THE UNIVERSITY OF CHICAGO
Department of Economics
Econ 304: Math Camp

Part I. : Real Analysis

Problems that are not explicitly formulated refer to Royden's Real Analysis, third edition. Pages and numbers may differ for earlier editions. Notation is either from Royden or lecture notes.

1. **Real Numbers**

a. Prove Corollary 4 on page 35.
See the proof on page 36.

b. Ch.2, p.34, Ex.1,5.
Before starting to do the exercises, let us prove two useful facts.
First, $0 \cdot a = 0$, because

$$\begin{aligned} b + 0 \cdot a &= b + a - a + 0 \cdot a = b + 1 \cdot a + 0 \cdot a - a \\ &= b + (1 + 0) \cdot a - a = b + 1 \cdot a - a = b + a - a = b + 0 = b. \end{aligned}$$

Second, $(-1) \cdot a = -a$ because

$$a + (-1) \cdot a = 1 \cdot a + (-1) \cdot a = (1 - 1) \cdot a = 0 \cdot a = 0.$$

Exercise 1. Show that $1 \in P$.

Assume $1 \notin P$. Then, by axioms A7 and B4, it must be true that $(-1) \in P$. Then, $1 \in P$, as a product of two positive numbers: $1 = -(-1) = (-1)(-1)$. So we arrive to the contradiction.

Exercise 5. $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$. Prove that

a. $|xy| = |x||y|$

Assume $x < 0, y > 0$. Then $|x| = -x$, $|y| = y$, and therefore $|x||y| = (-x)y = -xy$. By B4, $-x > 0$, and by B2, $(-x)y = -xy > 0$. Using this and B4 we conclude that $xy = -(-xy) < 0$. Therefore, $|xy| = -xy$, and this proves the result. Cases $x, y > 0$, $x, y < 0$, etc. are very similar.

b. $|x + y| \leq |x| + |y|$

$|x| \geq x$, $|y| \geq y$. Using this and B1 we conclude that $|x| + |y| \geq x + y$. Similarly, $|x| \geq -x$, $|y| \geq -y \Rightarrow |x| + |y| \geq (-x) + (-y) = -(x + y)$.Therefore, $|x| + |y| \geq |x + y|$.

c. $|x| = x\text{v}(-x)$.

$x > (-x) \Rightarrow x\text{v}(-x) = x$. Also, $x > (-x) \Rightarrow x > 0$, by axioms B1 (which implies transitivity of the relation $>$) and B4. Therefore, $|x| = x$,and we have the result. Cases $x < (-x)$, and $x = (-x)$ are analogous.

d. $x \text{v} y = \frac{1}{2}(x + y + |x - y|)$.

Assume $x \geq y$. Then $|x - y| = x - y$, and $\frac{1}{2}(x + y + |x - y|) = x$. Also, $x \geq y$ $\Rightarrow x \text{v} y = x$. This proves the result. The case $x < y$ is analogous.

e. if $-y \leq x \leq y$, *then* $|x| \leq y$

We know $x \leq y$, so it remains to show that $(-x) \leq y$. We use axiom B1: $-y \leq x \Rightarrow 0 \leq x + y \Rightarrow (-x) \leq y$. This proves the result.

c. Show that there exists a field which consists of three elements: $\{0, 1, 2\}$. *(This field is called* $\mathbb{F}_3$, *and you should think about it as the modulo* 3 *operations on* $\mathbb{Z}$.*)*

"modulo 3" means that instead of the actual sum or product, we take its remainder of integer division by 3. For example, $25 = 3 \times 8 + 1$, so instead of 25 we would take 1. Another example, $5 = 3 \times 1 + 2$, so instead of 5 we would take 2. Therefore, we define addition and multiplication as follows:

$$\begin{aligned} 1 + 1 &= 2,\ 1 + 2 = 0,\ 2 + 2 = 1 \\ 0 \times 0 &= 0,\ 0 \times 2 = 0,\ 2 \times 2 = 1 \end{aligned}$$

d. *Let* $\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\}$. *Let us define the addition and multiplication as follows (note that there is a typo in the problem set that was distributed in class):*

$$\begin{aligned} (a, b) + (c, d) &= (a + c, b + d)\,, \\ (a, b)\,(c, d) &= (ab - cd, ad + bc)\,. \end{aligned}$$

Show that $\mathbb{C}$ *is a field.* $\mathbb{C}$ *is called the set of complex numbers.*

It is straighforward to verify that field axioms hold.

e. *Show that* $\mathbb{R}$ *is isomorphic to a subfield of* $\mathbb{C}$ *with the following mapping*: $f : \mathbb{R} \to \mathbb{C}$.

$$f\,(x) = (x, 0)\,.$$

It is straightforward to verify that $f\,(x + y) = f(x) + f(y)$, and $f\,(x \times y) = f(x) \times f(y)$.

f. *Show that in* $\mathbb{C}$ *there exist solution for the* $x^2 + (1, 0) = (0, 0)$ *equation. Find all the solutions.*

We need to find x of the form (a, b) such that

$$\begin{aligned} a^2 - b^2 + 1 &= 0, \\ ab + ab &= 0. \end{aligned}$$

You can verify that the only two solutions are $(0, 1)$ and $(0, -1)$.

