as described before the statement of Proposition 3.5 since they all belong to a stable manifold.

**Lemma 3.6.** *Let  $\chi_1$  and  $\chi_2$  be linearly independent weights and  $W_1, \dots, W_m$  be the Weyl chambers of  $\alpha$  such that  $\chi_1$  and  $\chi_2$  are both negative on every  $W_i$ . Then if  $a_j \in W_j$  are regular elements in each chamber,  $D(\chi_1, \chi_2) = \{\beta : \beta(a_j) < 0\}$  for  $j = 1, \dots, m$ .*

*Proof.* One inclusion is obvious. For the other, suppose that  $\beta$  satisfies  $\beta(a_j) < 0$  for every  $j = 1, \dots, m$ . We first show  $\beta$  is a linear combination of  $\chi_1$  and  $\chi_2$ . Suppose that  $\beta$  is linearly independent of  $\chi_1$  and  $\chi_2$ . Then there exists  $b \in \mathbb{R}^k$  such that  $\beta(b) = 1$  and  $\chi_1(b) = \chi_2(b) = 0$ . Choose any  $a_j$ , and consider  $a_j + tb$ . Notice that any Weyl chamber this curve passes through must be one of the  $W_i$ , since the evaluations of  $\chi_1$  and  $\chi_2$  will still be negative. But for sufficiently large  $t$ ,  $\beta(a_j + tb) > 0$ . Hence there is a Weyl chamber such that  $\chi_1$  and  $\chi_2$  are both negative, but  $\beta$  is positive. This contradicts the fact that  $\beta$  is negative on each  $W_i$ , so by contradiction,  $\beta$  must be linearly dependent with  $\chi_1$  and  $\chi_2$ .

So we may write  $\beta = t\chi_1 + s\chi_2$ . We must show that  $t, s \geq 0$ . Indeed, assume  $t < 0$ . Since  $\chi_1$  and  $\chi_2$  are linearly independent, it follows that there exists  $b \in \mathbb{R}^k$  such that  $\chi_1(b) = 1$  and  $\chi_2(b) = 0$ . Choosing any  $a_j$  from the given elements, the curve  $a_j - rb$ ,  $r \in \mathbb{R}$  satisfies  $\chi_2(a_j - rb) = \chi_2(a_j) < 0$ ,  $\chi_1(a_j - rb) = \chi_1(a_j) - r < 0$  if  $r \geq 0$ , and  $\beta(a_j - rb) = t\chi_1(a_j) - tr + \chi_2(a_j)$ . Thus, if  $r$  is sufficiently large,  $\chi_1$  and  $\chi_2$  are negative but  $\beta$  is positive. Hence there is a Weyl chamber such that  $\chi_1$  and  $\chi_2$  are negative but  $\beta$  is positive. This is a contradiction, so  $t \geq 0$ . The argument for  $s$  is completely symmetric.  $\square$

**Lemma 3.7.** *If  $t, s \in \mathbb{R}$ , there exists a unique  $\rho^{\chi_1, \chi_2}(t, s, x) \in \mathcal{P}_{D \setminus \{\chi_1, \chi_2\}}$  such that:*

$$[t^{(\chi_1)}, s^{(\chi_2)}] * \rho^{\chi_1, \chi_2}(t, s, x) \in \mathcal{C}_D(x)$$

*and has the property that if  $\{\beta_1, \dots, \beta_n\}$  is a circular ordering of  $D(\chi_1, \chi_2)$ , there are unique functions  $\rho_{\beta_i}^{\chi_1, \chi_2} \in \mathbb{R}_{\beta_i}$  such that:*

$$\rho^{\chi_1, \chi_2}(t, s, x) = \rho_{\beta_n}^{\chi_1, \chi_2}(t, s, x) * \dots * \rho_{\beta_1}^{\chi_1, \chi_2}(t, s, x)$$

*Furthermore,  $\rho^{\chi_1, \chi_2}$  and  $\rho_{\beta_i}^{\chi_1, \chi_2}$  are continuous functions in  $t, s, x$ , and*

$$(3.2) \quad e^{\beta_i(a)} \rho_{\beta_i}^{\chi_1, \chi_2}(s, t, x) = \rho_{\beta_i}^{\chi_1, \chi_2}(e^{\chi_1(a)}s, e^{\chi_2(a)}t, a \cdot x).$$

*Proof.* Since  $\chi_1$  and  $\chi_2$  are not proportional, it follows from Proposition 3.5 and Lemma 3.6 that there exists some unique path  $\rho_x = v_n^{(\beta_n)} * \dots * v_1^{(\beta_1)} \in \mathcal{P}_D$  satisfying the equation. We wish to show that it has trivial  $\chi_1$  and  $\chi_2$  components. Notice that by linear independence

of  $\chi_1, \chi_2$ , we may choose  $a$  such that  $\chi_1(a) = 0$  and  $\chi_2(a) < 0$ . Then notice that by (2.1), for any  $x \in X$ :

$$(3.3) \quad \alpha(na) \left[ u_1^{(\chi_1)}, u_2^{(\chi_2)} \right] \alpha(-na)x = \left[ u_1^{(\chi_1)}, e^{n\chi_2(a)} u_2^{(\chi_2)} \right] x \xrightarrow{n \rightarrow \infty} x$$

Since the action of  $\tilde{\eta}$  is Hölder, the convergence is exponential since  $\chi_2(a) < 0$ . But if  $\rho_x$  has a nontrivial  $\chi_1$  component, that component does not decay, since it is isometric. Therefore it must be trivial. By a completely symmetric argument, the  $\chi_2$  component is trivial.  $\square$

*Remark 3.8.* A priori, the functions defined in Lemma 3.7 may depend on the circular ordering chosen, but any such ordering will agree with the the circular ordering induced from the embedding  $D(\chi_1, \chi_2) \ni \beta = u\chi_1 + v\chi_2 \mapsto (u, v) \in \mathbb{R}^2$ .

**Lemma 3.9.** *Let  $\mathcal{P}_\Omega \subset \mathcal{P}$  be a subgroup such that  $\Omega \subset \Delta$  and there exists  $a \in \mathbb{R}^k$  such that  $\alpha(a) < 0$  for all  $\alpha \in \Omega$ . Then if the action of  $\mathcal{P}_\Omega$  factors through a Lie group action  $H$ :*

- (1)  $H$  is nilpotent
- (2) if  $\rho_\chi^{\alpha, \beta}(s, t, x) \not\equiv 0$ , then  $\chi = u\alpha + v\beta$  with  $u, v \in \mathbb{Z}_+$
- (3)  $\rho_\chi^{\alpha, \beta}(s, t, x) = cs^u t^v$  for some  $c \in \mathbb{R}$

*Proof.* By Proposition 2.9, the automorphism  $\psi_a$  descends to an automorphism of  $H$ . Since the eigenvalues  $d\bar{\psi}$  are all less than 1,  $\bar{\psi}_a$  is a contracting automorphism of  $H$ , and  $H$  is nilpotent. If  $\rho_\chi^{\alpha, \beta}(s, t, x) \not\equiv 0$ , then the corresponding subgroups of  $H$  do not commute. Furthermore, the Baker-Campbell-Hausdorff formula, together with the end of Proposition 2.9, implies that  $\{u\alpha + v\beta : u, v \geq 0, u, v \in \mathbb{Z}\}$  is a closed subalgebra. In particular, if a weight  $\chi$  appears in  $[s^\alpha, t^{(\beta)}]$ , then  $\chi$  must have integral coefficients, as claimed. Finally, the Baker-Campbell-Hausdorff formula also implies that each  $\rho_\chi^{\alpha, \beta}$  is a polynomial whenever  $H$  is nilpotent, and (3.2) implies that that polynomial is  $u$ -homogeneous in  $s$  and  $v$ -homogeneous in  $t$ .  $\square$

**Lemma 3.10.** *If  $\mathbb{R}^k \curvearrowright X$  is a  $C^{1, \theta}$  Cartan action on a smooth manifold with  $k \geq 2$ ,  $\alpha, \beta \in \Delta$  are Lyapunov exponents, and  $\chi = u\alpha + v\beta$  with  $u, v < 1$ , then  $\rho_\chi^{\alpha, \beta}(s, t, x) \equiv 0$ .*

*Proof.* Choose  $a$  such that  $\alpha(a), \beta(a) < 0$ . Let  $\mathcal{W}$  be the foliation whose tangent bundle  $\bigoplus_{\chi \in D(\alpha, \beta)} T\mathcal{W}^\chi$ . By [13], there exists a family of charts  $\varphi_x : \mathbb{R}^l \rightarrow X$  such that  $\varphi_x(\mathbb{R}^l) = \mathcal{W}(x)$  and  $F_x = (\varphi_{a_0 \cdot x})^{-1} \circ a_0 \circ \varphi_x : \mathbb{R}^k \rightarrow \mathbb{R}^k$  is a subresonance polynomial for  $a_0$ . That is, if  $\{\chi_1, \dots, \chi_l\} = D(\alpha, \beta) \cup \{\alpha, \beta\}$ , then  $F_x(t_1, \dots, t_l) = (p_{x,1}(\mathbf{t}), \dots, p_{x,l}(\mathbf{t}))$  with  $p_{x,i}$  a polynomial whose monomial terms  $t_1^{k_1} \dots t_l^{k_l}$  all satisfy  $\sum k_j \chi_j(a) \leq \chi_i(a)$ .

Let  $\Omega = \{u\alpha + v\beta \in D(\alpha, \beta) : u, v < 1\}$ . Let us assume that we have ordered the weights  $\{\chi_i\}$  so that  $\alpha = \chi_1, \beta = \chi_2$  and  $\Omega = \{\chi_3, \dots, \chi_m\}$ . Observe that the subresonance condition implies that  $p_{x,i}$  cannot depend on  $t_1$  or  $t_2$  for all  $i = 3, \dots, m$ .

Now, again by [13], any diffeomorphism which commutes with  $a_0$  is also a resonant polynomial. Choose  $a$  close to  $\ker \beta$ . That is, pick  $a$  such that  $\alpha(a) = -1$  and  $-\varepsilon < \beta(a) < 0$ , with  $\varepsilon$  so small so that  $\alpha(a) < \chi(a)$  for every  $\chi \in \Omega$ . This implies that the  $\alpha$  Coarse lyapunov leaves



are contained in the translates of the hyperspace  $E = \{\mathbf{t} \in \mathbb{R}^l : t_i = 0, i = 3, \dots, m\}$ , since if the  $t_i$  coefficients differed, the points would not contract at sufficient speed. Similarly, by choosing an analogous  $b$  close to  $\ker \alpha$ , we know that in the normal form coordinates, the  $\beta$  coarse Lyapunov leaves are contained in the affine spaces parallel to  $E$ . Thus, the geometric bracket is contained in the hyperspace parallel to  $E$ . Since the  $\chi$  leaves are transverse to  $E$  for all  $\chi \in \Omega$ , this gives the result.  $\square$

In the remainder of this paper, we will prove that certain canonical cycles are all constant. For this we first prove that geometric brackets are constant. The arguments are much easier though different for Cartan actions of  $\mathbb{R}^k$  for  $k \geq 3$  as compared to  $\mathbb{R}^2$  actions. We will treat these cases in two separate parts below. They can be read independently of each other.

We note that Cartan actions for  $k \geq 3$  were treated by [21] for  $C^\infty$  actions, using transitivity of isometries on sums of coarse Lyapunov spaces and stronger ergodicity assumptions. Their argument does not generalize to  $C^{1,\theta}$  actions. More importantly, their method completely fails for  $\mathbb{R}^2$  Cartan actions as the transitive isometries arose from the intersection of two Lyapunov hyperplanes which are trivial in  $\mathbb{R}^2$ . Indeed, the main achievement of this paper consists in establishing the rank 2 case, and requires entirely new ideas and methods.

#### 4. IDEALS OF WEIGHTS

Our main result has a much simpler proof if all coarse Lyapunov foliations have dense leaves. However, this is not always the case, even for  $\mathbb{R}^2$  actions, see for examples twisted Weyl chamber flows (Section A.1.3). In such examples, the coarse Lyapunov foliations tangent to a fiber are at best dense in that fiber. This leads us to introduce natural quotients of Cartan actions obtained by collapsing whole coarse Lyapunov foliations. Such factors are closely tied with geometric brackets and we are naturally led to study ideals of weights. The main insight here is in Lemma 4.4.

**Definition 4.1.** *Given a subset  $E \subset \Delta$ , let  $\mathcal{A}^E(x)$  be the  $E$ -orbit closure of  $x$ . By this, we mean the set of all points  $y \in X$  which are endpoints of broken paths using the foliations from  $E$  starting at  $x$ , together with their limit points. Then set*

$$\mathcal{I}(E) = \{f \in C^0(X) : f(\eta_t^\chi(x)) = f(x) \text{ for every } x \in X, t \in \mathbb{R}, \chi \in E\}$$

and  $M^E(x) = \bigcap_{f \in \mathcal{I}(E)} f^{-1}(f(x))$ . The closure of  $E$  is the set

$$\bar{E} = \{\chi \in \Delta : \eta_t^\chi(M^E(x)) = M^E(x) \text{ for every } x \in X, t \in \mathbb{R}\}.$$

$E \subset \Delta$  is an ideal if  $\bar{E} = E$ . An ideal  $E$  is called proper if  $E \neq \emptyset$  and  $M^E(x) \neq M$ . We say that  $E$  is maximal if it is not properly contained in any proper ideal.

Notice that  $M^E(x)$  are all closed sets which contain the sets  $\mathcal{A}^E(x)$ .

**Lemma 4.2.** *If  $\Delta$  has no proper ideals, then for every weight  $\chi \in \Delta$ , the only continuous  $\eta^\chi$ -invariant functions are constant.*

*Proof.* Let  $\chi \in \Delta$ , and consider  $\overline{\{\chi\}}$ . Since  $\Delta$  has no proper ideals,  $E = \overline{\{\chi\}}$  satisfies  $M^E(x) = X$ . Then if  $f$  is  $\eta^\chi$ -invariant, it is invariant for all  $\beta \in E$ , and hence constant on  $M^E(x) = X$ .  $\square$

**Definition 4.3.** If  $\alpha, \beta \in \Delta$  are weights, let

$$[\alpha, \beta] = \{\chi \in \Delta : \rho_\chi^{\alpha, \beta}(s, t, x) \neq 0 \text{ for some } t, s \in \mathbb{R}, x \in X\}.$$

**Lemma 4.4.** If  $E$  is an ideal and  $\alpha \in E$ ,  $\beta \in \Delta$ , and  $\chi \in [\alpha, \beta]$ , then  $\chi \in E$ .

*Proof.* Choose  $s, t \in \mathbb{R}$  and  $x \in X$  such that  $\rho_\chi^{\alpha, \beta}(s, t, x) \neq 0$  (we without loss of generality assume it is positive). We first deal with the case in which  $\rho_\gamma^{\alpha, \beta}(s, t, x) = 0$  if  $\gamma \neq \chi$  (ie, there is only one weight in the commutator of  $\alpha, \beta$ ). Since the generic points for the Cartan action are dense and  $\rho_\chi^{\alpha, \beta}(s, t, x)$  is continuous, we may without loss of generality assume that  $x$  is generic for some ray in  $\{ra : r \in \mathbb{R}_+\} \subset \ker \chi$  such that  $\beta(a) = -1$ . Let  $f \in \mathcal{I}(E)$ , so that in particular,  $f$  is  $\eta^\alpha$  invariant. Fix any  $y \in X$ . Then choose  $r_k \rightarrow +\infty$  such that  $(r_k a) \cdot x \rightarrow y$ . Then if  $u = \rho_\chi^{\alpha, \beta}(s, t, x)$ ,

$$\begin{aligned} f(y) &= \lim_{k \rightarrow \infty} f((r_k a) \cdot x) \\ &= \lim_{k \rightarrow \infty} f(\eta_{se^{\alpha(a)r_k}}^\alpha((r_k a) \cdot x)) \\ &= \lim_{k \rightarrow \infty} f(\eta_{te^{-r_k}}^\beta \eta_{se^{\alpha(a)r_k}}^\alpha((r_k a) \cdot x)) \\ &= \lim_{k \rightarrow \infty} f(\eta_{-se^{\alpha(a)r_k}}^\alpha \eta_{te^{-r_k}}^\beta \eta_{se^{\alpha(a)r_k}}^\alpha((r_k a) \cdot x)) \\ &= \lim_{k \rightarrow \infty} f(\eta_{-te^{-r_k}}^\beta \eta_{-se^{\alpha(a)r_k}}^\alpha \eta_{te^{-r_k}}^\beta \eta_{se^{\alpha(a)r_k}}^\alpha((r_k a) \cdot x)) \\ &= \lim_{k \rightarrow \infty} f((r_k a) \cdot [\eta_t^\beta, \eta_s^\alpha]x) \\ &= \lim_{k \rightarrow \infty} f((r_k a) \cdot \eta_u^\chi(x)) \\ &= \lim_{k \rightarrow \infty} f(\eta_u^\chi((r_k a) \cdot x)) \\ &= f(\eta_u^\chi(y)) \end{aligned}$$

Notice that the equality on the second equation and the fourth equation follows from the  $\eta^\alpha$  invariance, and the third line and fifth equation follows from the uniform continuity of  $f$  and the fact that  $e^{-r_k} \rightarrow 0$ . Notice also that by sending  $s \rightarrow 0$ , the corresponding  $\rho_\chi^{\alpha, \beta}(s, t, x) = u \rightarrow 0$ . Therefore, we have  $f(y) = f(\eta_v^\chi(y))$  for all  $v \in [0, u)$  and all  $y \in X$ . Since it holds for every  $y$  and  $[0, u)$  generates  $\mathbb{R}_+$ , it holds for every  $u \in \mathbb{R}_+$ , and hence  $u \in \mathbb{R}$ . Therefore,  $f$  is  $\chi$ -invariant and  $\chi \in E$ .