2. **Sequences**

a. Ch.2, pp. 38-39, Ex. 7, 8, 10, 13.

Exercise 7. Show that a sequence can have at most one limit.

Assume x and y are two limits of the sequence $\langle x_n \rangle$. Take $\varepsilon = \frac{1}{2}|x-y|$. Then $\exists n_x \in \mathbb{N}$ such that $\forall n > n_x$ $|x - x_n| < \varepsilon$. Also, $\exists n_y \in \mathbb{N}$ such that $\forall n > n_y$ $|y-x_n| < \varepsilon$. Therefore, $|x-y| = \varepsilon+\varepsilon > |x-x_n|+|y-x_n| = |x-x_n|+|x_n-y| > |x-y|$, where the last inequality follows from Ex. 5b. So, we obtain the false result $|x-y| > |x-y|$, and this means our assumption was wrong.

Exercise 8. Show that l is a cluster point of $\langle x_n \rangle$ if and only if there is a subsequence $\langle x_{n_j} \rangle$ converging to l.

First we prove the "if" part: let there be a subsequence $\langle x_{nj} \rangle$ converging to l. That means, $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $\forall n_j > n$ $|l - x_{n_j}| < \varepsilon$. In plain words, for any ε there is such n that all elements of $\langle x_{n_j} \rangle$ starting from (n+1)th are in the ε-neighborhood of l. There are infinitely many such elements, and therefore l is a cluster point.

To prove the "only if" part we need construct a converging subsequence. At the first step, set $\varepsilon = 1$, and choose an arbitrary element x_n in the ε-neighborhood of l (there are infinitely many, since l is a cluster point). At the j-th step, set $\varepsilon = \frac{1}{j}$, and choose any element from $S_j = \{x_n | x_n \in C_\varepsilon(l)$ and $n > n_{j-1}\}$. There are still infinitely many elements in every S_j, because we withdraw only a finite number of elements at each step. Obviously, the sequence we have constructed converges to l, and this completes the proof.

Exercise 10. Sequence $\langle x_n \rangle$ is convergent if and only if there is exacly one extended real number of the sequence that is a cluster point of $\langle x_n \rangle$. Is this sequence true if we omit the word "extended"?

First of all, the second part is clearly false. To see that, consider the sequence $x_n = (-2)^n + 2^n$. It is easy to see that $x_n = 0$ for n odd, and x_n diverges to infinity for n even.

Let us now proof the "if" part Let x be the only cluster point of $\langle x_n \rangle$. Let us show first that for any $\varepsilon > 0$, there are only finitely many elements of $\langle x_n \rangle$ that are greater than $x + \varepsilon$. We show this by contradiction. Assume there are infinitely many elements of $\langle x_n \rangle$ that are greater than $x + \varepsilon$. Then, either $+\infty$ is a cluster point, or all of these elements are within some interval $[x + \varepsilon, N]$, where N is some natural number. In the latter case, the interval $[x+\varepsilon, N]$ must contain a cluster point by Bolzano theorem proven in class. So in either case we find another cluster point, which contradicts to the assumption we made. Similarly one can show that there are only finitely many elements of $\langle x_n \rangle$ that are smaller than $x - \varepsilon$. So, there are only finitely many elements of $\langle x_n \rangle$ outside of $(x - \varepsilon, x + \varepsilon)$ for an arbitrary $\varepsilon > 0$. Let $n_0(\varepsilon)$ be the highest n for which x_n is outside of $(x - \varepsilon, x + \varepsilon)$. It means $\forall n > n_0(\varepsilon)$, $|x - x_n| < \varepsilon$. Therefore, $\langle x_n \rangle$ converges to l.

We prove the "only if" part by contradiction again. We know that $\langle x_n \rangle$ converges to l. Clearly, l is a cluster point. Assume there is another cluster point, q. Take $\varepsilon = \frac{1}{4}|q - l|$. We know that $\langle x_n \rangle$ converges to l, and therefore $\exists n \in \mathbb{N}$ such that there are at most n elements of $\langle x_n \rangle$ outside of $C_\varepsilon(l)$. But it also must be true that there are infinitely many elements of $\langle x_n \rangle$ within $C_\varepsilon(q)$, and $C_\varepsilon(l) \bigcap C_\varepsilon(q) = \varnothing$. We have arrived to a contradiction.