Now address the general case, in which  $\rho^{\alpha, \beta}(s, t, x)$  is written as a product of nontrivial paths in weights  $\gamma_1, \dots, \gamma_k$ , ordered using the cyclical ordering in such a way that there exists  $a_i$  with  $\alpha(a_i) > 0$ ,  $\beta(a_i) < 0$ ,  $\gamma_i(a_i) = 0$ , and  $\gamma_j(a_i) < 0$  if  $j > i$  and  $\gamma_j(a_i) > 0$  if  $j < i$ . We will show that each  $\gamma_i \in E$ , and hence  $\chi \in E$  (since  $\chi = \gamma_i$  for some  $i$ ).

Notice that by an argument identical to the simplified case, choosing  $a_1$  will show that  $\gamma_1 \in E_0$ . Assume we have shown  $\gamma_1, \dots, \gamma_i \in E$ . By Lemma 2.4, the measure  $\mu$  disintegrates as Lebesgue on each coarse Lyapunov foliation. So we may choose  $x$  such that if  $\rho_{\gamma_j}^{\alpha, \beta}(s, t, x) = u_j$ ,  $x_0 = x$  and  $x_j = \eta_{u_j}^{\gamma_j} x_{j-1}$ , then  $x_i$  is generic. Now, the choice of  $a_i$  guarantees that in constructing the chain of equalities to obtain invariance under  $\gamma_i$ , the terms for  $\eta^{\gamma_j}$  with  $j < i$  will have their flow times become very large. However, by the inductive hypothesis,

$f$  is invariant under  $\gamma_j$ . The terms with  $j > i$  will have their flow times contracted, so by uniform continuity they disappear. The only surviving term is  $j = i$ , yielding the desired invariance for  $f$ , and hence that  $\gamma_i \in E$ .  $\square$

**Proposition 4.5.** *The sets  $M^E(x)$  can be obtained by the following transfinite induction: Let  $S_0(x) = \overline{\mathcal{A}^E(x)}$ ,  $\Sigma_{i+1}(x) = \{y \in M : S_i(x) \cap S_i(y) \neq \emptyset\}$  and  $S_{i+1}(x) = \overline{\Sigma_{i+1}(x)}$ . Then  $M^E(x)$  is the limit of this procedure for a transfinite induction, with the termination condition being that  $S_\omega(x) = S_{\omega+1}(x)$ .*

*Proof.* The limits of the  $S_i$ 's are closed because they terminate when  $S_i(x) = S_{i+1}(x)$ , so  $S_i(x) = \overline{\Sigma_{i+1}(x)}$ . Furthermore, they partition the space, since on the terminating  $\omega$ , if  $S_\omega(x) \cap S_\omega(y) \neq \emptyset$ , then  $y \in \Sigma_{\omega+1}(x) \subset S_{\omega+1}(x) = S_\omega(x)$ , by definition.

Because the sets  $S_\omega(x)$  partition the space, we may form a compact Hausdorff space from the quotient. By construction,  $S_\omega(x) \subset M^E(x)$ , since all continuous functions evaluate the same on each  $S_i$  by induction. Furthermore, the evaluation  $M \rightarrow M/\sim$ , with  $\sim$  being the equivalence relation  $x \sim y$  if and only if  $y \in S_\omega(x)$  yields a continuous function which separates the sets  $S_\omega(x)$ . Since each  $S_\omega(x)$  is invariant under the flows from  $E$ , we get that we may separate points from different such  $S_\omega$ 's with continuous functions into a compact Hausdorff space, and therefore continuous functions into  $\mathbb{R}$ . Hence,  $M^E(x) \subset S_\omega(x)$ .  $\square$

If  $E$  is an ideal, let  $X^E = X/\sim$  be the set of equivalence classes of  $\sim$ , where  $x \sim y$  if and only if  $M^E(x) = M^E(y)$ .

**Proposition 4.6.** *The Cartan action and flows  $\eta_\beta$  with  $\beta \notin E$  descend to actions on  $X^E$ .*

*Proof.* By definition, the sets  $M^E(x)$  are equivariant under the Cartan action, so it descends. We wish to show that  $\eta_\beta$  also acts equivariantly. That is, we wish to show that if  $y \in M^E(x)$ , then  $\eta_\beta(u)y \in M^E(\eta_\beta(u)x)$ . If  $y \in \mathcal{A}^E(x)$ , this follows from Lemma 4.4. Since the actions  $\eta_\beta$  are continuous, this passes to the closure,  $S_0(x)$  as defined in Proposition 4.5. Inductively and using continuity, we get that it is true for every  $S_i(x)$ , in particular for  $S_\omega(x) = M^E(x)$ . So the flows  $\eta_\beta$  act on  $X$ .  $\square$

We call the induced action  $\mathbb{R}^k \curvearrowright X^E$  an *ideal factor* of the Cartan action on  $X$ .

**Corollary 4.7.**  *$X^E$  is a compact metric space when given the Hausdorff metric.*

*Proof.* Since the points of  $X^E$  are compact sets, we only need to show that  $X^E$  is a closed subset of all compact subsets of  $X$ . We use the holonomies of the orbit of the Cartan action and the coarse Lyapunov flows  $\eta^\chi$ . Suppose that  $M^E(x_k)$  converges to some compact  $K \subset X$ , and note that we may choose  $x_k \rightarrow x$  for some  $x \in K$ . We first show  $M^E(x) \subset K$ . It suffices to show that any function invariant by  $\eta^\chi$ ,  $\chi \in E$  is constant on  $K$ . Any such function is constant on  $M^E(x_k)$ , so it is constant on the limit set,  $K$ .

We now show the opposite inclusion. Since  $x_l \rightarrow x$ , it is eventually in a small enough neighborhood for which the map  $(a, t_1, \dots, t_n) \mapsto a \cdot \eta_{t_1}^{\chi_1} \dots \eta_{t_n}^{\chi_n}(x)$  is a local homeomorphism from  $\mathbb{R}^k \times \mathbb{R}^n$  to  $X$ . Hence there exists  $(a_l, \mathbf{t}_l) \in \mathbb{R}^k \times \mathbb{R}^n$  converging to 0 such that  $x_k = a_l \cdot \eta_{\mathbf{t}_l}(x)$ . Then let  $F_l$  be the restriction of  $(a_l \cdot \eta_{\mathbf{t}_l})^{-1}$  to  $M^E(x_l)$ , which takes values in  $M^E(x)$  by Proposition 4.6. Notice that since  $F_l$  is a restriction of maps converging to the identity

on  $X$ , for every  $z_l \in M^E(x_l)$ ,  $d(z_l, F_l(z_l)) \rightarrow 0$ . Let  $y \in K$ , so that there exist  $y_l \in M^E(x_l)$  such that  $y_l \rightarrow y$ . Let  $z_l = F_l(y_l) \in M^E(x)$ . By construction,  $z_l \rightarrow y$ , so since  $M^E(x)$  is compact,  $y \in M^E(x)$ . That is,  $K \subset M^E(x)$ .  $\square$

*Remark 4.8.* While Proposition 4.6 shows that the flows descend to the quotient spaces  $X^E$ , it is not obvious that the quotient actions are topological Cartan actions as defined in Definition 3.1. Furthermore, the space  $X^E$  is not guaranteed to be finite-dimensional, even though  $X$  is. However, one may uniquely define continuous functions  $\rho^{\alpha, \beta}(s, t, x)$  by considering the definitions in the original space  $X$ , and ignoring any terms  $\rho_{\chi}^{\alpha, \beta}(s, t, x)$  with  $\chi \in E$ . Thus, there is a canonical unique choice for  $\rho^{\alpha, \beta}(s, t, x)$  as defined on  $X^E$ .

**Lemma 4.9.** *If  $\alpha, \beta \in \Delta$  are linearly independent, then  $\rho^{\alpha, \beta}(s, t, x)$  is independent of  $x$  on any maximal factor  $X^E$  of  $X$ .*

*Proof.* Let  $\Phi = \{\gamma_1, \dots, \gamma_k\}$  denote the weights which are positive linear combinations of  $\alpha$  and  $\beta$ , with the circular ordering. Notice that if  $\Phi = \emptyset$ , then  $\rho^{\alpha, \beta}(s, t, x)$  is the trivial path for every  $x$ , so we get the result trivially. In particular, the flows  $\eta^\alpha$  and  $\eta^\beta$  commute. Now proceed by induction, and assume we have shown the lemma for all  $\alpha, \beta$  for which there are at most  $k - 1$  weights in  $\Phi$ .

Choose a regular  $a \in \mathbb{R}^2$  such that  $\alpha(a), \beta(a) < 0$ , and let  $\gamma_{\max}$  be the weight of  $D(\alpha, \beta) \setminus \{\alpha, \beta\}$  which maximizes  $|\gamma_i(a)|$  over this set. Then  $\gamma_{\max}$  commutes with  $\alpha, \beta$  and all  $\gamma_i \in \Phi$ , since by Corollary 2.13, each of the flows  $\eta^{\gamma_i}$  fit into a Lie group action and the dynamics acts by automorphisms of this Lie group. In particular, if  $\gamma_{\max}$  failed to commute with one of the other  $\gamma_i, \alpha$  or  $\beta$ , then we would obtain a Lie subgroup with expansion  $\gamma_{\max}(a) + \gamma_i(a)$  by Proposition 2.9, contradicting the maximality.

But then the functions  $\rho_{\gamma_i}^{\alpha, \beta}(s, t, x)$  are all invariant under  $\eta^{\gamma_{\max}}$ , and are constant by maximality of  $E$ .  $\square$

## 5. HOMOGENEITY FROM PAIRWISE CYCLE STRUCTURES

Assuming constant commutator relations, we now construct a homogeneous structure of a Lie group for our Cartan actions. By the Gleason-Palais Theorem 2.12, we will show constant cycle relations for certain canonical subgroups corresponding to stable leaves for elements of the action. This will allow us to rearrange a general cycle to one in cyclical order, using the commutator relations and special “symplectic” relations for negatively proportional weights. We use the commutation relations between stable and unstable horocyclic subgroups provided by the classification in Proposition 3.4, which only works for a dense set of cycles containing an open neighborhood of the identity. This replaces a more complicated K-theoretic argument in other works.

These arguments allow us to give a fairly simple argument for homogeneity if the action satisfies the stronger ergodicity assumption that every codimension *two* subgroup acts ergodically (Section 5.2). In particular, the rank must be at least 3. The rank 2 case is much more complicated, and requires the use of the Gleason-Yamabe theorem on no small subgroups in Section 5.3. In the end we will have shown that factoring by a maximal ideal produces a homogeneous action. This sets up the induction on factoring out by a chain of ideals in Section 6.

**Definition 5.1.** Fix an ideal factor  $X^E$  of a topological Cartan action  $\mathbb{R}^k \curvearrowright X$ . We say that the action has constant pairwise cycle structure if

- (1) for each pair of nonproportional  $\alpha, \beta \in \Delta \setminus E$  and fixed  $s, t \in \mathbb{R}$   $\rho^{\alpha, \beta}(s, t, x)$  (as defined in Lemma 3.7) is independent of  $x$  and
- (2) for each  $\alpha \in \Delta \setminus E$  such that  $-\alpha \in \Delta$  for some  $c > 0$ , the action of  $\mathcal{P}_{\{\alpha, -c\alpha\}}$  factors through a Lie group action

The main goal of this section is to prove the following.

**Theorem 5.2.** If  $\mathbb{R}^k \curvearrowright X^E$  is an ideal factor of a topological Cartan action with constant pairwise cycle structure, then the action is topologically conjugate to a homogeneous action

We let  $\rho^{\alpha, \beta}(s, t)$  denote the common value of  $\rho^{\alpha, \beta}(s, t, x)$  (which is guaranteed to be constant by the pairwise group relations). Let  $\mathcal{P}$  be the topological group freely generated by a copy of  $\mathbb{R}$  for each  $\chi \in \Delta \setminus E$ , and  $\mathcal{C}'$  be the smallest closed normal subgroup containing all cycles of the form  $[s^{(\alpha)}, t^{(\beta)}] * \rho^{\alpha, \beta}(s, t)$  as described in Lemma 3.7 and any element of  $\mathcal{P}_{\{\alpha, -c\alpha\}}$  which factors through the identity of the Lie group action provided by (2). Since such cycles are cycles at every point by assumption,  $\mathcal{C}' \subset \mathcal{C}$  and  $\mathcal{C}'$  is normal. Consider the quotient group  $\mathcal{G} = \mathcal{P}/\mathcal{C}'$ .

**5.1. Stable-unstable-neutral presentations for elements of  $\mathcal{G}$ .** Fix  $a_0 \in \mathbb{R}^k$  a regular element. The goal of this subsection is to show that any  $\rho \in \mathcal{G}$  can be reduced (via the relations in  $\mathcal{C}'$ ) to some  $\rho_+ * \rho_- * \rho_0$ , with  $\rho_+$  having only terms  $t^{(\chi)}$  with  $\chi(a_0) > 0$ ,  $\rho_-$  having only terms  $t^{(\chi)}$  with  $\chi(a_0) < 0$ , and  $\rho_0$  being products of neutral elements as defined before Proposition 3.4. Indeed, we will instead show this for some subgroup obtained by combining  $\mathcal{G}$  with the Cartan action (see Proposition 5.11). We begin by identifying well-behaved subgroups of  $\mathcal{G}$ .

Let  $\Omega \subset \Delta \setminus E$  be any subset, and  $\mathcal{G}_\Omega$  denote the subgroup of  $\mathcal{G}$  generated by the corresponding  $\mathbb{R}$ -subgroups. If  $\Omega$  is such that there exists a regular  $a$  such that  $\chi(a) < 0$  for every  $\chi \in \Omega$  and which is closed under positive linear combinations, we call  $\Omega$  a *stable subset*. If  $\Omega$  is stable, we may order the elements of  $\Omega = \{\beta_1, \beta_2, \dots, \beta_r\}$  using a cyclical ordering after projecting to a generic 2-dimensional subspace. Then for each  $i$ , there exists an element  $a_i$  such that  $\beta_j(a_i) > 0$  if  $j < i$ ,  $\beta_i(a_i) = 0$  and  $\beta_j(a_i) < 0$  if  $j > i$ .

**Lemma 5.3.** If  $\Omega$  is stable,  $\rho \in \mathcal{G}_\Omega$  can be written as

$$u_1^{(\beta_1)} * \dots * u_r^{(\beta_r)}$$

Furthermore, the elements  $u_i$  are unique.

*Proof.* Any such  $\rho$  can be written as  $\rho = v_1^{(\beta_{i_1})} * \dots * v_k^{(\beta_{i_k})}$ . We may begin by pushing all of the terms from the  $\beta_1$  component to the left. We do this by looking at the first term to appear with  $\beta_1$ . Each time we pass it through, we may accumulate some  $[u^{\beta_1}, v^{\beta_j}] * \rho(u^{\beta_1}, v^{\beta_j})$  which consists of terms without  $\beta_1$  by Lemma 3.7, and are cycles at every  $x \in M$ . In particular, these cycles may be pushed freely to the right. So we have shown that  $\rho$  is equal to  $u_1^{(\beta_1)} * \rho' * \sigma'$ , where  $\rho'$  consists only of terms without  $\beta_1$  and  $\sigma'$  is a cycle at every  $x$ .

We now proceed inductively. We may in the same way push all  $\beta_2$  terms to the left. Notice now that each time we pass through, the “commutator”  $\rho(u^{\beta_2}, v^{\beta_j})$ ,  $j \geq 3$  has no  $\beta_1$  or  $\beta_2$  terms. Iterating this process yields the desired presentation of  $\rho$ .

Uniqueness will follow from an argument similar to Proposition 3.5. Suppose that  $u_1^{(\beta_1)} * \dots * u_r^{(\beta_r)} = v_1^{(\beta_1)} * \dots * v_r^{(\beta_r)}$ , so that

$$u_1^{(\beta_1)} * \dots * u_r^{(\beta_r)} * (-v_r)^{(\beta_r)} * \dots * (-v_1)^{(\beta_1)}$$

stabilizes every point of  $X^E$ . Picking some  $a \in \ker \beta_r$  such that  $\beta_i(a) < 0$  for all  $i = 1, \dots, r-1$  implies that

$$(e^{\beta_1(a)} u_1)^{(\beta_1)} * \dots * u_r^{(\beta_r)} * (-v_r)^{(\beta_r)} * \dots * (-e^{\beta_1(a)} v_1)^{(\beta_1)}$$

also stabilizes every point of  $X^E$ . Letting  $a \rightarrow \infty$  implies that  $(u_r - v_r)^{(\beta_r)}$  stabilizes every point of  $X^E$ . If  $u_r \neq v_r$ , applying an element which contracts and expands  $\beta_r$  yields an open set of  $t$  for which  $t^{(\beta_r)}$  stabilizes every point of  $X^E$ , so  $\eta^{\beta_r}$  descends to the trivial action on  $X^E$ . But this implies that  $\beta_r \in E$ , a contradiction, so we conclude that  $u_r = v_r$ . This allows cancellation of the innermost term, and one may inductively conclude that  $u_i = v_i$  for every  $i = 1, 2, \dots, r$ .  $\square$

**Corollary 5.4.** *If  $\Omega$  is stable,  $\mathcal{G}_\Omega$  is a nilpotent Lie group.*

*Proof.* Notice that Lemma 5.3 gives an injective map from  $\mathcal{G}_\Omega$  to  $\mathbb{R}^r$ . It will be continuous if its lift to  $\mathcal{P}_\Omega$  is continuous. In each combinatorial cell  $C_{\bar{\beta}}$ , the map is given by composition of addition of the cell coordinates and functions  $\rho^{\alpha, \beta}(\cdot, \cdot)$  evaluated on cell coordinates, which are continuous. Therefore, the lift is continuous, so the map from  $\mathcal{G}_\Omega$  is continuous. In particular, there is an injective continuous map from  $\mathcal{G}_\Omega$  to a finite-dimensional space, and  $\mathcal{G}_\Omega$  is a Lie group by Theorem 2.12. Fix  $a$  which contracts every  $\beta_i$ . The fact that  $\mathcal{G}_\Omega$  is nilpotent follows from proposition 2.9, since each Lie algebra commutator between  $\beta_i$  and  $\beta_j$  must end up in an eigenspace  $\beta_i + \beta_j$ , so the evaluation on  $a$  must strictly decrease. Since there are finitely many weights, it must eventually become trivial.  $\square$

Let  $\chi \in \Delta \setminus E$  be a weight such that  $-\chi \in \Delta$  for some  $c$ , and  $\beta \in \Delta$  be any linearly independent weight. Let  $\Omega = \{t\beta + s\chi : t \geq 0, s \in \mathbb{R}\} \cap \Delta$ , and  $\Omega' = \{t\beta + s\chi : t > 0, s \in \mathbb{R}\} \cap \Delta = \Omega \setminus \{\chi, -\chi\}$ .