Exercise 13. Show that the real number l is a limit superior of sequence $\langle x_n \rangle$ if and only if (i) given $\varepsilon > 0$, there exists n such that $x_k < l + \varepsilon$ for all $k \geq n$, and (ii) if given $\varepsilon > 0$, n there exists $k \geq n$ such that $x_k > l - \varepsilon$.

We start by proving the "if" part. First note that (ii) implies that $\forall \varepsilon > 0$, there are infinitely many elements of $\langle x_n \rangle$ *such that* $x_n > l - \varepsilon$ (otherwise we would set n to be the highest index of these elements, and (ii) would not hold for this n). This, together with (i) proves that l is a cluster point. Also, (i) guarantees that there is no cluster point greater than l (see Ex.10 for the idea of the proof). So, l is a highest cluster point, and therefore it is a limit superior. (the last connection has to be proven, strictly speaking, but I will leave it to you).

If l is a limit superior, then it is a largest cluster point (prove it as an exercise) and both (i) and (ii) trivially follow.

b. Where do the following sequences converge (if at all)? If they don't converge, what are their cluster points?

(i) $x_n = \sum_{i=1}^{n} (-1)^i$.

Does not converge, cluster points are 0 and (-1).

(ii) $x_n = \sum_{i=1}^{n} (1/2)^i$.

Converges to 1.

(iii) $x_n = \sum_{i=1}^{n} 1/i$.

This sequence diverges to infinity. This can be proven by induction, here is the idea:

$$\begin{aligned} x_2 &= 1 + \frac{1}{2}, \\ x_4 &= x_2 + \frac{1}{3} + \frac{1}{4} > x_2 + \frac{1}{4} + \frac{1}{4} = x_2 + \frac{1}{2}, \\ x_8 &= x_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > x_4 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = x_4 + \frac{1}{2}, \\ x_{2^k} &> \frac{k}{2} + 1 \text{ (prove as an exercise).} \end{aligned}$$

(iv) $x_n = \sum_{i=1}^{n} (-1)^i / i$.

Converges to 0.

(v) $x_n = \sum_{i=1}^{n} \alpha^i$, $\alpha < 1$.

Converges to $\frac{1}{1-a}$.

c. Suppose that $x_n > x_{n-1}$ for all $n > 1$, and $|x_n| < K \in R$ for all $n \in N$. Show that $\langle x_n \rangle_n$ is converging.

By completeness axiom, $\langle x_n \rangle_{n\in\mathbb{N}}$ has the smallest upper bound. Let us call it l.For any $\varepsilon > 0$, we know that $(l - \varepsilon)$ is not an upper bound on $\langle x_n \rangle$, and therefore there exists x_{n_0} such that $x_{n_0} > l - \varepsilon$. But since $\langle x_n \rangle$ is increasing, it must be true that for all $n > n_0$ we have $x_n > l - \varepsilon$. Therefore, $|l - x_n| < \varepsilon$ for all $n > n_0$. Therefore $\langle x_n \rangle$ converges to l.

d. A sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ is called Cauchy sequence if for all $\varepsilon > 0$ there exists an N such that $|x_k - x_m| < \varepsilon$ whenever $k, m > N$. Show that all Cauchy sequences are converging.

Let us first prove that any Cauchi sequence is bounded. Take $\varepsilon = 1$. Then, by definition, there exists N such that for any $k, m > N$ we have $|x_k - x_m| < 1$. Fix $k = N + 1$. Then for all $m > N$, we know that $x_m \in [x_{N+1} - 1, x_{N+1} + 1]$. Clearly, this interval can be extended to include the first N elements. Therefore, $\langle x_n \rangle$ is bounded

Applying Bolzano theorem we conclude that there is a cluster point of $\langle x_n \rangle$, $l \in [x_{N+1} - 1, x_{N+1} + 1]$. Now, take an arbitrary $\varepsilon > 0$, and let $\tilde{\varepsilon} = \frac{\varepsilon}{2}$. Let M be such that $|x_k - x_m| < \tilde{\varepsilon}$ *whenever* $k, m > M$. By definition of l, there are infinitely many elements of $\langle x_n \rangle$ in $C_{\tilde{\varepsilon}}(l)$ Therefore it must contain such x_k, that $k > M$. But then $\varepsilon = \tilde{\varepsilon} + \tilde{\varepsilon} > |x_k - x_m| + |l - x_k| > |l - x_m|$ is true for any $m > M$. Therefore, $\langle x_n \rangle$ converges to l.

e. Suppose that $\langle x_n \rangle_{n\in\mathbb{N}}$ has a cluster point l, and $x_n > x_{n-1}$ for all $n > 1$. Prove that $\langle x_n \rangle_{n\in\mathbb{N}}$ is convergent.