**Proposition 5.5.** *If  $\Omega$  is as above,  $\rho \in \mathcal{G}_\Omega$  is any element,  $\rho = \rho_\chi * \rho_{\Omega'}$ , where  $\rho_\chi \in \mathcal{G}_{\{\chi, -\chi\}}$ ,  $\rho_{\Omega'} \in \mathcal{G}_{\Omega'}$ .*

*Proof.* The proof technique is the same as that of Lemma 5.3. Using constancy of commutator relations, we may push any elements  $u^{(\chi)}$  or  $u^{(-\chi)}$  to the left, accumulating elements of  $\mathcal{G}_{\Omega'}$  as the commutator, as well as cycles on the right (on the right because they are cycles at every point, so we may conjugate them by whatever appears to their right).  $\square$

**Corollary 5.6.** *If  $\Omega$  is as above,  $\mathcal{G}_\Omega$  is a Lie group. Furthermore,  $\mathcal{G}_\Omega$  has the semidirect product structure  $\mathcal{G}_{\{\chi, -\chi\}} \ltimes \mathcal{G}_{\Omega'}$ , with  $\mathcal{G}_{\{\chi, -\chi\}}$  isomorphic to  $\mathbb{R}^2$ , the Heisenberg group, or (some cover of)  $PSL(2, \mathbb{R})$ , and  $\mathcal{G}_{\Omega'}$  a nilpotent group.*

*Proof.* We show that if  $\sigma \in \mathcal{G}_\Omega$  fixes  $x$ , then  $\sigma = \sigma_\chi * \sigma_{\Omega'}$ , where  $\sigma_\chi$  is a cycle in  $\mathcal{G}_{\{\chi, -c\chi\}}$  and  $\sigma_{\Omega'} \in \mathcal{G}_{\Omega'}$ . By Proposition 5.5, it can always be written as such with  $\sigma_\chi, \sigma_{\Omega'}$  not necessarily being cycles, instead only paths. Assume for contradiction that  $\sigma_\chi$  is not a cycle. Pick  $a$  such that  $a \in \ker \chi$  and  $\beta(a) < 0$ , so that  $\lambda(a) < 0$  for all  $\lambda \in \Omega'$ . Then on the one hand,  $a^n \sigma$  is a cycle at  $a^n x$  for every  $n$ , but it becomes closer to  $\sigma_\chi \cdot (a^n x)$  since  $a \in \ker \chi$ . By choosing a convergent subsequence we obtain that  $\sigma_\chi$  is a cycle at some point and hence also at  $x$  (since the cycles in  $\{\chi, -c\chi\}$  are independent of  $x$ ). Hence  $\sigma_\chi$  is a cycle and so is  $\sigma_{\Omega'}$ . Therefore, if  $\sigma$  fixes  $x$ ,  $\sigma = e \in \mathcal{G}_\Omega$ , and the evaluation map shows that  $\mathcal{G}_\Omega$  is a Lie group. The semidirect product structure is clear from the proof of Proposition 5.5.  $\square$

Fix a regular element  $a_0 \in \mathbb{R}^k$ , and let

$$\Delta_+ = \{\chi \in \Delta \setminus E : \chi(a_0) > 0\}, \quad \Delta_- = \{\chi \in \Delta \setminus E : \chi(a_0) < 0\}.$$

Furthermore, let  $\mathcal{G}_+ = \mathcal{G}_{\Delta_+}$  and  $\mathcal{G}_- = \mathcal{G}_{\Delta_-}$ .

The next lemma is just a calculation but it is crucial to the main proposition below.

**Lemma 5.7.** *Let  $H$  cover  $PSL(2, \mathbb{R})$  and denote by  $h_s^+$  and  $h_s^-$  the unipotent flows in  $H$  covering  $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$  respectively. Also let  $a_t = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$ .*

*If  $st \neq -1$  then*

$$h_s^+ h_t^- = h_{\frac{t}{1+st}}^- h_{s(1+st)}^+ a_{1+st} z(s, t)$$

*where  $z(s, t)$  belongs to the center of  $H$ .*

*Proof.* If  $st \neq -1$  then

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t/(1+st) & 1 \end{pmatrix} \begin{pmatrix} 1 & s(1+st) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+st & 0 \\ 0 & 1/(1+st) \end{pmatrix}$$

Therefore the claim holds in  $PSL(2, \mathbb{R})$ , and thus in  $H$  up to some central element  $z(s, t)$ .  $\square$

The main result of this section is the following proposition.

Consider a  $\mathcal{G}_{\{\chi, -c\chi\}}$  locally isomorphic with  $SL(2, \mathbb{R})$ . Let  $\mathcal{D}_\chi$  be the subgroup of  $\mathcal{G}_{\{\chi, -c\chi\}}$  which normalize both  $\eta^\chi$  and  $\eta^{c\chi}$ . Let  $\mathcal{D} \subset \mathcal{G}$  be the product of all such  $\mathcal{D}_\chi$ .

**Lemma 5.8.** *Suppose  $x_0$  is a generic point for the  $\mathbb{R}^k$  action. Then if  $d \in \mathcal{D}$  is a cycle at  $x_0$ , then  $d$  is a cycle everywhere.*

*Proof.* Let  $g \in \mathcal{G}_{\{\chi, -c\chi\}}$  and  $a \in \mathbb{R}^k$ . Then  $ag = \phi_a(g)$  for some automorphism  $\phi_a$  of  $\mathcal{G}_{\{\chi, -c\chi\}}$  in the connected component of this automorphism group. Hence  $\phi_a$  is an inner automorphism. Note that  $\mathbb{R}^k$  and hence  $\phi_a$  normalize  $\eta^\chi$ , and  $\eta^{-c\chi}$ . Hence  $\phi_a(g) = d_0 g d_0^{-1}$  for some  $d_0 \in \mathcal{D}_\chi$ . As  $\mathcal{D}_\chi$  is abelian,  $\phi_a(d) = d$  for  $d \in \mathcal{D}_\chi$  and in fact for all  $d \in \mathcal{D}$ . Hence  $ad = da$ . Hence if  $d$  is a cycle at  $x_0$ ,  $d$  is a cycle at any point in  $\mathbb{R}^k(x_0)$ , hence everywhere as  $x_0$  is generic for the  $\mathbb{R}^k$  action.  $\square$

Denote by  $\mathcal{C}_\mathcal{D}$  the group of cycles in  $\mathcal{D}$ , and set  $G = \mathcal{G}/\mathcal{C}_\mathcal{D}$  and  $D = \mathcal{D}/\mathcal{C}_\mathcal{D}$ . Furthermore, let  $G_\pm$  denote the projections of the groups  $\mathcal{G}_\pm$  to  $G$ . Notice that by proposition 3.4, the action of  $D$  coincides with some restriction of the Cartan action. This immediately implies the following lemma

**Lemma 5.9.** *Let  $\rho^\pm \in G_\pm$  and  $\rho^0 \in D$ . Then*

$$\rho^0 \rho^\pm (\rho^0)^{-1} \in G_\pm.$$

**Lemma 5.10.** *For an open of elements  $\rho^+ \in G_+$ ,  $\rho^- \in G_-$  containing  $\{e\} \times \{e\}$  there exist  $(\rho^+)' \in G_+$ ,  $(\rho^-)' \in G_-$  and  $\rho^0 \in D$  such that*

$$\rho^+ * \rho^- = (\rho^-)' * (\rho^+)' * \rho^0.$$

*Furthermore,  $(\rho^+)', (\rho^-)'$  and  $\rho^0$  depend continuously on  $\rho^+$  and  $\rho^-$ .*

*Proof.* Order the weights of  $\Delta_+$  and  $\Delta_-$  using a fixed circular ordering as  $\Delta_+ = \{\alpha_1, \dots, \alpha_n\}$  and  $\Delta_- = \{\beta_1, \dots, \beta_m\}$ . Since  $\Delta_+$  is a stable subset,  $G_+$  is a nilpotent group. Therefore, we may write  $\rho_+ = r_n^{(\alpha_n)} * \dots * r_1^{(\alpha_1)}$  for some  $t_1, \dots, t_n \in \mathbb{R}$ . We will inductively show that we may write the product  $\rho^+ * \rho^-$  as  $t_n^{(\alpha_n)} * \dots * t_k^{(\alpha_k)} * (\rho^-)' * s_{k-1}^{(\alpha_{k-1})} * \dots * s_1^{(\alpha_1)} * \rho^0$  for some  $t_i s_i \in \mathbb{R}$ ,  $(\rho^-)' \in \mathcal{G}_-$  and  $\rho^0 \in D$  (all of which depend on  $k$ ). Our given expression is the base case  $k = 1$ .

Suppose we have this for  $k$ . If  $-c_k \alpha_k \in \Delta$ , then it must be in  $\Delta_-$ . Let  $l(k)$  denote the index for which  $\beta_{l(k)} = -c_k \alpha_k$  if  $-c_k \alpha_k$  is a weight. Otherwise, set  $\beta_{l(k)} = -\alpha_k$  (it will not be a weight) with  $l(k)$  a half integer so that  $-\alpha_k$  appears between  $\beta_{l(k)-1/2}$  and  $\beta_{l(k)+1/2}$ . Then decompose  $\Delta$  into six (possibly empty) subsets:  $\{\alpha_k\}$ ,  $\{-c_k \alpha_k\}$ ,  $\Delta_1 = \{\alpha_l : l < k\}$ ,  $\Delta_2 = \{\alpha_l : l > k\}$ ,  $\Delta_3 = \{\beta_l : l < l(k)\}$  and  $\Delta_4 = \{\beta_l : l > l(k)\}$ . We let  $G_{\Delta_i}$  denote the subgroup of  $G$  generated by the  $\Delta_i$ . Notice that  $\Delta_- = \Delta_3 \cup \{-c_k \alpha_k\} \cup \Delta_4$  (with  $\{-c_k \alpha_k\}$  omitted if there is no weight of this form) is stable, so again, since  $G_-$  is nilpotent,  $(\rho^-)'$  may be expressed uniquely as  $q_3 * u^{(-c_k \alpha_k)} * q_4$  with  $q_3 \in G_{\Delta_3}$  and  $q_4 \in G_{\Delta_4}$  (if  $-c_k \alpha_k$  is not a weight, we omit this term). Now,  $\{\alpha_k\} \cup \Delta_2 \cup \Delta_3$  is a stable set with associated group nilpotent. So  $t_k^{(\alpha_k)} * q_3 = q_2 * (q_3)' * (s'_k)^{(\alpha_k)}$  for some  $q_2 \in G_{\Delta_2}$ ,  $(q_3)' \in G_{\Delta_3}$  and  $s'_k \in \mathbb{R}$ . The term  $s'_k$  is determined by an invertible polynomial in  $t_k$  and the lengths of the legs of  $q_3$ . In particular, properties which hold for an open dense set of the  $s'_k$  hold for an open dense set of the  $t_k$ . Thus, we have put our expression in the form:

$$t_n^{(\alpha_n)} * \dots * t_{k+1}^{(\alpha_{k+1})} * q_2 * (q_3)' * (s'_k)^{(\alpha_k)} * u^{(-c_k \alpha_k)} * q_4 * s_{k-1}^{(\alpha_{k-1})} * \dots * s_1^{(\alpha_1)} * \rho^0$$

Now, there are three cases: there is no weight of the form  $-c_k \alpha_k$  in which case  $u^{(-c_k \alpha_k)}$  does not exist. Or this term exists but commutes with  $(s'_k)^{(\alpha_k)}$  and we simply switch the order. The last possibility is that  $\alpha_k$  and  $-c_k \alpha_k$  generate a cover of  $PSL(2, \mathbb{R})$ . In the latter case, if  $s'_k u \neq -1$ , which is an open and dense property, we may use Lemma 5.7 to commute  $(s'_k)^{(\alpha_k)}$  and  $u^{(-c_k \alpha_k)}$  up to an element of  $D$  (if they do not span a cover of  $SL(2, \mathbb{R})$ , then they commute). Furthermore, notice that  $q_2$  and the terms appearing before  $q_2$  all belong to  $\Delta_2$ , so we may combine them to reduce the expression to:

$$(t'_n)^{(\alpha_n)} * \dots * (t'_{k+1})^{(\alpha_{k+1})} * (q_3)' * (u')^{(-c_k \alpha_k)} * (s''_k)^{(\alpha_k)} * d * q_4 * s_{k-1}^{(\alpha_{k-1})} * \dots * s_1^{(\alpha_1)} * \rho^0$$

for some collection of  $t'_i, u', s''_k \in \mathbb{R}$ , and  $d \in D$ . But by Lemma 5.9,  $d$  may be pushed to the right preserving the form of the expression and being absorbed into  $\rho^0$ . We abusively do not change these terms and drop  $d$  from the expression.



Now, we do the final commutation by commuting  $(s_k'')^{(\alpha_k)}$  and  $q_4$ . Notice that  $\{\alpha_k\} \cup \Delta_1 \cup \Delta_4$  is a stable subset. Therefore, we may write  $(s_k'')^{(\alpha_k)} * q_4$  as  $(q_4)' * (s_k''')^{(\alpha_k)} * q_1$  with  $q_1 \in G_{\Delta_1}$ ,  $(q_4)' \in G_{\Delta_4}$  and  $s_k''' \in \mathbb{R}$ . Inserting this into the previous expression, we see that the  $q_1$  term can be absorbed into the remaining product of the  $s_i^{(\alpha_i)}$  terms. This yields the desired form.  $\square$

Recall that the action of  $D$  coincides with the Cartan action, let  $f : D \rightarrow \mathbb{R}^k$  be the homomorphism which associates an element of  $D$  with the corresponding element of the  $\mathbb{R}^k$  action. Then let  $\hat{G}$  be the quotient of  $\hat{\mathcal{P}}$  (see Definition 2.8) by the group generated by  $\ker(\mathcal{P} \rightarrow G)$  and elements of the form  $f(d)d^{-1}$ .

Recall that  $\mathcal{P}$  is the free product of copies of  $\mathbb{R}$ , and has a canonical CW-complex structure as described in Section 2.2. The cell structure can be seen by considering sub-complexes corresponding to a sequence of weights  $\bar{\chi} = (\chi_1, \dots, \chi_n)$  and letting  $C_{\bar{\chi}} = \left\{ t_1^{(\chi_1)} * \dots * t_n^{(\chi_n)} : t_i \in \mathbb{R} \right\} \cong \mathbb{R}^n$ . Then a neighborhood of the identity is a union of neighborhoods in each cell  $C_{\bar{\chi}}$  containing 0.

**Proposition 5.11.** *There exists an open neighborhood  $U$  of  $e \in \hat{\mathcal{P}}$  and a continuous map  $\Phi : U \rightarrow G_+ \times G_- \times \mathbb{R}^k$  such that if  $\Phi(u) = \Phi(v)$ , then  $u$  and  $v$  represent the same element of  $\hat{G}$ .*

*Proof.* We describe the map  $\Phi$ , whose domain will become clear from the definition. Let  $\Delta_+ = \{\alpha_1, \dots, \alpha_n\}$  and  $\Delta_- = \{\beta_1, \dots, \beta_m\}$  be the weights as described in the proof of Lemma 5.10. Given a word  $\rho = t_1^{(\chi_1)} * \dots * t_n^{(\chi_n)}$ , we begin by taking all occurrences of  $\alpha_n$  in  $\rho$  and pushing them to the left, starting with the leftmost term. When we commute it past another  $\alpha_i$ , we get only other  $\alpha_i$  in  $\rho^{\alpha_1, \alpha_i}$ , which we may canonically present in increasing order on the right of the commutation. Similarly for the commutation of  $\alpha_1$  with  $\beta_i$ . The procedure is a similar iterated application of Lemma 5.10. Since the commutation operations involved are determined by the combinatorial type, the resulting presentation is continuous from the cell  $C_{\bar{\chi}}$ . Furthermore, if one of the terms  $t_i^{(\chi_i)}$  happens to be 0, the procedure yields the same result whether it is considered there or not. Thus, it is a well-defined continuous map from  $\mathcal{P} \rightarrow G_+ \times G_- \times \mathbb{R}^k$  (it is continuous from  $\hat{\mathcal{P}}$  because it is continuous from each  $C_{\bar{\chi}}$ ).

Notice that in the application of Lemma 5.10, we require that  $st \neq 1$  whenever we try to pass  $t^{(\chi)}$  by  $s^{(-\chi)}$ . This is possible if  $s$  and  $t$  are sufficiently small. Thus, in each combinatorial pattern, since the algorithm is guaranteed to have a finite number of steps and swaps appearing, and each term appearing will depend continuously on the initial values of  $t_i^{(\chi_i)}$ , we know that for each  $\bar{\chi}$ , some neighborhood of 0 will be in the domain of  $\Phi$ , by the neighborhood structure described above.  $\square$

**5.2. A special case of theorem 1.4.** We have developed enough tools to prove theorem 1.4 under stronger ergodicity assumptions on the Cartan action. It is important to note that no rank 2 action will satisfy the stronger assumptions. The general case will be more difficult because we cannot prove constant pairwise cycle structures directly. Instead, we

obtain homogeneous structures for the ideal factor actions developed in Section 4, and a priori the factor spaces are only compact metric spaces, adding extra complications.