It is enough to prove that $\langle x_n \rangle$ is bounded, because then we can use the result in (c.). Assume that l is not an upper bound on $\langle x_n \rangle$. Then there is x_n such that $x_n > l$. Then, for any $m > n$, $x_m > x_n > l$, and therefore, there can be only finitely many elements of $\langle x_n \rangle$ in $C_\varepsilon(l)$ for any $\varepsilon < x_n - l$. Therefore l cannot be a cluster poit. We arrived to a contradiction, and therefore l must be an upper bound on $\langle x_n \rangle_{n\in\mathbb{N}}$.

f.If $\lim_{n\to\infty} x_n = a$, what can we say about $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$?

We can use the rule $\lim_{n\to\infty} \frac{y_n}{z_n} = \frac{\lim_{n\to\infty} y_n}{\lim_{n\to\infty} z_n}$ when $\lim_{n\to\infty} z_n \neq 0$ (you can find the proof in any calculus textbook). Therefore, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$ whenever $a \neq 0$. If $a = 0$, we cannot say much. For example, for $x_n = \frac{1}{2^n}$, we have $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 2$, and for $x_n = \frac{1}{n!}$, we know that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ is zero.

3. **Open Sets**

a. Prove Proposition 16 on page 45.

Proposition 16. Let $\mathcal{C}$ be a collection of closed sets (of real numbers) with the property that every finite subcollection has a nonempty intersection, and suppose that one of the sets in $\mathcal{C}$ is bounded. Then

$$\cap_{F\in\mathcal{C}} F \neq \emptyset.$$

Proof: Let $B \in \mathcal{C}$ be the bounded set in $\mathcal{C}$ and assume that $\cap_{F\in\mathcal{C}} F = \emptyset$. Then it must be that $\cup_{F\in\mathcal{C}} \widetilde{F} = \mathbb{R}$ (by $\widetilde{F}$ we denote a complement of F). This, in particular, means that $B \in \cup_{F\in\mathcal{C}} \widetilde{F}$. Note that $\widetilde{F}$ is open, whenever F is closed. Therefore, we can apply Heine-Borel theorem to conclude that there exists a finite subcollection $\mathcal{F}$ of sets in $\mathcal{C}$ such that $B \subset \cup_{F\in\mathcal{F}} \widetilde{F}$. But then, $B \cap \tilde{} \left(\cup_{F\in\mathcal{F}} \widetilde{F}\right) = \emptyset$, and $\tilde{} \left(\cup_{F\in\mathcal{F}} \widetilde{F}\right) = \cap_{F\in\mathcal{F}} F$. This gives us an empty finite intersection, $(\cap_{F\in\mathcal{F}} F) \cap B$. We have arrived to a contradiction.

b. Ch.2, p.46, Ex. 24-27, 30, 31, 34.

Exercise 24. Is the set of rational numbers open or closed?
We know that $\mathbb{Q}$ is not closed, because there are sequences of rational numbers that converge to irrationals. For example, $x_n = (1+\frac{1}{n})^n$, where $\lim x_n = e$. Also, $\mathbb{Q}$ is not open, because in any neighborhood of a rational number there is an irrational number. In particular, you can easily verify that if x is rational, then $x + \frac{e}{n}$ is irrational for any $n \in \mathbb{N}$.

Exercise 25. What are the sets of real numbers that are both open and closed?
The only two sets that are both open and closed are $\emptyset$ and $\mathbb{R}$. It is straightforward to verify the statement.

Exercise 26. Find two sets A and B such that $A \cap B = \emptyset$, and $\overline{A} \cap \overline{B} \neq \emptyset$.
Consider $(-\infty, a)$ and (a, ∞). It is straightforward to verify that the condition holds.

Exercise 27. Show that x is a point of closure of E if and only if there is a sequence $\langle y_n \rangle$ with $y_n \in E$ and $x = \lim y_n$.
The "if" part follows directly from the definition of the limit. To prove the "only if" part we need to construct a converging sequence for an arbitrary closure point. The algorythm is exactly the same as in Ex. 8 solved in section 2.

Exercise 30. A set is called isolated if $E \cap E' = \emptyset$.Show that every isolated set of real numbers is countable.
If $x \in E$ is not an accumulation point of E, then there must be an interval $(x - \varepsilon, x + \varepsilon)$ that contains no other element of E.Also, these intervals can be constructed in such a way that they do not intersect. One way would be to take $(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})$ instead of $(x - \varepsilon, x + \varepsilon)$ in every case. Each of these intervals contains a rational number, and therefore, we can index them by a subset of $\mathbb{Q}$, which is necessarily countable.

Exercise 31. A set D is called dense in $\mathbb{R}$, if $\overline{D} = \mathbb{R}$. Show that the set of rational numbers is dense in $\mathbb{R}$.
There is a rational number between any two real numbers. This implies that any real number is a closure point of $\mathbb{Q}$. Therefore, $\overline{\mathbb{Q}} = \mathbb{R}$.