**Theorem 5.12.** *If  $\mathbb{R}^k \curvearrowright X$  is a topological Cartan action with  $k \geq 3$  such that for every pair of weights  $\alpha, \beta \in \Delta$ , the action of  $\ker \alpha \cap \ker \beta$  has a dense orbit, then it is topologically conjugate to a homogeneous action*

*Proof.* Notice that if  $a \in \ker \alpha \cap \ker \beta$ , then  $a$  commutes with  $\eta^\gamma$  for any  $\gamma = s\alpha + t\beta$ ,  $s, t \in \mathbb{R}$ . In particular:

$$[t^{(\alpha)}, s^{(\beta)}]\rho^{\alpha, \beta}(s, t, a \cdot x) \cdot x = a^{-1}[t^{(\alpha)}, s^{(\beta)}]\rho^{\alpha, \beta}(s, t, a \cdot x)a \cdot x = x$$

Therefore,  $\rho^{\alpha, \beta}(s, t, a \cdot x) = \rho^{\alpha, \beta}(s, t, x)$  for all  $a \in \ker \alpha \cap \ker \beta$  orbits. So  $\rho^{\alpha, \beta}(s, t, \cdot)$  is constant on the orbits of  $\ker \alpha \cap \ker \beta$  orbits, and hence constant everywhere by our dense orbit assumption. Similarly,  $\mathcal{C}_{\{\alpha, -\alpha\}}$  is constant on  $\ker \alpha$  orbits. Therefore, we get pairwise constant cycle structures, and by Theorem 5.2, the Cartan action is topologically conjugate to a homogeneous action.  $\square$

*Remark 5.13.* In the proof of Theorem 5.12 we used that  $X$  is finite dimensional metric space as per definition of topological Cartan actions. When we consider ideal factor actions on the spaces  $X^E$  however, we no longer know that  $X^E$  is a finite dimensional metric space as the topological dimension may go up when taking the quotient. Thus we cannot use the Gleason-Palais result directly. Instead we argue below in subsection 5.3 that the group  $\hat{G}$  is locally compact and has the no small subgroups property.

Theorem 5.12 immediately implies the following

**Corollary 5.14.** *If  $\mathbb{R}^k \curvearrowright X$  is a  $C^{1, \theta}$  Cartan action on a smooth manifold  $X$  preserving an ergodic measure of full support, and the action of  $\ker \alpha \cap \ker \beta$  is ergodic for every pair  $\alpha, \beta \in \Delta$ , then it is topologically conjugate to a homogeneous action. Furthermore, if the action is  $C^\infty$ , the conjugacy is  $C^\infty$ .*

To see how to obtain the regularity of the conjugacy, see Section 7.

**5.3. No Small Subgroups Argument.** In this subsection, we prove that  $X^E$  is the homogeneous space of a Lie group. First we note that  $\hat{G}$  is locally compact in Lemma 5.16. Then we show that small compact subgroups of  $\hat{G}$  act trivially on  $X^E$  employing the hyperbolic dynamics of the  $\mathbb{R}^k$  action. This allows us to use standard results from Hilbert's 5th problem to the image  $\tilde{G}$  of  $\hat{G}$  in the homeomorphisms of  $X^E$ .

**Proposition 5.15.** *The action of  $\hat{G}$  on  $X^E$  factors through a connected Lie group  $\tilde{G}$ . In particular,  $X^E$  is the homogeneous space of  $\tilde{G}$ , and  $\mathbb{R}^k$  acts via the image of  $\mathbb{R}^k$  in  $\tilde{G}$ .*

**Lemma 5.16.**  *$\hat{G}$  is locally compact.*

*Proof.* This is immediate from the stable unstable cycle decomposition, Proposition 5.11.  $\square$

Since  $\hat{G}$  is connected, locally compact and acts transitively on  $X^E$ , Proposition 5.15 immediately follows from the following result:

**Proposition 5.17.** *Let  $N^E \subset \hat{G}$  be the kernel of the action of  $\hat{G}$  on  $X^E$ . Then  $\hat{G}^E := \hat{G}/N^E$  is locally compact and has the no small subgroups property, and hence is a Lie group.*

We will prove this proposition by the chain of arguments below. We will need to proceed very carefully with the order in which we choose various gadgets. That  $\hat{G}^E$  is locally compact follows from local compactness of  $\hat{G}$ .

First recall that  $\hat{G}_+$  and  $\hat{G}_-$  are Lie groups by Corollary 5.4, and hence do not have small subgroups. We denote by  $\hat{G}_+^E$  and  $\hat{G}_-^E$  their images in the homeomorphism group of  $X^E$ , and the image of  $\mathbb{R}^k$  by  $A^E$ . Then they are nilpotent Lie groups as quotients of Lie groups by closed normal subgroups are Lie groups again. Also note that  $\hat{G}_+^E$ ,  $\hat{G}_-^E$  and  $A^E$  generate  $\hat{G}^E$ .

We first describe the kernel  $\ker_+^E$  of the projection  $\hat{G}_+ \rightarrow \hat{G}_+^E$ .

**Lemma 5.18.**  $\ker_+^E = \{e\}$ .

*Proof.* Suppose  $g^+ \in \hat{G}_+$  acts trivially on  $X^E$ , and recall that  $\hat{G}_+$  is a nilpotent Lie group whose Lie algebra is given as a sum of 1-dimensional Lie algebras for each  $\chi \in \Delta_+ \subset \Delta \setminus E$ . Write  $g^+ = \exp X$  with  $X = \sum_{\lambda \notin E} c_\lambda X^\lambda$ . For any  $b \in \mathbb{R}^k$ ,  $b^{-1}g^+b^{-1} \in \ker_+^E$  as well. Pick  $b$  so that conjugation by  $b$  contracts  $\hat{G}_+$ , and so that all  $\lambda(b)$  are distinct and not 0. Note that the cyclic group generated by  $g^+$  conjugated by  $b^n$  will converge to the image of the one-parameter group  $\eta_s^\lambda$  where  $\lambda$  is the weight with  $\lambda(b)$  the smallest  $\lambda(b)$  with  $c_\lambda \neq 0$ . Hence  $\eta_s^\lambda \subset \ker_+^E$ . But this implies that  $\eta_s^\lambda$  fixes each  $M^E(x) \subset X$ , so  $\lambda \in E$ . This contradicts to our assumption that  $\lambda \notin E$ .  $\square$

**Lemma 5.19.** *There is a periodic point  $p \in X^E$  such that no element  $\neq 1$  in  $\hat{G}_+^E$  fixes  $p$ .*

*Proof.* Suppose first that  $p \in X^E$  is periodic for the  $\mathbb{R}^k$  action, and that  $g^+p = p$  for some  $g^+ \in \hat{G}_+^E$ . By Lemma 5.18,  $g^+ = \exp X$  with  $X = \sum_{\lambda \notin E} c_\lambda X^\lambda$ . Pick  $b \in \mathbb{R}^k$  in the stabilizer of  $p$  which shrinks  $\hat{G}_+^E$  and for which all the weights  $\lambda(b)$  are all distinct. Then the cyclic group generated by  $g^+$  conjugated by  $b^n$  will converge to the image of a one-parameter group  $\eta_s^\lambda \subset \hat{G}_+^E$  in  $\hat{G}_+^E$ , with  $\lambda \notin E$ . Hence we get a one parameter subgroup of  $\hat{G}_+^E$  which fixes  $p$ .

Now suppose that every periodic point  $p$  is fixed by a one-parameter subgroup  $\eta^\beta$  for  $\beta \notin E$ . Let  $F^\beta$  denote the fixed point set of the whole one-parameter group  $\eta^\beta$ . Since the periodic points are dense, the union  $\cup_\beta F^\beta = X^E$  of these finitely many closed sets is all of  $X^E$ . Hence at least one of them has non-empty interior. Since the  $\mathbb{R}^k$  action has dense orbits and normalizes each  $\eta^\beta$ ,  $F^\beta = X^E$  for some weight  $\beta$ . This implies that  $\beta \in E$ , in contradiction to our choice of  $\beta$ .

Therefore there is an  $\mathbb{R}^k$  periodic point  $p$  on which  $\hat{G}_+^E$  acts freely.  $\square$

**Lemma 5.20.** *Let  $d_E$  be a metric on  $X^E$ , and  $d_E^G$  on  $\hat{G}_+^E$ . Let  $p$  be as in Lemma 5.19. Suppose  $c \in \mathbb{R}^k$  fixes  $p$  with  $\lambda(c) > 0$  for all  $\lambda \in \Delta_+$ ,  $\lambda(c) < 0$  for all  $\lambda \in \Delta_-$ , and that conjugation of  $\hat{G}_+^E$  by  $c$  expands  $\hat{G}_+^E$  with Lipschitz constant  $C > 1$ .*

*Then for all  $\delta > 0$  there is a constant  $\epsilon > 0$ , and a neighborhood  $V$  of  $p$  such that for  $g$  in the compact annulus  $A_\delta$  in  $\hat{G}_+^E$  defined by  $\delta \leq d_E^G(g, 1) \leq C\delta$  and all  $x \in V$ ,  $d_E(x, gx) > \epsilon$ .*

*Moreover,  $\hat{G}_+^E$  acts locally freely on  $V$ .*

*Proof.* Suppose this is false. Let  $V$  be a compact neighborhood of  $p$  on which the ball of radius  $C\delta$  in  $\hat{G}_+^E$  acts freely, as provided by Lemma 5.19. Then for  $\epsilon = \frac{1}{n}$ , we get  $g_n \in A_\delta$  and  $v_n \in V$  with  $d_E(g_n v_n, v_n) < \frac{1}{n}$ . Taking limits we get a contradiction to local freeness of the action of the ball of radius  $C\delta$  on  $V$ .

Local freeness of the action of  $\hat{G}_+^E$  in a neighborhood of  $p$  now follows easily. Since  $\hat{G}_+^E$  is a Lie group, if  $h$  is very close to 1, then  $h^l \in A_\delta$  for some  $l \in \mathbb{N}$ . Hence if  $hv = v$  for some  $v \in V$ , also  $h^l v = v$  in contradiction to the local freeness of  $A_\delta$  acting on  $V$ . It follows that  $\hat{G}_+^E$  acts locally freely on  $V$ , as desired.  $\square$

Now Proposition 5.17 follows immediately from

**Lemma 5.21.** *There exists a neighborhood  $U \subset \hat{G}$  of  $e$  such that if  $K$  is a compact normal subgroup contained in  $U$ , then  $K$  acts trivially on  $X^E$ .*

*Proof.* Pick a periodic point  $p$ ,  $\delta > 0$ ,  $\epsilon > 0$ , a neighborhood  $V$  of  $p$ ,  $c \in \mathbb{R}^k$  and Lipschitz constant  $C > 1$  as in Lemma 5.20. We will also assume that  $U$  is the image under  $\Phi$  (cf. 5.11) of three balls (w.r.t. the exponential maps) in  $\hat{G}_+$ ,  $\hat{G}_-$  and  $\mathbb{R}^k$ , and that  $d(up, p) < \epsilon/100$  for all  $u \in U^3$ .

Now suppose  $K$  is a compact normal subgroup contained in  $U$ . Let  $k \in K$  and write  $k = k^+ k^- a$  as above for any element in  $U$ . If  $k^+ \neq 1$ , pick  $l \in \mathbb{N}$  such that  $c^l k^+ c^{-l} \in A_\delta$ . Then

$$c^l k^+ c^{-l} = (c^l k c^{-l}) a^{-1} (c^l (k^-)^{-1} c^{-l}).$$

The left hand side moves  $p$  by at least  $\epsilon$  since  $c^l k^+ c^{-l} \in A_\delta$ . For the expression on the right hand side, first note that  $c^l k c^{-l} \in K \subset U$ ,  $a^{-1} \in U$  since  $U$  is symmetric, and finally  $c^l (k^-)^{-1} c^{-l} \in U$  since conjugation by  $c$  contracts balls in  $U^-$ . By the assumption on the neighborhood  $U$  of 1 in  $\hat{G}$ ,  $d(a^{-1} c^l (k^-)^{-1} c^{-l} p, p) < \epsilon/100$ . This is a contradiction, and therefore, for all  $k \in K$ ,  $k^+ = 1$ . By a similar argument, also  $k^- = 1$ , and hence  $K \subset \mathbb{R}^k$ . Since  $\mathbb{R}^k$  does not have small subgroups,  $K = \{1\}$ .  $\square$

*Proof of Proposition 5.17.* Let  $U$  be as in Lemma 5.21. By the Gleason-Yamabe theorem (Theorem 2.11), there exists a compact normal subgroup  $K \subset U$  such that  $\tilde{G} = \hat{G}/K$  is a Lie group. But  $K \subset N^E$  by Lemma 5.21. Therefore,  $\hat{G}^E$  is a factor of a Lie group, and hence a Lie group.  $\square$

Theorem 5.2 is now immediate from the following:

**Proposition 5.22.** *Fix a topological Cartan action with local product structures, and let  $\eta$  be the induced action of  $\hat{\mathcal{P}}$  on  $X$ . If there exists some subgroup  $\mathcal{C} \subset \mathcal{P}$  such that  $\mathcal{C} \subset \text{Stab}(x)$  for every  $x \in X$  and an injective continuous map from some open subset of  $\hat{\mathcal{P}}/\mathcal{C}$  to a finite-dimensional Euclidean space, then there is a Lie group  $G$ , cocompact lattice  $\Gamma$ , a homomorphism  $f: \mathbb{R}^k \rightarrow G$ , and a homeomorphism  $h: G/\Gamma \rightarrow X$  such that:*

$$a \cdot h(x) = h(f(a)x)$$

*Proof.* By Corollary 2.13, we get that  $G = \hat{\mathcal{P}}/\mathcal{C}$  is a Lie group, and the action of  $G$  is transitive on  $X$ . Therefore,  $X = G/\text{Stab}(x)$  is a  $G$ -homogeneous space, and the action is

embedded into the  $G$  action by construction of  $\hat{\mathcal{P}}$ . By Theorem A.1, we may assume that  $\Gamma = \text{Stab}(x)$  is discrete, and hence a cocompact lattice. Since  $G/\Gamma$  is compact, and  $h$  is a continuous bijection, it is a homeomorphism □

## 6. EXTENSIONS BY MAXIMAL IDEALS

We work in the following setting: Assume that  $\mathbb{R}^k \curvearrowright X$  is a  $C^{1,\theta}$  Cartan action with local product structure,  $E_0 \subset \Delta$  is some ideal for which the induced action on  $X^{E_0}$  is homogeneous, and  $E \subset E_0 \subset \Delta$  is the largest ideal strictly contained in  $E_0$ , and set  $\Delta_b = \Delta \setminus E_0$  and  $\Delta_f = E_0 \setminus E$ . We call  $\Delta_b$  the *base weights* and  $\Delta_f$  the *fiber weights*. Then  $X^{E_0}$  is a quotient space of  $X^E$ , and  $\Delta_f$  are all flows along the “fibers” (ie, sets  $M^{E_0}(x)$ ), and  $\Delta_b$  are flows on  $X^E$  covering the homogeneous flows on  $X^{E_0}$ . Furthermore,  $[\Delta, \Delta_f] \subset \Delta_f$ , and since  $E$  was chosen maximally, any continuous function which is invariant under a flow  $\eta^\chi$  for some  $\chi \in \Delta_f$  is constant along fibers.

The main result of this section is the following.

**Theorem 6.1.** *The induced Cartan action on  $X^E$  has constant pairwise cycle structure*

*Remark 6.2.* In this section, we require that the action is  $C^{1,\theta}$  rather than a topological Cartan action. The only place in which this is used is in an application of Lemma 3.10. Under the assumption that  $\chi = u\alpha + v\beta \in [\alpha, \beta]$  implies  $u \geq 1$  or  $v \geq 1$ , one can prove Theorem 6.1 for topological Cartan actions, as well.

Part (2) of Definition 5.1 follows from ergodicity of the action of  $\ker \alpha$ , so we must show (1). That is, we must show that if  $\alpha, \beta \in \Delta$  are linearly independent, then  $\rho^{\alpha,\beta}(s, t, x)$  is independent of  $x$ . We do this in three parts: if  $\alpha, \beta \in \Delta_f$  (Section 6.2), if  $\alpha \in \Delta_b$  and  $\beta \in \Delta_f$  (Section 6.3), and if  $\alpha, \beta \in \Delta_b$  (Section 6.4). We begin by showing that the action on the weights in  $\Delta_f$  are transitive on each fiber.

**6.1. Structure of fibers.** Consider a topological Cartan action of  $\mathbb{R}^k$  on a space  $X$  with local product structure as in 3.1. We will show that the fibers of the natural projection maps  $\pi_E : X \rightarrow X^E$  are endowed with Cartan actions themselves. This allows us to use our earlier results on the fibers in subsection 6.2. This is the only part of this paper where we use local product structures.

Given an ideal  $E$  in the set of weights, we form the subgroup  $\mathcal{P}^E \subset \mathcal{P}$  generated by the legs of the weights in  $E$ .

**Proposition 6.3.** *Let  $E$  be an ideal, and  $X^E$  the associated ideal factor, with natural projection map  $\pi = \pi^E : X \rightarrow X^E$ . Suppose that the flows  $\eta_s^\lambda$  for  $\lambda \in \Delta - E$  on  $X^E$  generate a Lie group  $H^E$ . Then the preimages  $\pi^{-1}(x)$  are the orbits of  $\mathcal{P}^E$ .*

*Moreover, let  $p$  be a periodic point of the  $\mathbb{R}^k$ -action on  $X^E$ . Then the stabilizer of  $p$  in  $\mathbb{R}^k$  is isomorphic to  $\mathbb{Z}^l \times \mathbb{R}^{k-l}$ , for some  $0 \leq l \leq k$ . Its action on the fiber  $\pi^{-1}(p)$  is a topological Cartan action with local product structures.*

We note that topological Cartan actions always have periodic points as the arguments from the Anosov closing lemma apply.