Exercise 34. A point x is called an interior point of a set A if there is a $\delta > 0$ such that the interval $(x-\delta, x+\delta)$ is contained in A. The set of interior points of A is denoted by A°. Show that

a. A is open if and only if $A = A^\circ$.

b. $A^\circ = \tilde{}(\overline{\tilde{A}})$.

a. The "only if" part follows from the definition of and open set, and the "if" part follows from the definition of the set of interior points immediately

b. First, rewrite the statement as $\widetilde{(A^\circ)} = \overline{\tilde{A}}$. Now, note that the definitions for these two sets are the same: $\{x | \nexists \delta > 0 : (x-\delta, x+\delta) \subset A\}$.

4. **Continuous Functions**

a. Ch.2, p.49, Ex. 42-44, 46, 47.

Exercise 42. Let $\langle f_n \rangle$ be a seqence of continious functions defined on a set E. Prove that if $\langle f_n \rangle$ converges uniformly to f on E. then f is continious on E.

Choose an arbitrary ε.Uniform convergence implies that there exist a function f_{n_0} such that for all $x \in E$, we have $|f(x) - f_{n_0}(x)| < \frac{\varepsilon}{3}$. Continuity of f_{n_0} at an arbitrary point x implies that there exists $\delta > 0$ such that $|f_{n_0}(x) - f_{n_0}(y)| < \frac{\varepsilon}{3}$ whenever $|x-y| < \delta$. Combining these facts we obtain the following:

$$|f(x)-f(y)| \leq |f(x)-f_{n_0}(x)|+|f_{n_0}(x)-f_{n_0}(y)|+|f_{n_0}(y)-f(y)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

For an arbitrary $\varepsilon > 0$ we have found $\delta > 0$ such that $|f(x)-f(y)| < \varepsilon$ whenever $|x-y| < \delta$.Therefore f is continious.

Exercise 43. Let f be the function defined by setting

$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational,} \\ p \sin \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is f continious?

Clearly this function is continious at $x = 0$. To see this note that $|\sin x| \leq |x|$, and therefore $|p \sin \frac{1}{q}| \leq |\frac{p}{q}|$. Therefore we can take $\delta(\varepsilon) = \varepsilon$ in the definiton of continuity. Also, the function is discontinious at any nono-zero rational number. Indeed, if $\varepsilon \equiv \frac{1}{2}|\frac{p}{q} - p \sin \frac{1}{q}| > 0$, then for any irrational number $x \in (\frac{p}{q} - \varepsilon, \frac{p}{q} + \varepsilon)$ it must be true that $|f(x) - f(\frac{p}{q})| = |x - p \sin \frac{1}{q}| \geq |\frac{p}{q} - p \sin \frac{1}{q}| - |x - \frac{p}{q}| \geq 2\varepsilon - \varepsilon = \varepsilon$. So, in any neighborhood of $\frac{p}{q}$ there will be a number x such that $|f(x) - f(\frac{p}{q})| \geq \varepsilon$. A challenging question: is the above argument sufficient to claim that the function is discontinious on irrationals?

b. The following function is called the Dirischlet function.

$$d(x) = \begin{cases} 1 & \text{if } x \text{ is irrational,} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

(i) Find all the points where d is continuous!

The function is discontinious everywhere. Take $\varepsilon = 1/2$. We know that both rationals and irrationals are dense, i.e., there is a rational number in any neighborhood of an irrational number, and also there is an irrational number in any neighborhood of a rational number. The argument is the same as in Ex.24 solved in section 3.

(ii) Find all the points where $xd(x)$ is continuous!

The only continuity point is $x = 0$. See Ex.43 for the proof. For any other point take $\varepsilon = x/2$ to prove discontinuity.

(iii) Find all those numbers, r, for which $d(x+r)\,d(x)$ is continuous everywhere!

First r cannot be rational, because $x + r$ is rational whenever x is rationall, and irrational whenever x is irrational. Therefore, $d(x+r)\,d(x) = d(x)$, whenever r is rational. Neither can r be irrational. To prove that consider the point $x = r$.

5. **Differentiation**

a. What are the derivatives of the following functions their domains?

(i) $f(x) = e^{3x}$.
$f'(x) = 3e^{3x}$

(ii) $f(x) = \left(x^2+2x+3\right)/\left(x^3+4\right)$.
$f'(x) = \frac{\left(x^3+4\right)(2x+2)-\left(x^2+2x+3\right)3x^2}{(x^3+4)^2}$

(iii) $f(x) = \ln\left(x^2+1\right)$.
$f'(x) = \frac{2x}{x^2+1}$.

(iv) $f(x) = \frac{\ln x}{x}$.
$f'(x) = \frac{1-\ln x}{x^2}$.