*Proof.* Order both the weights in  $E$  by  $\bar{\eta}$  and  $\Delta - E$  by  $\bar{\delta}$ . This induces an order  $\bar{\beta} = (\bar{\eta}, \bar{\delta})$  of all weights by putting the weights in  $E$  first. Since the action has local product structure, the map from  $\Psi_{\bar{\beta}} : C_{\bar{\beta}}\mathbb{R}^k \rightarrow X$  is a local homeomorphism. Note that  $C_{\bar{\beta}} = C_{\bar{\eta}} \times C_{\bar{\delta}}$  and that  $\pi$  is constant on the first factor  $C_{\bar{\eta}}$ . The assumption that  $H^E$  is a Lie group implies that  $\Psi_{\bar{\beta}}|_{C_{\bar{\delta}}}$  is a local homeomorphism. Hence the fiber  $\pi^{-1}(x)$  is homeomorphic to  $C_{\bar{\eta}}$ , and thus the orbit of  $\mathcal{P}^{E_0-E}$ .

Next consider a periodic point  $p \in X^E$ . The stabilizer of  $p$  in  $\mathbb{R}^k$  is a subgroup  $\mathbb{Z}^l \times \mathbb{R}^{k-l}$  which acts on the fiber  $\pi^{-1}(p)$ . The latter is a compact metric space of finite dimensions as it is a closed subspace of the finite dimensional space  $X$ . The other conditions of 3.1 are easy to check.  $\square$

**Corollary 6.4.** *Let  $E \subset E_0 \subset \Delta$  be two ideals, and  $X^E$  and  $X^{E_0}$  the associated ideal factors, with natural projection map  $\pi = \pi_{E_0}^E : X^E \rightarrow X^{E_0}$ . Suppose that the flows  $\eta_s^\lambda$  for  $\lambda \in \Delta \setminus E_0$  on  $X^{E_0}$  generate a Lie group  $H^{E_0}$ . Then the preimages  $(\pi_{E_0}^E)^{-1}(x)$  are the orbits of  $\mathcal{P}^{E_0}$ .*

*Moreover, let  $p$  be a periodic point of the  $\mathbb{R}^k$ -action on  $X^{E_0}$ . Then the stabilizer of  $p$  in  $\mathbb{R}^k$  is isomorphic to  $\mathbb{Z}^l \times \mathbb{R}^{k'}$ , for some  $k'$ . Its action on the fiber  $(\pi_{E_0}^E)^{-1}(p)$  is a topological Cartan action with local product structures.*

**6.2. Fiber-Fiber Relations.** First we prove constancy of fiber-fiber relations. Given two weights  $\alpha$  and  $\beta$ , then we prove constancy of the function  $\rho^{\alpha,\beta}(s, t, \cdot)$  along each fiber in a manner similar to the maximal factor, by an induction on  $\#(D(\alpha, \beta))$ . This will give group structures on each fiber, and by using the intertwining property of the action and ergodicity of the factor action on the base, we will conclude that the functions are constant everywhere. We also will obtain a classification of the groups which may appear, which will be important in establishing constancy of  $\rho^{\alpha,\beta}(s, t, \cdot)$  when  $\alpha$  and/or  $\beta$  are in  $\Delta_b$ .

**Proposition 6.5.**  *$\Delta_f$  generates the action of a fixed Lie group  $H$  which acts transitively on each set  $M^{E_0}(x) \subset X^E$ . Furthermore,  $H$  is either abelian, 2-step nilpotent or  $\mathbb{R}$ -split semisimple, and for every  $\alpha \in \Delta_f$  the only  $\eta^\alpha$ -invariant functions are also  $H$ -invariant.*

*Proof.* To get a Lie group action on each  $M^{E_0}(x)$ , observe that  $\rho^{\alpha,\beta}(s, t, x)$  is constant along  $M^E(x)$  by the arguments of Lemma 4.9, since each  $\chi \in \Delta_f$  has  $M^{\{\chi\}}(x) = M^{E_0}(x)$  by maximality of the ideal  $E \subset E_0$ . We also know from Lemma 3.3 that the symplectic relations factor through a Lie group action. Hence, we obtain a Lie group action on each fiber (possibly depending on  $x$ ) using the arguments of Section 5. Let  $H_x$  denote the Lie group which acts on  $M^E(x)$ .

We now claim that if  $x_0$  is generic,  $H_{x_0}$  acts on every fiber (and in particular,  $H_x$  is a factor of  $H_{x_0}$  for every  $x$ ). Notice that if  $\alpha \in \Delta_f$ , there exists  $\varepsilon > 0$  such that the set of all points for which the  $\eta^\alpha$  action is free on  $X^E$  when restricted to  $(-\varepsilon, \varepsilon)$  contains an open dense set, since it is invariant under the  $\ker \alpha$  action and open for  $\varepsilon$  sufficient small. Therefore, generically, every  $\eta^\alpha$  is locally free. Therefore, generically, each coarse Lyapunov flow  $\eta^\alpha$ ,  $\alpha \in \Delta_f$  gives an element of  $\text{Lie}(H_x)$  by setting  $X_\alpha$  to be the canonical unit generator of the one parameter subgroup corresponding to  $\eta^\alpha$ . Notice that the group relations  $\mathcal{S}_x = \ker(\mathcal{P}^E \rightarrow H_x)$  are determined by the Lie algebra relations among the  $X_\alpha$ ,  $\alpha \in \Delta_f$  and generators of the orbit contained in the fiber,  $Y \in \mathfrak{o}$  (where  $\mathfrak{o}$  is the part of the orbit generated by the  $X_\alpha$ ). Then  $[X_\alpha, X_\beta] = c_{\alpha\beta}(x)X_{\alpha+\beta}$  and  $[X_\alpha, X_{-c\alpha}] = 0$  if  $c \neq 1$

and  $[X_\alpha, X_{-\alpha}] = Y_x$  for some  $Y_x \in \mathbb{R}^k$ . Pushing these algebra relations forward, it is clear that  $c_{\alpha\beta}(x)$  is constant along  $\mathbb{R}^k$  orbits (since both sides rescale by  $e^{\alpha(a)+\beta(a)}$ ) and  $Y_x$  is constant along  $\mathfrak{o}$  orbits for similar reasons. Therefore,  $\text{Lie}(H_x)$  has canonically generating one-parameter subgroups, with constant infinitesimal brackets, so  $H_x$  is constant along  $\mathbb{R}^k$  orbits of generic points, and hence  $H_{x_0}$  extends to an action everywhere.

This gives a fixed Lie group  $H$  acting transitively on each set  $M^{E_0}(x) \subset X^E$ . Notice that  $H_x$  is generated by the one parameter subgroups  $U_\beta$  corresponding to the coarse Lyapunov flows  $\eta^\beta$ ,  $\beta \in \Delta_f$ . Let  $x \in X^{E_0}$  be a periodic orbit of the homogeneous Cartan action, and  $V \cong \mathbb{R}^{k-l} \times \mathbb{Z}^l$  be the stabilizer of the point. Then  $V$  acts on the fiber of  $x$  by a Cartan action, and for every  $\beta \in \Delta_f$ , any  $\eta^\beta$ -invariant function from  $X^E$  is constant along the fibers  $M^{E_0}(x)$ . Furthermore,  $V$  will preserve the conditional measure at almost every point. Furthermore, so do the flows  $\eta^\beta$ , which generate  $H$ . Hence, the conditional measures are  $H$ -invariant, and must be Haar. In particular, we may apply Theorem A.1. If  $H$  has a semisimple factor, then the closure of the coarse Lyapunov subgroups in such a factor are contained in some section, since the lattice splits as a semidirect product as well. Therefore, since each coarse Lyapunov subgroup must have a dense orbit in the fiber by maximality of the ideal chosen,  $H$  is semisimple and  $\text{Stab}_H(x)$  must be an irreducible lattice for every  $x \in X_E$ .

If it does not have a semisimple factor, then  $H$  is a solvable group. But each coarse Lyapunov flow is contained in the nilradical, which is a closed subspace. So since the coarse Lyapunov foliations generate the action of  $H$ , the group must be nilpotent. Furthermore,  $[\mathfrak{h}, \mathfrak{h}]$  has closed orbits and is generated by coarse Lyapunov subspaces. By maximality of the ideal chosen,  $[\mathfrak{h}, \mathfrak{h}]$  does not contain a coarse Lyapunov subspace. In particular,  $[\mathfrak{h}, \mathfrak{h}]$  must be contained in the orbit of  $\mathbb{R}^k$ , and is generated by commutators of symplectic pairs of weights. Since  $\mathfrak{h}$  is nilpotent, we know the group generated by any given pair of symplectic weights must be either abelian or the Heisenberg group by Proposition 3.4. Furthermore, since  $\mathfrak{h}$  is nilpotent,  $[[\mathfrak{h}, \mathfrak{h}], \mathfrak{h}] \subset [\mathfrak{h}, \mathfrak{h}]$ . For each coarse Lyapunov subalgebra  $\mathfrak{g}^\beta$ ,  $[[\mathfrak{h}, \mathfrak{h}], \mathfrak{g}^\beta] \subset \mathfrak{g}^\beta$ . But  $(\bigoplus_{\beta \in \Delta_f} \mathfrak{g}^\beta) \cap [\mathfrak{h}, \mathfrak{h}] = \{0\}$ , so  $[\mathfrak{h}, \mathfrak{h}]$  is central in  $\mathfrak{h}$  and  $\mathfrak{h}$  is a 2-step nilpotent group.  $\square$

*Remark 6.6.* It is easy to see that if any part of the  $\mathbb{R}^k$  action is contained in the fiber, then it cannot be ergodic on the total space. Therefore, if one assumes that every one-parameter subgroup is ergodic, one may conclude that  $H$  is abelian.

**6.3. Base-Fiber Relations.** We will now analyze commutator cycles between weights from the base and the fiber.

**Proposition 6.7.** *If  $\alpha \in \Delta_b$  and  $\beta \in \Delta_f$  are linearly independent, then  $\rho^{\alpha,\beta}(s, t, x)$  is independent of  $x$ .*

Recall that by Proposition 6.5, there exists a Lie group  $H$  generated by the fiber subgroups acting transitively on each fiber, and that  $H$  is either semisimple, abelian or 2-step nilpotent. We prove Lemma 6.7 by treating these cases separately. First no

**Case:  $H$  is semisimple.** We first address the case when  $H$  is semisimple. In this case, the flows  $\eta^\beta$  with  $\beta \in \Delta_f$  generate the root subgroups of  $H$ . Furthermore, as in Proposition 3.4,

the restriction of the Cartan action to the elements preserving the fiber  $H \cdot x$  coincides with the translation action by some Cartan subgroup of  $H$ . Let  $V \subset \mathbb{R}^k$  be the subspace which preserves the fibers, so that  $\dim V = \text{rank}_{\mathbb{R}}(H)$ .

Now if  $a \in \mathbb{R}^k$  is any element, it still induces an automorphism of  $H$  homotopic to the identity which preserves the root spaces by Proposition 2.9. The space of such automorphisms can be identified with  $V$ , since it is exactly the inner automorphisms of these elements. Therefore, if  $W = \bigcap_{\beta \in \Delta_f} \ker \beta$ , then  $W \cap V = \{0\}$  and  $\dim(W) = k - \text{rank}_{\mathbb{R}}(H)$ . Therefore,  $\mathbb{R}^k = V \oplus W$ . Now notice that if  $a \in V$  and  $\alpha \in \Delta_b$ , then  $\alpha(a) = 0$ , since  $a$  only moves the position in the fiber and  $\alpha$  is transverse to the fiber. Similarly, if  $\beta \in \Delta_f$ ,  $\beta(a) = 0$  for every  $a \in W$ , by definition of  $W$ .

Assume for contradiction that  $\alpha \in \Delta_b$ ,  $\beta \in \Delta_f$ , but  $\rho^{\alpha, \beta}(s, t, x) \not\equiv 0$ . Then there exists some  $\chi \in \Delta$  such that  $\chi = s\alpha + t\beta$  with  $s, t > 0$ . But neither  $\chi|_V$  nor  $\chi|_W$  is identically 0, so it can't be in either  $\Delta_b$  or  $\Delta_f$ . This is a contradiction, so we may conclude  $\rho^{\alpha, \beta}(s, t, x) \equiv 0$ .

**Case:  $H$  is abelian.** In this case, we will prove the following stronger claims:

$$(6.1) \quad \text{if } [\alpha, \beta] \neq \emptyset, [\alpha, \beta] = \{\beta + \alpha, \dots, \beta + m\alpha\} \text{ for some } m \geq 1$$

$$(6.2) \quad \rho_{\beta+k\alpha}^{\alpha, \beta}(s, t, x) = cs^k t \text{ for all } s, t \in \mathbb{R} \text{ and } x \in X^E$$

We prove these by an induction on  $|D(\alpha, \beta) \cap \Delta_f|$ . Before starting this induction, we need to develop some preparatory material.

We first claim that if  $\alpha \in \Delta_b$  and  $\beta \in \Delta_f$ , then  $\rho^{\alpha, \beta}(s, t, x) = \rho^{\alpha, \beta}(s, t, y)$  for all  $x, y$  in the same fiber. Notice that if  $\chi \in \Delta_b$ , then  $\rho_{\chi}^{\alpha, \beta} \equiv 0$ , since  $\beta \in \Delta_f = E_0 \setminus E$  and  $E_0$  is an ideal. Therefore, as in Lemma 4.9, it suffices to show that there exists  $\chi_0 \in D(\alpha, \beta) \cup \{\beta\}$  such that  $\eta^{\chi_0}$  commutes with  $\eta^{\chi}$  for every  $\chi \in (D(\alpha, \beta) \cap \Delta_f) \cup \{\alpha\} \cup \{\beta\}$  (such a  $\chi_0$  will satisfy  $M^{\{\chi_0\}}(x) = M^{E_0}(x)$  by maximality of  $E \subset E_0$ ). Choose  $\chi_0$  to be the weight closest to  $\alpha$  in  $(D(\alpha, \beta) \cup \{\beta\}) \cap \Delta_f$ . Then since the fiber is abelian, it commutes with everything in  $(D(\alpha, \beta) \cup \{\beta\}) \cap \Delta_f$ , and since it is the closest weight to  $\alpha$ , it commutes with  $\alpha$ . Thus,  $\rho^{\alpha, \beta}(s, t, x)$  is constant along the fiber.

**Lemma 6.8.** *In the case of  $H$  abelian,  $\rho^{\alpha, \beta}(s, \cdot, x) : \mathbb{R} \rightarrow H$  is a homomorphism (which may depend on  $s, x$ ).*

*Proof.* From the definition of geometric commutators and the fact that  $[\chi_0, \alpha] = \emptyset$ , we get that  $\rho^{\alpha, \beta}(s, t_1 + t_2, x) = \rho^{\alpha, \beta}(s, t_1, x) + \rho^{\alpha, \beta}(s, t_2, y)$  for some  $y$  in the same fiber of  $x$  (c.f. Figure 1). Notice that in the figure, the arrow points in the  $\alpha$  direction, which is the only direction along the base. The blue curves represent the value of the commutators evaluated at their respective points. Since the fiber is abelian, and the “rectangle” of blue segments and black segments based at  $y$  is contained in the fiber, it is exactly a rectangle. Therefore, we get the desired additivity. □

Recall that if  $\chi \in D(\alpha, \beta)$ ,  $\chi = u\alpha + v\beta$  with  $u, v > 0$ . By (3.2), if  $a \in \ker \alpha$ ,  $e^{v\beta(a)} \rho_{\chi}^{\alpha, \beta}(s, t, a \cdot x) = \rho_{\chi}^{\alpha, \beta}(s, e^{\beta(a)} t, x)$ . Choose  $x$  to be a periodic orbit. Since the induced map on the fiber of a periodic point is an irreducible Cartan action on a torus,  $\{\beta(a) : a \cdot x = x\}$



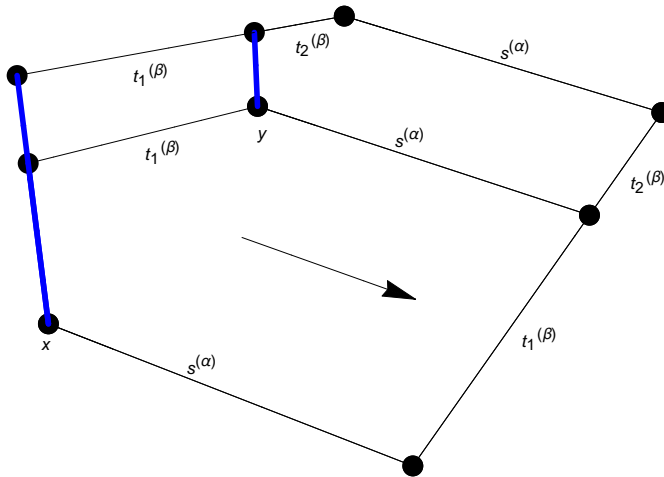


FIGURE 1. The cocycle property for  $\rho_{\chi_0}^{\alpha, \beta}(s, \cdot, x)$

is dense in  $\mathbb{R}$ . Therefore, by continuity,  $e^{vr} \rho_{\chi}^{\alpha, \beta}(s, t, x) = \rho_{\chi}^{\alpha, \beta}(s, e^r t, x)$ . By Lemma 6.8, we conclude that either  $v = 1$  or  $\rho_{\chi}^{\alpha, \beta} \equiv 0$ .

This implies that  $[\alpha, \beta] \subset \{u\alpha + \beta : u > 0\} \cap \Delta$ . We inductively show that each  $\rho_{\chi}^{\alpha, \beta}(s, t, x)$  is a polynomial of the form  $cs^k t$  for some  $c$  independent of  $x$  for all  $\chi \in [\alpha, \beta]$ . Write  $[\alpha, \beta] = \{\chi_0, \chi_1, \dots, \chi_m\} = \{u_0\alpha + \beta, u_1\alpha + \beta, \dots, u_m\alpha + \beta\}$ , with  $u_0 < \dots < u_m$ .