(v) $f(x) = x^x$.
$f'(x) = (e^{x\ln x})' = (\ln x + 1)x^x$

b. Find the following higher-order derivatives:

(i) $f(x) = \sqrt{x}$, find $f^{(10)}(x)$.
(ii) $f(x) = \frac{1+x}{\sqrt{1-x}}$, find $f^{(100)}(x)$.
(iii) $f(x) = x^2e^{2x}$, find $f^{(20)}(x)$.
(iv) $f(x) = \frac{\ln x}{x}$, find $f^{(5)}(x)$.
(v) $f(x) = x^2\sin 2x$, find $f^{(50)}(x)$.

6. **Extreme Value Computation**

a. Show a function for which $f'(0) = 0$, and f is strictly increasing at zero.
Consider the function $f(x) = x^2$ at zero.

b. Prove the following proposition!

Suppose that f is differentiable on $[a,b]$. Then

(i) f is increasing on $[a,b]$ iff $f' \geq 0$ on $[a,b]$.

(ii) f is constant on $[a,b]$ iff $f' = 0$ on $[a,b]$.

(iii) f is strictly increasing iff $f' \geq 0$ on $[a,b]$ and $f' \neq 0$ on any subinterval of $[a,b]$.

These propositions have been proven on the lecture, they are all immediate corollaries of Lagrange Theorem, also called Mean Value Theorem. It is formulated as follows. Let f be continious on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

c. Compute the local and global extreme values of the following functions.

(i) $f(x) = x^2 - 3x + 3$ on $[-10, 10]$.

Global minimum at $x = 3/2$, where $f'(x) = 0$, $f''(x) > 0$. Global maximum at one of the end points, $x = -10$.

(ii) $f(x) = e^x - 10x^2$ on $[0, 1000]$.

Taking the first two derivatives we obtain $f'(x) = e^x - 20x$, $f''(x) = e^x - 20$. We cannot obtain exact solutions to $e^x - 20x = 0$, but we know there are two such points. One of them is in $(0,1)$ and corresponds to a local maximum. We know that because $f'(0) = 1 > 0$, $f'(1) = e - 10 < 0$, and f is concave on $(0,1)$. Another extremum is somewhere in $(4,5)$, and you can verify this is a global minimum. The global maximum is achieved at $x = 10000$.

7. **Convex Functions**

a. Ch.5, p.116, Ex. 23-25.

Exercise 23.

a. Let φ be a convex function on a finite interval $[a,b)$. Then φ is bounded from below.

Take an arbitrary $x_0 \in [a,b)$. Let $l(x) = m(x - x_0) + \varphi(x_0)$ be a supporting line of $\varphi(x)$ at x_0. We know that $l(x) \leq \varphi(x)$, and, as it follows from Lemma 16 in the book, such a line exists for all $x \in [a,b)$. But this implies that $\varphi(x) \geq l(x) \geq \min_{x \in [a,b)} l(x) = \min m(a - x_0) + \varphi(x_0), m(b - x_0) + \varphi(x_0) \equiv B$. So we have found B, the lower bound on $\varphi(x)$.

b.Show that if φ is convex on (a,b), then $\varphi(x)$ has a limit (possibly infinite) as x approaches a (or b) from within (a,b). If a (or b) is finite, then the limit at a (or b) can be $+\infty$, but not $-\infty$.

We will prove the statement for point b. Let us first consider the case when φ is monotonically decreasing on (a,b). If b is finite, then φ is bounded from below, as we know from Ex.23.a. So, for any $\langle x_n \rangle_{n=1}^{\infty}$ converging to b, the sequence $\langle \varphi(x_n) \rangle_{n=1}^{\infty}$ is monotone and bounded, and therefore has a limit. Now, let us consider the case when φ is not monotonically decreasing on (a,b), i.e. there exist points $p, q \in (a,b)$, such that $p < q$, and $\varphi(p) < \varphi(q)$. Tnen, convexity implies that $\varphi(x)$ is increasing on (q,b). Indeed, by Lemma 16, for any $x_0, x_1 \in (q,b)$ such that $x_0 < x_1$ we have the following

$$\frac{\varphi(x_1)-\varphi(x_0)}{x_1-x_0} \geq \frac{\varphi(q)-\varphi(p)}{q-p} > 0.$$

Therefore, $\varphi(x_1) > \varphi(x_0)$. Now, if φ is unbounded from above on (q,b), we automatically have $\langle\varphi(x_n)\rangle$ converge to $+\infty$ whenever $\langle x_n\rangle$ converges to b. If φ is bounded from above on (q,b), it converges too, since it is monotone.