Fix any  $\chi \in [\alpha, \beta]$ , and let  $\varphi_{\chi}(s, x) = \rho_{\chi}^{\alpha, \beta}(s, 1, x)$ . Then  $e^{\chi(a)} \varphi_{\chi}(s, x) = \rho_{\chi}^{\alpha, \beta}(e^{\alpha(a)} s, e^{\beta(a)}, a \cdot x)$  and

$$(6.3) \quad \varphi_{\chi}(s, x) = e^{-u\alpha(a)} \varphi_{\chi}(e^{\alpha(a)} s, a \cdot x),$$

for every  $a \in \mathbb{R}^k$  by linearity of  $\rho_{\chi}^{\alpha, \beta}$  in the second variable.

**Lemma 6.9.** *If  $\varphi : \mathbb{R} \times X \rightarrow \mathbb{R}$  is any continuous function satisfying (6.3),  $\varphi(s, x) = cs^u$  for all  $x \in X^E$  and  $s > 0$  and some constant  $c$  independent of  $x$ .*

*Proof.* Let  $V_0 = \ker \alpha \subset \mathbb{R}^k$ , and  $A_0 = \{x \in X^E : V_0 \cdot x \text{ is dense in } X^E\}$ . If  $a_q \in \mathbb{R}^k$  is any element such that  $\alpha(a) = q$ , then  $V_q = \{a \in \mathbb{R}^k : \alpha(a) = q\} = V_0 + a_q$  is a translate of  $V_0$ , and  $A_q = \{x \in X^E : V_q \cdot x \text{ is dense in } X^E\}$  has  $A_q = a_q \cdot A_0$ . Then since the action of  $\ker \alpha$  is ergodic,  $A_0$ , and hence  $A_q$ , has full measure. Let  $A = \bigcap_{q \in \mathbb{Q}} A_q$ . Then  $A$  also has full measure, and is dense.

If  $x \in A$ , then for every  $q$ , there exists a sequence  $a_n \in V_q$  such that  $a_n \cdot x \rightarrow x$ . Then if  $q = -\log s$ , by (6.3):

$$\varphi(s, x) = e^{-u\alpha(a_n)}\varphi(e^{\alpha(a_n)}s, a_n \cdot x) = s^u\varphi(1, a_n \cdot x) \rightarrow s^u\varphi(1, x).$$

Since this holds for a dense set of  $x \in X^E$  and all  $q \in \mathbb{Q}$ , it holds for all  $x \in X^E$  and  $s > 0$ . If  $c_x = \varphi(1, x)$ , then  $\varphi(s, x) = c_x s^u$  for all  $s > 0$ . Then notice that by (6.3),

$$c_{a \cdot x}(e^{\alpha(a)}t)^u = \varphi(e^{\alpha(a)}t, a \cdot x) = e^{u\alpha(a)}\varphi(t, x) = e^{u\alpha(a)}c_x t^u.$$

So  $c_{a \cdot x} = c_x$  for all  $a \in \mathbb{R}^k$ . By ergodicity,  $c_x$  is independent of  $x$ .  $\square$

**Corollary 6.10.**  $u \in \mathbb{Z}_+$  and  $\rho_\chi^{\alpha, \beta}(s, t, x) = c_\chi s^u t$  for some  $c_\chi$  independent of  $x$ .

*Proof.* First observe that  $\rho_\chi^{\alpha, \beta}(s, t, x) = t\rho_\chi^{\alpha, \beta}(s, 1, x) = t\varphi_\chi(s, x) = ts^u c_\chi$  by Lemmas 6.8 and 6.9.

Let  $\mathcal{Q}$  be the group freely generated by a copy of  $\mathbb{R}$  for each  $\alpha, \beta$  and  $\chi \in [\alpha, \beta] \cap \Delta_f$ . Let  $Q$  be the factor of  $\mathcal{Q}$  by the relations  $[s^{(\alpha)}, t^{(\beta)}] * \rho^{\alpha, \beta}(s, t, x)$  and  $[s^{(\chi_1)}, t^{(\chi_2)}]$  for  $\chi_1, \chi_2 \in \Delta_f \cap [\alpha, \beta]$ , which are independent of  $x$  by the first part of the Corollary. These relations allow us to express any element in the form  $t^{(\alpha)} * h$ , with  $h \in H$  ( $H$  is the abelian group which acts transitively on the fiber). This expression is unique, since if  $t^{(\alpha)} * h = s^{(\alpha)} * h'$ , then  $(t - s)^{(\alpha)} * h''$  is a cycle, and  $t = s$ . Therefore,  $h = h'$ . This means that  $Q$  is a Lie group since it is locally path-connected and has a neighborhood which injects into Euclidean space. Therefore, the coefficients go up by at least one when taking commutators by Proposition 2.9 and  $u \in \mathbb{Z}_+$ .  $\square$

**Case:  $H$  is 2-step nilpotent.** Observe that in the abelian case, the only weights  $\chi$  for which  $\rho_\chi^{\alpha, \beta} \neq 0$  are those in  $D(\alpha, \beta)$ . But  $-\beta \notin D(\alpha, \beta)$  for any pair  $\alpha \in \Delta_b$  and  $\beta \in \Delta_f$ , so the weights of  $D(\alpha, \beta)$  generate an abelian subgroup of  $H$  which does not generate the center of  $H$ . In particular, the arguments for the abelian case work verbatim in the 2-step nilpotent case.

**6.4. Base-base relations.** For the final step in proving constant pairwise cycle structure, consider two weights  $\alpha$  and  $\beta$  from the base. While we already know that the geometric commutator is constant when taken along the base, the same is not at all clear on the space  $X^E$  as the geometric commutators may and indeed will involve weights from the fibers.

The following proposition is the main result in this section, and will finish the proof of Theorem 6.1.

**Proposition 6.11.** *If  $\alpha, \beta \in \Delta_b$  are nonproportional, then  $\rho^{\alpha, \beta}(s, t, x)$  is independent of  $x$ .*

As in Section 6.3, the proof of this proposition is split into the semisimple and abelian case.

**Case:  $H$  is semisimple.** We use the results and notations from the semisimple case in 6.3. Recall that  $W \subset \mathbb{R}^k$  is the intersection of all the kernels of the fiber weights, and that the homogeneous action of  $W$  is effective on the base  $X^{E_0}$ . Given nonproportional base weights  $\alpha, \beta \in \Delta_b$ , find  $a \in W$  with  $\alpha(a), \beta(a) < 0$ . Then any leg  $\lambda$  in the geometric bracket  $[\alpha, \beta]$  in  $X^E$  gets contracted by  $a^n, n \rightarrow \infty$ . Since  $a$  is in the kernel of all fiber weights, no  $\lambda \in [\alpha, \beta]$  can be a fiber weight. If  $\beta = -c\alpha$  are proportional, then we know actions of  $\alpha$  and  $\beta$  factor through a Lie group in any factor by Lemma 3.3.

**Case:  $H$  is abelian.**

The proof of this case involves several inductions. The outermost induction is on  $\#(D(\alpha, \beta) \cap \Delta_b)$ . We unify the proof of the inductive and base cases, and point out the part of the proof where  $\#(D(\alpha, \beta) \cap \Delta_b) = 0$  is used for the base case. In each step we will show

$$(6.4) \quad \text{if } \chi \in [\alpha, \beta], \text{ then } \chi = k\alpha + l\beta \text{ for some } k, l \in \mathbb{Z}_+$$

$$(6.5) \quad \text{if } \chi = k\alpha + l\beta \in [\alpha, \beta], \text{ then } \rho_\chi^{\alpha, \beta}(s, t, x) = c_\chi s^k t^l$$

First note that if  $\chi \in \Delta_b$ , then  $\rho_\chi^{\alpha, \beta}(s, t, x)$  is independent of  $x$ , since the bracket relations on the total space  $X^E$  project to bracket relations on  $X^{E_0}$ . Since  $X^{E_0}$  is homogeneous, Claims 6.4 and 6.5 follow immediately. Therefore, it suffices to show the claims for every  $\chi \in [\alpha, \beta] \cap \Delta_f$ .

We now begin the second induction. Let  $\Omega_l = \{u\alpha + v\beta : u + v = l, u, v \geq 0\} \cap [\alpha, \beta] \cap \Delta_f$ , so that there are finitely many values  $l_0 < l_1 < \dots < l_m$  such that  $[\alpha, \beta] \cap \Delta_f = \bigcup_{i=0}^m \Omega_{l_i}$ .

**Lemma 6.12.** *If  $\chi \in [\alpha, \Omega_{l_i}]$  or  $[\beta, \Omega_{l_i}]$ , then  $\chi \in \Omega_{l_i+t}$  for some  $t \geq 1$ .*

*Proof.* Without loss of generality consider  $\chi \in [\alpha, \Omega_{l_i}]$ . If  $\chi \in \Delta_b$ , let  $\mathcal{Q}$  be the group freely generated by copies of  $\mathbb{R}$  corresponding to  $\alpha, \chi$ , any base weight which is a positive combination of  $\alpha$  and  $\chi$ , as well as  $[\alpha, \beta] \cap \Delta_f$ . Let  $Q$  be the quotient of  $\mathcal{Q}$  by the normal subgroup generated by the commutator relations between each weight appearing (by Section 6.3 and the outer induction hypothesis, we know the corresponding  $\rho$ -functions are independent of the basepoint). Using a circular ordering to present elements gives a unique presentation as in Lemma 5.3, so  $Q$  is a Lie group. By Proposition 2.9, the coefficients of any weight in  $\chi \in [\alpha, \Omega_{l_i}]$  appear with linear integral terms, so the  $l$ -value increases.

If  $\chi \in \Delta_f$ , this follows from Claim 6.1 of Section 6.3.  $\square$

Since  $[\alpha, \beta] \cap \Delta_f = \bigcup \Omega_{l_i}$ , it suffices to show Claims 6.4 and 6.5 for each weight  $\chi \in \Omega_{l_i}$ . We will do this by an induction on  $i$ , starting from  $i = -1$ . The base of the induction will be  $i = -1$ , with  $\Omega_{l_{-1}} = \emptyset$ . Assume Claims 6.4 and 6.5 hold for  $\chi \in \Omega_{l_j}$ ,  $j < i$ . We use this induction hypothesis together with the induction hypothesis on the forms of commutators of  $\alpha$  and  $\beta$  with  $\gamma \in [\alpha, \beta] \cap \Delta_b$  (the outer induction hypothesis) and  $\chi \in \Omega_{l_j}$  with  $j < i$  (the inner induction hypothesis) to conclude the following.

**Lemma 6.13.**  *$\varphi_\chi(s_1 + s_2, x) = \varphi_\chi(s_1, x) + \varphi_\chi(s_2, \eta_{s_1}^\alpha x) + s_2 p(s_1, s_2)$  for some polynomial  $p$  whose coefficients are independent of  $x$ .*

*Proof.* We assume that  $\chi \in \Omega_{l_i}$ . Recall that  $\varphi_\chi(s, x) = \rho_\chi^{\alpha, \beta}(s, 1, x)$  is the unique path, written in circular ordering of the weights in  $D(\alpha, \beta)$ , which connects  $[s^{(\alpha)}, 1^{(\beta)}] \cdot x$  and  $x$ . Notice that using only the free group relations, we get that:

$$\begin{aligned}
[(s_1 + s_2)^{(\alpha)}, t^{(\beta)}] &= (-t)^{(\beta)} * (-s_1 - s_2)^{(\alpha)} * t^{(\beta)} * (s_1 + s_2)^{(\alpha)} \\
&= (-t)^{(\beta)} * (-s_1 - s_2)^{(\alpha)} * t^{(\beta)} * s_2^{(\alpha)} \\
&\quad * (((-t)^{(\beta)} * s_1^{(\alpha)} * t^{(\beta)}) * ((-t)^{(\beta)} * (-s_1)^{(\alpha)} * t^{(\beta)})) * s_1^{(\alpha)} \\
&= (-t)^{(\beta)} * (-s_1)^{(\alpha)} * (t^{(\beta)} * (-t)^{(\beta)}) * (-s_2)^{(\alpha)} * t^{(\beta)} * s_2^{(\alpha)} * \\
&\quad * ((-t)^{(\beta)} * s_1^{(\alpha)} * t^{(\beta)}) * [s_1^{(\alpha)}, t^{(\beta)}] \\
&= ((-t)^{(\beta)} (-s_1)^{(\alpha)} * t^{(\beta)}) * [s_2^{(\alpha)}, t^{(\beta)}] * ((-t)^{(\beta)} * s_1^{(\alpha)} * t^{(\beta)}) * [s_1^{(\alpha)}, t^{(\beta)}]
\end{aligned}$$

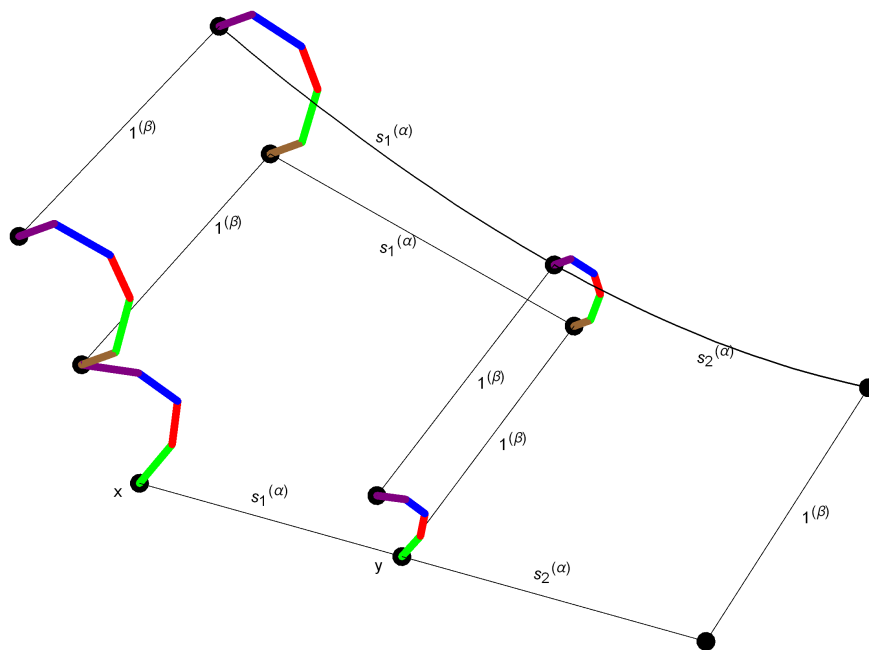


FIGURE 2. The cocycle-like property

Figure 2 illustrates the equality of  $[(s_1 + s_2)^{(\alpha)}, t^{(\beta)}]$  and the last term of the string of equalities above. We may replace the last line with

$$(6.6) \quad \rho^{\alpha, \beta}(s_1, t, x) * \left( (t^{(\beta)} * s_1^{(\alpha)} * (-t)^{(\beta)}) * \rho^{\alpha, \beta}(s_2, t, y) * (t^{(\beta)} * s_1^{(\alpha)} * (-t)^{(\beta)})^{-1} \right)$$

where  $y = \eta_{s_1}^\alpha(x)$ . We now use the induction hypothesis. Notice that we would like to compute the  $\chi$  term, so we must try to use the group relations known to put the expression into its circular ordering. Let us summarize the figure. Notice that in Figure 2, there are 4 curves representing each  $\rho^{\alpha, \beta}$ -term in (6.6) which terminate at  $x$  and  $y$ , respectively. The first, purple curve represents the weights in  $[\alpha, \beta] \cap \Delta_b$ . Since we know the base-fiber

relations from Section 6.3 take polynomial forms independent of  $x$ , we may assume that we have reordered the usual circular ordering to have all base weights appearing first. The blue curve represents the legs from the collection of weights  $\gamma \in \Omega_{l_j}$  with  $j < i$ , for which we know that  $\rho_\gamma^{\alpha,\beta}$  take the form of Claim 6.5.<sup>1</sup> The red curve represents the leg  $\chi \in \Omega_{l_i}$  which we now consider. Finally, the green legs represent the other legs in  $\Omega_{l_i}$ , together with any legs of  $\Omega_{l_j}$ ,  $j > i$ .

Conjugating the second commutator term corresponds to “sliding” it along the  $1^{(\beta)}$ ,  $s_1^{(\alpha)}$  and in the opposite direction  $1^{(\beta)}$  legs. Notice that all terms, with the possible exception of the purple base legs, commute with one another, since the fiber is abelian. If  $\gamma$  is a green leg, then  $\rho_\chi^{\alpha,\gamma} = \rho_\chi^{\beta,\gamma} \equiv 0$  by Lemma 6.12. Therefore, the nonlinear parts of the  $\chi$  terms come from the purple legs ( $[\alpha, \beta] \cap \Delta_b$ ) and blue legs ( $\Omega_{l_j}$ ,  $j < i$ ), which are known to have polynomial commutator forms by induction. The new brown color curves correspond to the possibility of weights appearing  $[\alpha, [\alpha, \beta]] \setminus [\alpha, \beta]$  and  $[\beta, [\alpha, \beta]] \setminus [\alpha, \beta]$ , which we also know take polynomial forms by induction.

Let us also argue algebraically using (6.6). We must commute the  $\rho^{\alpha,\beta}(s_2, t, x)$  term with the  $(t^{(\beta)} * s_1^{(\alpha)} * (-t)^{(\beta)})$  term. By induction, if  $\gamma \in [\alpha, \beta] \cap \Delta_b$ , since  $\#D(\alpha, \gamma), \#D(\gamma, \beta) < \#D(\alpha, \beta)$ , we know that commuting  $(-t)^{(\beta)}$  past any  $\gamma$  gives polynomials, possibly in other base weights or fiber weights, but all of which lie strictly between  $\alpha$  and  $\beta$ . Commuting past the fiber weights, we know also by induction that any fiber weight  $\chi'$  for which  $\chi \in [\alpha, \chi']$  or  $[\beta, \chi']$  (or any such chain of commutators) is already known to be of polynomial form. By Section 6.3, commuting  $(-t)^{(\beta)}$  past each leg of  $\rho^{\alpha,\beta}(s_2, t, x)$  in the fiber also gives a polynomial in  $s_2$  and  $t$ . We may similarly pass  $s_1^{(\alpha)}$  and  $t^{(\beta)}$  through to arrive at a product of terms such that every  $\gamma \in [\alpha, \beta] \cap \Delta_b$  is a polynomial, and every  $\chi' \in \Omega_{l_j}$  with  $j < i$  is a polynomial. The  $\chi$  terms coming from directly from  $\rho^{\alpha,\beta}(s_i, t, x)$  are therefore the only nonpolynomial terms which could appear after putting the weights in a circular ordering, and must appear linearly. □

By Lemma 3.10, for every  $\chi \in [\alpha, \beta]$ , either  $u \geq 1$  or  $v \geq 1$ . Without loss of generality assume that  $u \geq 1$ .