c. Let φ be continious on an interval I (open, closed, half-open).and convex on the interior of I. Then we have

$$\varphi(tx+(1-t)y) \leq t\varphi(x)+(1-t)\varphi(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Note that the above relation holds on the interior of I due to convexity of φ, so we only have to consider the end points of I. Let y be the end point of I. Consider a sequence $\langle y_n\rangle_{n=1}^{\infty}$ converging to y, such that $y_n \in I\ \forall n$. Due to continuity of φ we know that $\langle\varphi(tx+(1-t)y_n)\rangle_{n=1}^{\infty}$ converges to $\varphi(tx+(1-t)y)$, and we obtain the result:

$$t\varphi(x)+(1-t)\varphi(y) = \lim_{n\to\infty}\left[t\varphi(x)+(1-t)\varphi(y_n)\right] \leq \lim_{n\to\infty}\varphi(tx+(1-t)y_n) = \varphi(tx+(1-t)y).$$

Exercise 24. Prove Corollary 19. Let φ have a second derivative at each point of (a,b). Then φ is convex on (a,b) if and only if $\varphi''(x) \geq 0$ for each $x \in (a,b)$.

The"if" part follows directly from Proposition 18, since non-negative second derivative implies that both derivates are non-decreasing. The "only if" part was proven on the lecture.

Exercise 25.

a. Suppose $a \geq 0$ and $b > 0$. Then the function $\varphi(t) = (a+bt)^p$ is convex on $[0,\infty)$ for $1 \leq p < \infty$, and concave for $0 < p \leq 1$.

b. Show that φ is strictly convex for $p > 1$, and strictly concave for $0 < p < 1$.

b. If $f(x)$ is convex on $[a,b]$, then it is also continious on $[a,b]$. Prove or disprove.

First let us prove the statement for the case when the interval is open: if $f(x)$ is convex on (a,b), then it is also continious on (a,b). Take an arbitrary point $x \in (a,b)$. Now, choose a closed subinterval $[c,d] \subset (a,b)$ that contains x. We can always do this by picking $c \in (a,x)$ and $d \in (x,b)$. According to Proposition 17, $f(x)$ is uniformly continious on $[c,d]$, and since continuity is a weaker property, we conclude that f is continious at x. When the interval is closed the statement is false. Consider for example the following function:

$$f(x) = \begin{cases} x^2, & \text{if } x \in (-1,1), \\ 2, & \text{if } x \in \{-1,1\}. \end{cases}$$

We have discontinuities in the end points, but the function is convex on $[-1,1]$.

8. **Riemann Integral**

Find the following integrals:

(i) $\int \frac{dx}{x+a} =^{(y=x+a)} \int \frac{dy}{y} = \ln y + C = \ln(x+a) + C.$

(ii) $\int (2x-3)^{10} dx =^{(y=2x+3)} \int \frac{y^{10}}{2} dy = \frac{y^{11}}{22} + C = \frac{(2x-3)^{11}}{22} + C.$

(iii) $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x}+1} =^{(y=e^x)} \int \frac{dy}{y^2+1} = \arctan x^2 + C.$

(iv) $\int \frac{\ln^2 x}{x} dx = \ln^3 x - \int \frac{2\ln^2 x}{x} dx = \frac{\ln^3 x}{3} + C.$

(v) $\int \ln x dx = x \ln x - \int \frac{1}{x} x dx = x \ln x - x + C.$

(iv) $\int x^3 e^{-x^2} dx =^{(y=x^2)} \int \frac{y}{2} e^{-y} dy = -\frac{ye^y}{2} + \int \frac{e^{-y}}{2} dy.$
$= -\frac{ye^y}{2} - \frac{e^{-y}}{2} + C = -\frac{e^{-x^2}}{2}\left[x^2+1\right] + C.$

Part III. : Optimization

Most problems in this part are taken from the cook-book I once used as a student. You can find it at the following address:

http://www.cerge-ei.cz/pdf/lecture_notes/LN01.pdf

It contains solutions to these problems too, and has most of the basic mathematical techniques used in economics summarized. So have a look!

1 The Profit-Maximizing Firm

Consider a firm that has the profit function $F(l;k) = pf(l;k) - wl - rk$, where f is the firm's (neoclassical) production function, p is the price of output, l, k are the amount of labor and capital employed by the firm (in units of output), w is the real wage and r is the real rental price of capital. The firm takes p, w and r as *given. Assume that the Hessian matrix of f is negative definite.*

a) Show that if the wage increases by a small amount, then the firm decides to employ less labor.

b) Show that if the wage increases by a small amount and the firm is constrained to maintain the same amount of capital k_0, then the firm will reduce l by less than it does in part a).

2 Utility Maximization

The preferences of a consumer over two goods x and y are given by the utility

function
$U(x;y) = (x+1)(y+1) = xy + x + y + 1$
The prices of goods x and y are 1 and 2, respectively, and the consumer's income is 30. What bundle of goods will the consumer choose?
See page in the cook-book.