**Corollary 6.14.**  $\varphi_\chi(s, x) = cs^u$  for some  $c \in \mathbb{R}$  independent of  $x$

*Proof.* We claim that for every  $x \in X^E$  and almost every  $s_1 \in \mathbb{R}$ ,  $\frac{\partial}{\partial s} \Big|_{s=s_1} \varphi(s, x)$  exists. It suffices to show that  $\varphi(s, x)$  is locally Lipschitz in  $s$ . By Lemma 6.13,

$$\begin{aligned} |\varphi(s, x) - \varphi(s_1, x)| &= |\varphi(s - s_1, \eta_{s_1}^\alpha x) + (s - s_1) \cdot p(s, s - s_1)| \\ &\leq |s - s_1|^u |\varphi(1, a \cdot \eta_{s_1}^\alpha x)| + |s - s_1| \cdot |p(s, s - s_1)| \end{aligned}$$

---

<sup>1</sup>Note that in the case of  $\Omega_{l_0}$ , there are no blue legs, so one may consider this to be the base case, as we do not require any inductive hypothesis. We elect to write the proof this way as the proof of the base and inductive steps are virtually identical.

Since  $p$  is a polynomial and  $u \geq 1$ ,  $\varphi(s, x)$  has a Lipschitz constant in any neighborhood of  $s$ . Therefore, for almost every  $s_1 \in \mathbb{R}$ ,  $\varphi$  is differentiable in  $s$  at  $s_1$ . By Lemma 6.13, for every  $x \in X^E$ , almost every  $y \in W^\alpha(x)$  has  $\varphi(s, y)$  differentiable at 0. Therefore, for almost every  $x$ ,  $f(x) = \frac{\partial}{\partial s}\big|_{s=0} \varphi(s, x)$  exists. If  $a \in \ker \beta$ , then

$$f(a \cdot x) = \frac{\partial}{\partial s}\bigg|_{s=0} \varphi(s, a \cdot x) = \frac{\partial}{\partial s}\bigg|_{s=0} e^{u\alpha(a)} \varphi(e^{-\alpha(a)}s, x) = e^{(u-1)\alpha(a)} f(x).$$

$f$  is a measurable eigenfunction for every  $a \in \ker \beta$ , with eigenvalue  $e^{(u-1)\alpha(a)}$ . Therefore, either  $u = 1$  or  $f \equiv 0$ . In either case,  $f$  is constant almost everywhere. Since  $\varphi$  is a continuous function whose derivative at  $s = 0$  is constant at almost every point, the derivative exists everywhere and is constant. Let  $m$  denote the common value. Therefore,

$$\frac{\partial}{\partial s}\bigg|_{s=s_1} \varphi(s, x) = \frac{\partial}{\partial s}\bigg|_{s=0} \varphi(s + s_1, x) = \frac{\partial}{\partial s}\bigg|_{s=0} [\varphi(s_1, x) + \varphi(s, \eta_{s_1}^\alpha x) + p(s_1, s)] = m + q(s_1)$$

where  $q$  is some polynomial independent of  $x$ . Since  $\varphi(0, x) = 0$ , one may integrate  $c+q(s_1)$  to get a polynomial form independent of  $x$  for  $\varphi$ . Therefore,  $c = \varphi(1, x)$  is independent of  $x$  and  $\varphi(s, x) = cs^u$ . □

*Proof of Claims 6.4 and 6.5.* Since  $\varphi_\chi$  takes a polynomial form independent of  $x$ , we may analyze the function  $t \mapsto \rho_\chi^{\alpha, \beta}(s, t, x)$  in the same manner as Lemma 6.9, since

$$e^{u\alpha(a)+v\beta(a)} \rho_\chi^{\alpha, \beta}(1, t, x) = \rho_\chi^{\alpha, \beta}(e^{\alpha(a)}, e^{\beta(a)}t, a \cdot x) = e^{u\alpha(a)} \rho_\chi^{\alpha, \beta}(1, e^{\beta(a)}t, a \cdot x)$$

is the equation analogous to (6.3) which is the principal ingredient of the proof. That is,  $\rho_\chi^{\alpha, \beta}(s, t, x) = c_x s^u t^v$ . Notice that  $u, v \in \mathbb{Z}_+$  because  $c_x s^u t^v$  is a polynomial. □

**Case: 2-step nilpotent.** As in the base-fiber case, there cannot be negatively proportional weights appearing in  $D(\alpha, \beta)$ , so the corresponding fiber weights all commute, and the argument in the abelian case works verbatim in the case of a 2-step nilpotent group.

## 7. PROOFS OF THE MAIN RESULTS

*Proof of Theorem 1.4.* Let  $\mathbb{R}^k \curvearrowright X$  be a Cartan action satisfying the assumptions of Theorem 1.4, and  $\Delta$  denote its set of weights. Then Section 4 allows us to construct a sequence of subsets  $\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = \Delta$  such that each  $E_i$  is an ideal, and there are no ideals between  $E_i$  and  $E_{i+1}$  for  $i = 1, \dots, n-1$ . Then  $E_{n-1}$  is a maximal ideal of  $\Delta$ , and  $X^{E_{n-1}}$  has constant pairwise cycle structures by Lemma 4.9. Hence by Theorem 5.2,  $X^{E_{n-1}}$  is a homogeneous space and the induced Cartan action is a translation action.

Inductively, we will show that if the induced action on  $X^{E_j}$  is homogeneous, then so is the induced action on  $X^{E_{j-1}}$ . Notice that these are exactly the assumptions laid out at the start of Section 6. Then Propositions 3.3, 6.5, 6.7 and 6.11 imply that the action on  $X^{E_{j-1}}$  has constant pairwise cycle structures. Hence by Theorem 5.2, the induced action on  $X^{E_{j-1}}$  is homogeneous. Therefore, after finitely many steps, we conclude that the action on  $X^\emptyset = X$  is homogeneous.

Let  $h$  denote the conjugating map from  $X$  to  $G/\Gamma$ . We claim that  $h$  is  $C^1$ .

For any  $\alpha \in \Delta$ ,  $h$  intertwines the actions of the one parameter groups  $\eta_t^\alpha$  as well as the  $\mathbb{R}^k$  actions on  $X$  and  $G/\Gamma$  by Proposition 4.6. Notice that  $dh$  maps the generator of  $\eta_t^\beta$  on  $X$  to the one on  $G/\Gamma$ . The latter is smooth, the first Hölder by the assumption that the Cartan action is  $C^{1,\theta}$ . In consequence,  $h$  is  $C^{1,\theta}$ ,  $\theta$  as above, along all  $\eta_t^\beta$  orbits. Similarly,  $h$  is  $C^{1,\theta}$  along the  $\mathbb{R}^k$  orbit foliation.

We recall the main result of Journé in [18]: Given two continuous transverse foliations  $F_1$  and  $F_2$  with uniformly smooth leaves on a manifold  $M$ . Suppose  $f$  is a function uniformly  $C^{1,\theta}$  along  $F_1$  and  $F_2$ . Then  $f$  is  $C^1$ . Note that the theorem in [18] states a version for  $C^\infty$  functions along  $F_1$  and  $F_2$ . The proof however works for uniformly  $C^{1,\theta}$  functions as the author explicitly states.

Recall that  $a_0$  is the regular element in  $\mathbb{R}^k$  which determines the positive and negative weights  $\Delta_+$  and  $\Delta_-$ . Order the weights  $\lambda \in \Delta_-$  cyclically:  $\lambda_1, \dots, \lambda_l$ . Then inductively we see from Journé's theorem that  $h$  is  $C^{1,\theta}$  along the foliations tangent to  $E^{\lambda_1}, E^{\lambda_1} \oplus E^{\lambda_2}, \dots, \oplus_{i=1, \dots, l} E^{\lambda_i}$ . Clearly, the last foliation is nothing but the stable foliation of  $a_0$ ,  $\mathcal{W}_{a_0}^s$ . Hence,  $h$  is  $C^{1,\theta}$  along  $\mathcal{W}_{a_0}^s$ , and similarly along the unstable foliation  $\mathcal{W}_{a_0}^u$  of  $a_0$ . Using Journé's theorem again, we get that  $\pi$  is  $C^{1,\theta}$  along the weak stable foliation, by combining stable and orbit foliations, and then on  $X$  combining weak stable and unstable foliations.

In the  $C^\infty$  setting, one may repeat the arguments with the  $C^\infty$  version of the regularity lemma [18] to obtain that  $h$  is  $C^\infty$ , provided that the Hölder metrics of Section 2.1.3 are  $C^\infty$  when restricted to each coarse Lyapunov leaf.

Fix a Lyapunov exponent  $\alpha$  and some smooth Riemannian metric on  $X$  (not necessarily the one of Section 2.1.3). Let  $\varphi_a : X \rightarrow \mathbb{R}$  be defined to be the derivative of  $a$  restricted to  $E^\alpha$ . That is,  $\varphi_a(x) = da|_{E^\alpha}(x)$  and  $\varphi_a$  is a Hölder. Notice that if the ratio of the dynamical norm of Section 2.1.3 to the smooth metric is given by  $\psi(x)$ , then  $\psi(x)$  is Hölder (since the dynamical metric is Hölder) and  $\varphi_a(x) = e^{\alpha(a)}\psi(a \cdot x)\psi(x)^{-1}$ .

Take  $\mathcal{L} = X \times \mathbb{R}$  to be the trivial line bundle over the manifold  $X$ , and let  $\pi(x, t)$  be the point which has (signed) distance  $t$  from  $x$  in the smooth Riemannian metric on  $X$ . For each  $a \in \mathbb{R}^k$ , there exists a unique lift  $\tilde{a}$  to  $\mathcal{L}$  such that  $\pi(\tilde{a}(x, t)) = a \cdot \pi(x, t)$ . Notice that since each fiber corresponds to a  $C^\infty$  submanifold,  $\tilde{a}$  is a  $C^\infty$  extension as in Section A.3. Let  $H : \mathcal{L} \rightarrow \mathcal{L}$  be the map  $H(x, t) = (x, H_x(t))$ , with  $H_x$  uniquely satisfying  $\eta_{H_x(t)}^\alpha(x) = \pi(x, t)$ . Then observe that  $H\tilde{a}H^{-1}(x, t) = (a \cdot x, e^{\alpha(a)}t)$  by (2.1).

Notice that the derivative of  $\pi(x, t)$  with respect to  $t$  is the unit vector of  $E^\alpha$  with respect to the smooth Riemannian metric, and the derivative of  $\eta_{H_x(t)}^\alpha(x)$  is the derivative of  $H_x$  times the unit vector of  $E^\alpha$  with respect to the dynamical norm. Therefore,  $H'_x(t) = \psi(\pi(x, t))$ , so  $H_x \in C^1(\mathbb{R}, \mathbb{R})$  varies continuously with  $x$ . Now, finally we modify  $H$  slightly by defining  $G(x, t) = H(x, \psi(x)^{-1}t)$ . Then  $G'_x(0) = 1$  and  $G$  is still a linearization. Therefore,  $G$  is a system of  $C^\infty$  normal form coordinates by Corollary A.3, and  $H_x(t)$  is  $C^\infty$  in  $t$ . □

## APPENDIX A. EXAMPLES AND CLASSIFICATION OF AFFINE CARTAN ACTIONS

**A.1. Examples of Cartan Actions.** We summarize some well-known classes of homogeneous Cartan actions, as well as some lesser-known examples. There are two principle “building blocks,” (suspensions of) affine  $\mathbb{Z}^k$  actions on nilmanifolds and Weyl chamber flows.

*A.1.1.  $\mathbb{Z}^k$  affine actions and their suspensions.* Let  $A_1, A_2, \dots, A_k \in SL(n, \mathbb{Z})$  be a collection of commuting matrices such that  $A_1^{m_1} \dots A_k^{m_k} \neq \text{Id}$  unless  $m_i = 0$  for every  $i$ . Then  $\mathbb{R}^n$  splits as a sum of common eigenspaces for each matrix  $A_i$ . If every eigenspace is one-dimensional and the corresponding eigenvalues are positive and not identically 1 for every element of the action, we say that the action is *Cartan*.<sup>2</sup> Then there is an automorphism action  $\mathbb{Z}^k \curvearrowright \mathbb{T}^n$  defined by  $\mathbf{m} \cdot (v + \mathbb{Z}^n) = A_1^{m_1} A_2^{m_2} \dots A_k^{m_k} v + \mathbb{Z}^n$ . Analogous actions can be constructed when replacing  $\mathbb{T}^n$  by a nilmanifold, see [31]. We will restrict our discussion to tori for simplicity.

We may produce an  $\mathbb{R}^k$  action from this  $\mathbb{Z}^k$  action by defining a solvable group  $S = \mathbb{R}^k \ltimes \mathbb{R}^n$ . We define the semidirect product structure of  $S$  by fitting the  $\mathbb{Z}^k$  subgroup of  $SL(n, \mathbb{Z})$  into an  $\mathbb{R}^k$  subgroup. Since the eigenvalues of each  $A_i$  are all positive real numbers, each  $A_i$  fits into a one-parameter subgroup,  $A_i = \exp(tX_i)$  with  $X_i \in \mathfrak{sl}(n, \mathbb{R})$ . Furthermore, since  $[A_i, A_j] = e$ ,  $[X_i, X_j] = 0$ .<sup>3</sup> Thus, there exists a homomorphism  $f : \mathbb{R}^k \rightarrow SL(n, \mathbb{Z})$  such that  $f(e_i) = A_i$  for every  $i$ .

We are ready to define the semidirect product structure of  $S$ . Let  $(a_i, x_i) \in \mathbb{R}^k \times \mathbb{R}^n$  for  $i = 1, 2$ , and define

$$(a_1, x_1) \cdot (a_2, x_2) = (a_1 + a_2, f(a_2)^{-1}x_1 + x_2)$$

Then  $\Gamma = \mathbb{Z}^k \ltimes \mathbb{Z}^n$  is a cocompact subgroup of  $\mathbb{R}^k \times \mathbb{R}^n$  and  $\mathbb{R}^k$  is an abelian subgroup. The translation action is a Cartan action, since the common eigenspaces in  $\mathbb{R}^n$  are the coarse Lyapunov subspaces. Furthermore, the stabilizer of the subgroup  $\mathbb{T}^n \subset S/\Gamma$  is exactly  $\mathbb{Z}^k$ , and by construction, if  $v \in \mathbb{R}^n$  and  $a \in \mathbb{Z}^k$ :

$$a \cdot v = f(a)v \cdot a \sim f(a)v$$

Therefore, the translation action on  $S/\Gamma$  is the suspension of the  $\mathbb{Z}^k$  action on  $\mathbb{T}^n$ .

*A.1.2. Weyl Chamber Flows.* Let  $\mathfrak{g}$  be a semisimple Lie algebra and  $\mathfrak{a} \subset \mathfrak{g}$  be the *split Cartan subalgebra*, a canonically defined abelian subalgebra, unique up to automorphism of  $\mathfrak{g}$ . The subalgebra  $\mathfrak{a}$  satisfies that for every  $X \in \mathfrak{a}$ ,  $\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$  diagonalizable over  $\mathbb{R}$ , and is the maximal abelian subalgebra satisfying this property. A semisimple algebra  $\mathfrak{g}$  is called ( $\mathbb{R}$ -)split if the centralizer of  $\mathfrak{a}$  (ie, the common zero eigenspace of  $\text{ad}_X$  for  $X \in \mathfrak{a}$ ) is  $\mathfrak{a}$  itself.

The semisimple split Lie algebras are well-classified, the most classical example being  $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{R})$ , with Cartan subalgebra  $\mathfrak{a} = \{\text{diag}(t_1, t_2, \dots, t_d) : \sum t_i = 0\} \cong \mathbb{R}^{d-1}$ , which we will address directly now. Other examples include  $\mathfrak{g} = \mathfrak{so}(m, n)$  with  $|m - n| \leq 1$  and  $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$ , as well as split forms of the exotic algebras. In what follows,  $SL(d, \mathbb{R})$  may be replaced with a split Lie group  $G$ , with its corresponding objects.

<sup>2</sup>This differs from the notion of Cartan defined by other authors, where it is assumed that  $k = n - 1$ . Our condition is implied by  $k = n - 1$ .

<sup>3</sup>This is why  $k = n - 1$  implies the action is Cartan: this implies that the  $X_i$  generate a Cartan subgroup of  $\mathfrak{sl}(n, \mathbb{R})$



Let  $\Gamma \subset SL(d, \mathbb{R})$  be a cocompact lattice, and define the *Weyl chamber flow* on  $SL(d, \mathbb{R})/\Gamma$  to be the translation action of  $A = \{\text{diag}(e^{t_1}, \dots, e^{t_d}) : \sum t_i = 0\} \cong \mathbb{R}^{d-1}$ . Notice that if  $Y \in \mathfrak{sl}(d, \mathbb{R}) = T_e SL(d, \mathbb{R})$ , and  $a = \exp(X) \in A$  with  $X \in \mathfrak{a}$ , then:

$$da(Y) = \text{Ad}_a(Y) = \exp(\text{ad}_X)Y$$

Therefore, if  $Y$  is an eigenvector of  $\text{ad}_X$  with eigenvalue  $\alpha(X)$ ,  $da(Y) = e^{\alpha(X)}Y$ . The eigenvectors are exactly the elementary matrices  $Y_{ij}$ , with all entries equal to 0 except for the  $(i, j)^{\text{th}}$  entry, which is 1. For a general split group, it is classical that the eigenspaces are 1-dimensional. By direct computation, if  $X = \text{diag}(t_1, \dots, t_d)$ , then  $\text{ad}_X(Y) = XY - YX = (t_i - t_j)Y$ . Therefore, the eigenvalue functionals  $\alpha$  are exactly  $\alpha(X) = t_i - t_j$ . These functionals  $\alpha$  are called the *roots* of  $\mathfrak{g}$ , and are the weights of the Cartan action as considered above.