Suppose that consumer's utility function takes the form $u(x,y) = [\alpha x^\rho + \beta y^\rho]^{1/\rho}$. *This utility function is known as constant elasticity of substitution (or CES) utility function. Assume the budget constraint of hte consumer is* $p_x x + p_y y \le w$.

(a) Write down the utility maximization problem and solve for Walrasian demand, i.e., express optimal x and y as functions of p_x, p_y, *and* w.

The utility maximization problem is

$$\begin{aligned} \max_{x,y} \quad & [\alpha x^\rho + \beta y^\rho]^{1/\rho} \\ s.t. \quad & p_x x + p_y y \le w \\ & x, y \ge 0 \end{aligned}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial L}{\partial x} &= \frac{1}{\rho}[\alpha x^\rho + \beta y^\rho]^{\frac{1-\rho}{\rho}} \alpha \rho x^{\rho-1} - \lambda p_x = \alpha [\alpha x^\rho + \beta y^\rho]^{\frac{1-\rho}{\rho}} x^{\rho-1} - \lambda p_x \le 0 \\ \frac{\partial L}{\partial y} &= \frac{1}{\rho}[\alpha x^\rho + \beta y^\rho]^{\frac{1-\rho}{\rho}} \beta \rho y^{\rho-1} - \lambda p_y = \beta [\alpha x^\rho + \beta y^\rho]^{\frac{1-\rho}{\rho}} y^{\rho-1} - \lambda p_y \le 0 \end{aligned}$$

Let us firs look for a solution when both goods are consumed in positive amount, and therefore both conditions hold at equality.Then we have the following system

$$\begin{aligned} \alpha [\alpha x^\rho + \beta y^\rho]^{\frac{1-\rho}{\rho}} x^{\rho-1} &= \lambda p_x, \\ \beta [\alpha x^\rho + \beta y^\rho]^{\frac{1-\rho}{\rho}} y^{\rho-1} &= \lambda p_y. \end{aligned}$$

After some manipulations we obtain the following expression

$$x = y \left[\frac{\beta p_x}{\alpha p_y}\right]^{\frac{1}{\rho-1}}.$$

Plugging this into the budget constraint we get the Walrasian demands

$$\begin{aligned} x(p_x, p_y, w) &= w \left[p_x + p_y \left[\frac{\beta p_x}{\alpha p_y}\right]^{\frac{1}{1-\rho}}\right]^{-1} \\ y(p_x, p_y, w) &= w \left[p_x \left[\frac{\beta p_x}{\alpha p_y}\right]^{\frac{1}{\rho-1}} + p_y\right]^{-1} \end{aligned}$$

(b) Verify that these Walrasian demand functions are homogeneous of dergee zero in (p_x, p_y, w), i.e. $x(\alpha p_x, \alpha p_y, \alpha w) = x(p_x, p_y, w)$, $\forall \alpha > 0$.

It is straightforward to verify this property.

(c) The elasticity of substitution between x and y is defined as $\xi_{xy} = -\partial \ln [x(p,w)/y(p,w)] / \partial \ln [p_x/p_y]$. Compute this expression for the CES function.

$$\begin{aligned} \ln \frac{x}{y} = \ln \left[\frac{\beta p_x}{\alpha p_y}\right]^{\frac{1}{\rho-1}} &= \frac{1}{\rho-1}\left[\ln \frac{\beta}{\alpha} + \ln \frac{p_x}{p_y}\right] \\ -\partial \ln \left[\frac{x}{y}\right] / \partial \ln \left[\frac{p_x}{p_y}\right] &= \frac{1}{1-\rho} \end{aligned}$$

3 An Application of the Implicit-Function Theorem

Consider the problem of maximizing the function $f(x;y) = ax + y$ subject to the constraint $x^2 + ay^2 = 1$ where $x > 0$, $y > 0$ and a is a positive parameter. Given that this problem has a solution, find how the optimal values of x and y change if a increases by a very small amount.

4 More problems

4.1

Solve graphically

$$\begin{aligned} \min C &= (10 - x_1)^2 + (20 - x_2)^2 \\ \text{subject to} \quad & 5x_1 + 3x_2 \leq 40 \\ & x_1 \leq 5,\ x_2 \leq 10,\ x_1, x_2 \geq 0 \end{aligned}$$

Find the conditional extrema of the following functions and classify them:

1. $u = x + y + z^2$, $s.t. z - x = 1$, $xz = 1$
2. $u = x + y$, $s.t.$ $1/x^2 + 1/y^2 = 1/a^2$
3. $u = (x + y)z$, $s.t. 1/x^2 + 1/y^2 + 2/z^2 = 4$
4. $u = xyz$, $s.t.$ $x^2 + y^2 + z^2 = 3$
5. $u = xyz$, $s.t.$ $x^2 + y^2 + z^2 = 1$; $x + y + z = 0$