**A.1.3. Twisted Weyl Chamber Flows.** This example is a combination of the previous two examples. Let  $G$  be an  $\mathbb{R}$ -split semisimple Lie group, and  $\rho : G \rightarrow SL(N, \mathbb{R})$  be a representation of  $G$ , which has an induced representation  $\bar{\rho} : \mathfrak{g} \rightarrow \mathfrak{sl}(N, \mathbb{R})$ . Then  $\bar{\rho}$  has a *weight*  $\alpha : \mathfrak{a} \rightarrow \mathbb{R}$  for each common eigenspace  $E \subset \mathbb{R}^N$  for the transformations  $\{\rho(X) : X \in \mathfrak{a}\}$ , which assigns to  $X$  the eigenvalue of  $\rho(X)$  on  $E$ . We call  $\rho$  a *Cartan* representation if:

- (1) (No zero weights)  $\mathbf{0}$  is not a weight of  $\bar{\rho}$
- (2) (Non-resonant) no weight  $\alpha$  of  $\bar{\rho}$  is proportional to a root of  $\mathfrak{g}^4$
- (3) (One-dimensional) the eigenspaces  $E_\alpha$  for each weight  $\alpha$  are one-dimensional

Given an  $\mathbb{R}$ -split semisimple group with Cartan representation  $\rho$ , we may define a semidirect product group  $G_\rho = G \ltimes \mathbb{R}^N$  which is topologically given by  $G \times \mathbb{R}^N$ , with multiplication defined by:

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, \rho(g_2)^{-1} v_1 + v_2)$$

We now assume that  $\Gamma$  is a (cocompact) lattice in  $G$  such that  $\rho(\Gamma) \subset SL(N, \mathbb{Z})$  (this severely restricts the possible classes of  $\rho$  and  $\Gamma$  one may take). Then  $\Gamma_\rho = \Gamma \ltimes \mathbb{Z}^N \subset G_\rho$  is a (cocompact) lattice in  $G_\rho$ , and the translation action of the Cartan subgroup  $A \subset G$  on  $G_\rho/\Gamma_\rho$  is the *twisted Weyl chamber flow*. The weights of the action are exactly the roots of  $\mathfrak{g}$  and weights of  $d\bar{\rho}$ , which by the non-resonance condition implies that the coarse Lyapunov distributions are all one-dimensional.

When  $\mathbb{R}^N$  is replaced by a nilpotent Lie group, one may replace the toral fibers with certain nilmanifold fibers. See [31].

**A.1.4. Some Exotic Examples.** The main examples of Cartan  $\mathbb{R}^k \times \mathbb{Z}^l$  actions on solvable groups  $S$  are Cartan actions by automorphisms and suspensions of such actions. Indeed, (4) of Theorem A.1 implies that the orbit must generate a subgroup transverse to  $\mathfrak{n}$ , the nilradical of  $S$ . However, the orbit may also intersect  $\mathfrak{n}$ . This may happen in a trivial way, by taking any  $\mathbb{R}^k \times \mathbb{Z}^l$  action and its direct product with a transitive translation action on a torus. There are other interesting examples as well. For instance, one may construct an  $\mathbb{R}^2$  action in the following way: let  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  by a hyperbolic toral automorphism. Then

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<sup>4</sup>In fact, this is implied by (1)

$A$  also induces a diffeomorphism of the nilmanifold  $H/\Lambda$ , where  $H$  is the Heisenberg group  $H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$  and  $\Lambda$  are the integer points. The automorphism is given by  $(x, y, z) \mapsto (A(x, y), z)$ . While the automorphism is not Cartan, letting one copy of  $\mathbb{R}$  act by translation along the center and the other by a suspension (as in Section A.1.1) gives an  $\mathbb{R}^2$  Cartan action, since the action will still be normally hyperbolic with respect to the orbit for a dense set of elements.

In an analogous way, assume that a Cartan representation of a split Lie group  $G$  preserves a symplectic form  $\omega$  on  $\mathbb{R}^N$ . Then one may define a Heisenberg extension  $1 \rightarrow \mathbb{R} \rightarrow H \rightarrow \mathbb{R}^N \rightarrow 1$  of  $\mathbb{R}^N$  by letting  $H = \mathbb{R} \times \mathbb{R}^N$  topologically, with multiplication  $(t_1, v_1) \cdot (t_2, v_2) = (t_1 + t_2 + \omega(v_1, v_2), v_1 + v_2)$ . Then let  $\bar{G}_\rho = G \times_\rho H$ , where  $\rho(g)(t, v) = (t, \rho(g)v)$ . Since  $\rho$  preserves the symplectic form  $\omega$ , the extension of  $\rho$  to  $H$  acts by automorphisms. Then by construction, the Cartan subgroup of  $G$  commutes with the center of  $H$ , and the action of  $A \times \mathbb{R}$  is a Cartan action on any cocompact factor of  $\bar{G}_\rho$ .

What makes these examples special is that there exists elements  $a$  of the action for which  $\alpha(a) = 0$  for every weight  $\alpha$ . However, they are still totally Cartan, because the set of Anosov elements for homogeneous actions is exactly the complement of  $\bigcup_{\alpha \in \Delta} \ker \alpha$ , which is always dense. As remarked earlier, one can produce such isometric directions by taking the direct product with the transitive  $\mathbb{R}^l$  action on  $\mathbb{T}^l$ . The examples described here are extensions by such isometric actions, but are not direct products. In fact, such isometric directions are exactly the ones generated by any center of a Heisenberg group appearing in Proposition 3.4.

**A.2. Classification of Cartan Actions.** Here we consider affine actions of  $\mathbb{R}^k$  by left translations or more generally  $\mathbb{R}^k \times \mathbb{Z}^l$  by left translations and automorphisms on homogeneous spaces  $G/\Gamma$  where  $G$  is a connected Lie group, and  $\Gamma \subset G$  a closed subgroup of  $G$ . Unlike the rest of the paper, we do not require  $k + l \geq 2$  to obtain classification under the homogeneity assumption. Below we describe the structure of the homogeneous space  $G/\Gamma$  and the action  $\alpha$  in terms of the Levi decomposition. We also show that the connected component of the identity of  $\Gamma$  is normal in  $G$ . Passing to the quotient by  $\Gamma^0$ , we may assume w.l.o.g. that  $\Gamma$  is discrete and hence a lattice in  $G$ . The latter follows easily from Borel's density theorem for  $G$  semisimple but requires more work for a general connected Lie group  $G$ .

**Theorem A.1.** *Suppose that  $G$  is a connected Lie group, and  $\Gamma \subset G$  a closed subgroup. Suppose  $\mathbb{R}^k \times \mathbb{Z}^l \curvearrowright G/\Gamma$  is an affine volume preserving Cartan action, where  $\mathbb{R}^k \subset G$ . Let  $S$  and  $N$  denote the solvradical and nilradical of  $G$ , respectively, and  $G \cong L \times S$  be a Levi decomposition of  $G$  for some semisimple group  $L$ . Then*

- (1) *the connected component  $\Gamma^0$  of  $\Gamma$  is normal in  $G$ . Thus  $\Gamma$  may and will be assumed to be discrete henceforth.*
- (2) *If  $\sigma : G \rightarrow L$  is the canonical projection, then  $\sigma(L)$  is a lattice in  $L$ .*
- (3)  *$S \cap \Gamma$  is a lattice in  $S$ .*

- (4) *The restriction of the Cartan action to  $\mathbb{R}^k$  covers a Weyl chamber flow on  $L/\sigma(\Gamma)$ , and the action of  $\mathbb{Z}^l$  factors through a finite groups action by automorphisms.*
- (5)  *$N \cap \Gamma$  is a lattice in  $N$  and if  $\chi \in \Delta$  is not a root of  $L$ , then  $\mathfrak{g}_\chi \subset \mathfrak{n}$ .*

*Proof.* Note that the induced action from  $\mathbb{R}^k \times \mathbb{Z}^l$  to  $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$  is affine. Thus it suffices to argue the case of an affine action by  $\mathbb{R}^k$ .

(1): We prove (1) using the following strengthening by Y. Shalom of the classical Borel density theorem [36, Theorem 3.11]: if a real algebraic group acts  $\mathbb{R}$ -regularly on a variety  $V$ , and  $\mu$  is any probability measure on  $V_{\mathbb{R}}$ , then the stabilizer of  $\mu$  in  $G_{\mathbb{R}}$  is algebraic and has a normal cocompact subgroup which fixes every point in the support of  $\mu$ .

We apply Shalom's result to the adjoint representation of  $G$ . Recall that the adjoint group  $Ad G$  of  $G$  is always an  $\mathbb{R}$  group, and acts  $\mathbb{R}$ -regularly on the Grassmannian  $\mathcal{G}_l$  of  $l$ -dimensional subspaces of the Lie algebra of  $G$  where  $l = \dim \Gamma^0$ . Note that  $\Gamma$  fixes the Lie algebra of  $\Gamma^0$  under this action. Hence the probability Haar measure on  $G/\Gamma$  projects to a probability measure  $\nu$  on  $\mathcal{G}_l$ . By Shalom's result,  $Ad G$  contains a normal subgroup  $N$  which is cocompact and fixes the support of  $\nu$ . We deduce that the preimage  $\bar{N}$  of  $N$  in  $G$  normalizes  $\Gamma^0$ .

Finally, recall that the Cartan action comes from the subgroup  $\mathbb{R}^k \subset G$  which forces  $\mathbb{R}^k$  to be a split Cartan subgroup of  $G$ , i.e. a largest abelian subgroup of  $G$  whose adjoint action diagonalizes over  $\mathbb{R}$ . Then  $\bar{N}$  is normalized by  $\mathbb{R}^k$ . Hence the Oseledets splitting of  $\mathbb{R}^k$  is subordinate to the Lie algebra of  $\bar{N}$ . Hence  $G/\bar{N}$  has an induced Oseledets splitting. Since  $G/\bar{N}$  is compact, this cannot be.

(2), (3) and the first part of (5): These follow from Corollary 8.28 of [33], provided  $L$  has no compact factors. To see that there are no such factors, observe that since the Cartan action is affine,  $a \cdot (g\Gamma) = (f(a)\varphi_a(g)\Gamma)$ , where  $f : \mathbb{R}^k \times \mathbb{Z}^l \rightarrow G$  is some homomorphism and  $\varphi_a$  is an automorphism of  $G$  preserving  $\Gamma$ . This implies that  $da = Ad(f(a)) \circ d\varphi_a$ , which is an automorphism of  $\mathfrak{g} = Lie(G)$ . The sum of the eigenspaces of modulus 1 for this automorphism must be  $Lie(f(\mathbb{R}^k))$ , since the action is Cartan. Every automorphism of a compact semisimple Lie group has only eigenvalues of modulus 1, but cannot be abelian. Therefore, we may apply the result.

(4): We claim that  $a \cdot Sx = S(a \cdot x)$ . Since  $S$  is a characteristic subgroup, it is invariant under the automorphism  $\varphi_a$  as well as conjugation by an element of  $G$ . Since the action of  $\mathbb{R}^k \times \mathbb{Z}^l$  is a composition of such transformations, we get the intertwining property. In particular, the Cartan action descends to an action on  $L/\sigma(\Gamma)$ . For any semisimple group, the identity component of  $Aut(L)$  is the inner automorphism group. Therefore, the  $\mathbb{R}^k$  component of  $\mathbb{R}^k \times \mathbb{Z}^l$  must act by translations, since there are no small  $g \in L$  such that  $g\sigma(\Gamma)g^{-1} = \sigma(\Gamma)$ . Furthermore, the induced action must have some continuous part if  $L \neq \{e\}$ . Indeed, since the outer automorphism group of  $G$  is finite, if the action were by  $\mathbb{Z}^l$ , some finite index subgroup of the  $\mathbb{Z}^l$  action would be a translation action. But the elements of  $G$  may be written as  $z \cdot a \cdot u$ , where  $z$  is in the center of  $G$ ,  $a$  is in some Cartan subgroup and  $u$  is a nilpotent element commuting with  $a$ . In particular, the one-parameter subgroup generating  $a$  would be a 1-eigenspace of the action which is not contained in the orbit, violating the Cartan condition. Therefore, the action must contain at least one one-parameter subgroup which acts by translations. By the Cartan assumption,  $da$  is diagonalizable for every  $a$  in

this subgroup, so the translations must be by semisimple elements. Furthermore, the action must contain all elements that commute with  $a$ , and therefore an  $\mathbb{R}$ -split Cartan subgroup. That is, the action of the continuous part of  $\mathbb{R}^k \times \mathbb{Z}^l$  is a Weyl chamber flow and the  $\mathbb{Z}^l$  factors through a finite group action by automorphisms fixing the corresponding Cartan subgroup.

To see the second part of (4), we note that the arguments of Proposition 3.13 in [10] can be applied to show that if  $\mathfrak{g}_\chi \subset \mathfrak{s}$ , then  $\mathfrak{g}_\chi \oplus \mathfrak{n}$  is a nilpotent ideal of  $\mathfrak{s}$ . This is sufficient for our claim. The setting there is that of an affine Anosov diffeomorphism, rather than an action which may have an orbit sitting inside the group. This leads to some exotic examples (see Section A.1.4).  $\square$

**A.3.  $C^1$ -uniqueness of normal forms.** In this section, we generalize a uniqueness result for normal forms to the  $C^1$  setting. Let  $f : X \rightarrow X$  be a uniformly continuous transformation of a compact space  $X$ ,  $\mathcal{L} = X \times \mathbb{R}$  be the trivial line bundle over  $f$ , and  $F : \mathcal{L} \rightarrow \mathcal{L}$  be a uniformly contracting  $C^r$  extension of  $f$ . That is,  $F$  takes the form  $F(x, t) = (f(x), F_x(t))$ , where  $x \mapsto F_x \in C^r(\mathbb{R}, \mathbb{R})$  is continuous in the  $C^r$  topology and  $\|F'_x(t)\| < 1 - \varepsilon$  for all  $(x, t) \in \mathcal{L}$ . A  $C^r$ -normal form coordinate system for  $F$  is a map  $G : \mathcal{L} \rightarrow \mathcal{L}$  of the form  $G(x, t) = (x, G_x(t))$  such that:

- (1)  $G_x \in C^r(\mathbb{R}, \mathbb{R})$  for every  $x \in X$ ,
- (2)  $G_x(0) = 0$  and  $G'_x(0) = 1$  for every  $x \in X$ ,
- (3)  $x \mapsto G_x$  is continuous from  $X$  to  $C^r(\mathbb{R}, \mathbb{R})$ , and
- (4)  $GF(x, t) = \hat{F}G(x, t)$

where  $\hat{F}(x, t) = (f(x), F'_x(0)t)$ . Katok and Lewis showed that every  $C^\infty$  extension has  $C^\infty$ -normal form coordinates [22]

**Lemma A.2.** *If  $G_1$  and  $G_2$  are both  $C^1$ -normal form coordinate systems, then  $G_1 = G_2$ .*

*Proof.* Let  $H = G_2^{-1}G_1$ , and  $\hat{F}$  be as in the definition of normal forms coordinates. Then

$$H\hat{F} = G_2^{-1}G_1\hat{F} = G_2^{-1}FG_1 = \hat{F}G_2^{-1}G_1 = \hat{F}H$$

Therefore  $H$  commutes with  $\hat{F}$  and  $H(x, t) = (x, H_x(t))$ , with  $x \mapsto H_x$  continuous in the  $C^1$ -topology and  $H'_x(0) = 1$  for all  $x \in X$ . By the commutativity:

$$(A.1) \quad H_{f^{n(x)}}((F_x^{(n)})'(0)t) = (F_x^{(n)})'(0)H_x(t)$$

where  $F_x^{(n)} = F_{f^n(x)} \circ \dots \circ F_{f(x)} \circ F_x$ . By continuity of  $x \mapsto F_x$  in the  $C^1$  topology,  $\left| (F_x^{(n)})'(x) \right| < \lambda^n$  for some fixed  $\lambda < 1$ . Choose some convergent subsequence  $f^{n_k}(x) \rightarrow y$ . Then there exists  $t_k \in [0, ((F_x^{(n)})'(0)t)] \subset [0, \lambda^n]$  such that

$$H'_{f^{n_k}(x)}(t_k) = \frac{H_{f^{n_k}(x)}((F_x^{(n)})'(0)t)}{(F_x^{(n)})'(0)t}$$

Since  $f^{n_k}(x) \rightarrow y$  and  $t_k \rightarrow 0$ , since  $x \mapsto H_x$  is continuous in the  $C^1$  topology,  $H'_{f^{n_k}(x)}(t_k) \rightarrow H'_y(0) = 1$ . Then by (A.1),

$$H_x(t) = \lim_{k \rightarrow \infty} \frac{H_{f^n(x)}((F_x^{(n)})'(0)t)}{(F_x^{(n)})'(0)} = \lim_{k \rightarrow \infty} H'_{f^{nk}(x)}(t_k)t = t$$

so  $H = \text{Id}$  and  $G_1 = G_2$ . □

**Corollary A.3.** *If  $F$  is a  $C^\infty$  extension, then any  $C^1$ -normal form coordinate system is a  $C^\infty$  coordinate system.*

*Proof.* If  $F$  is a  $C^\infty$  extension, it has a  $C^\infty$ -normal form coordinate system  $G$  by [22]. Obviously, any  $C^\infty$  coordinate system is a  $C^1$  coordinate system. Therefore, any  $C^1$  coordinate system must agree with  $G$  by Lemma A.2, and is therefore  $C^\infty$ . □

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