1. (a) Linearly Independent
Consider some linear combination which is equal to zero:

$$\alpha \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the second equation we have that $\alpha = 0$ and hence from the first we have that $\beta = 0$.

(b) Linearly dependent. Consider linear combination with weights a, b, c respectively:

$$a \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus we have that:

$$\begin{aligned} 4a &= b \\ 2a &= -c \\ 2a + 4b + 9c &= 0 \end{aligned}$$

Expressing everything as a function of a and substituting in the lst equation we get:

$$2a + 16a - 19a = 0$$

Take e.g. $a = 1, b = 4, c = -2$ this will work.

(c) Linearly independent from (b) in the last equation we wil get instead that $a = 0$ and hence $b = c = 0$

2. Vectors in (a) and (c) are bases since they are linearly independent and their number conincides with the dimension of the space. System in (b) is not basis.

3. (a) Matrix does not have full rank, see (1) (b)

(b) Matrix have full rank:
Take linear conbination of the columns we will get:

$$a \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + c \begin{pmatrix} 0 \\ 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We will get:

$$\begin{aligned} 3a &= b \\ a &= -2c \\ 2a + 4b + 9c &= 0 \end{aligned}$$

Expressing everything as a function of c we will get: $(b = -6c)$

$$-4c - 24c + 9c = 0$$

Hence $c = 0$ and $a = b = 0$.

5.

$r(AB) \leq \min\{R(A), R(B)\}$

Let $R(B) < R(A)$

Let $m = r(AB) > r(B)$. Consider those m columns of AB (for simplicity let them be the firsts m columns $ab_1, ab_1, ...ab_m$. In fact we know that $ab_i = Ab_i$ Thus we have that:

$$c_1 Ab_1 + c_2 Ab_2 + ... + c_m Ab_m = 0 => c_1 = c_2 = ... = c_m = 0$$

Consider m columns of b: $b_1, ..., b_m$. Since $r(B) < m$ we have that: those columns are linearly dependent, i.e. there exists $c_1, c_2, ..., c_m$ not all equal to zero such that:

$$c_1 b_1 + c_2 b_2 + ... + c_m b_m = 0$$

Applying A to the equality above we obtain a contradiction.

What if $r(A) < r(B)$ then apply 4 and our result above to matrices $B_1 = A', A_1 = B'$

We will have that $r(AB) = r((AB)') = r(B'A') = r(A_1 B_1) \leq r(B_1) = r(A') = r(A) < r(A_1) = r(B)$.

6. Take e.g. the basis to be $ae_1, ae_2, ..., ae_n$ for some $a \neq 0$ evidently for each a this would be a basis.

7. (a)

$$x \cdot y = -3 - 4 - 2 = -9$$

(b)

$$x \cdot y = c_1^2 + c_2^2 + ... + c_n^2$$

8. From the definition of determinant each tern in sum would be multiplied by the same number hence

$$\det(\alpha A) = \alpha^n \det(A)$$

$$\det A^2 = (\det A)^2$$

10.

Subtracting the first row from all other rows we will get:

$$\det\begin{pmatrix}1&1&1&1\\1&1&1&2\\1&1&3&1\\1&4&1&1\end{pmatrix}=\det\begin{pmatrix}1&1&1&1\\0&0&0&1\\0&0&2&0\\0&3&0&0\end{pmatrix}=(-1)^{1+1}\det\begin{pmatrix}0&0&1\\0&2&0\\3&0&0\end{pmatrix}=-6$$

9.

Consider matrix from question 10, lets divide it in 2x2 matrices. Then we would have:

$$\begin{aligned}A&=\begin{pmatrix}1&1\\0&0\end{pmatrix},B=\begin{pmatrix}1&1\\0&1\end{pmatrix}\\C&=\begin{pmatrix}0&0\\0&3\end{pmatrix},D=\begin{pmatrix}2&0\\0&0\end{pmatrix}\end{aligned}$$

Thus we would have:

$$\det A=0,\det B=1,\det C=0,\det D=0$$

Thus

$$\det A\det D-\det B\det C=0$$

11.

$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix},B=2A$$

$$\det(A+B)=\det(3A)=9$$

$$\det A+\det B=1+4=5$$

12.

Generic entry of matrix A^{-1} is:

$$\frac{1}{\det A}(-1)^{i+j}M_{i,j}$$

Where $M_{i,j}$ is minor to i,j element. Since $\det A$ is +1 or -1 and minor is simply a sum of products of integer numbers we get that this element is also integer

13

Since all elements of these matrices are integer then so are the determinants, which are equal to the sume of the products of integers. Thus $\det A$, $\det(A^{-1})$ are integer, but since $\det(A^{-1})=\frac{1}{\det A}$ the only case is that $\det A$ is either 1 or -1.

14

(a)
We know that

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

where $A_{i,j} = (-1)^{i+j} M_{i,i}$ and $M_{i,j}$ is (i,j) minor. Thus we have that:

$$\begin{aligned}
A_{11} &= \begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} = 20 \\
A_{21} &= -\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -10 \\
A_{31} &= \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = -12 \\
A_{12} &= -\begin{vmatrix} 0 & 0 \\ 0 & 5 \end{vmatrix} = 0 \\
A_{22} &= \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5 \\
A_{32} &= -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0 \\
A_{13} &= \begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix} = 0 \\
A_{23} &= -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0 \\
A_{33} &= \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4
\end{aligned}$$

$$\det A = 20$$

Thus inverse matrix is:

$$A^{-1} = \frac{1}{20} \begin{pmatrix} 20 & -10 & -12 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

you can verify the result by direct multiplication.

(b)
In this case $\det A = 1$

$$
\begin{aligned}
A_{11} &= \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \\
A_{21} &= -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \\
A_{31} &= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \\
A_{12} &= -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0 \\
A_{22} &= \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \\
A_{32} &= -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \\
A_{13} &= \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \\
A_{23} &= -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0 \\
A_{33} &= \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1
\end{aligned}
$$

$$
A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}
$$

(c) By similar formula we immediately get that:

$$
A^{-1} = \frac{1}{-3} \begin{pmatrix} 9 & -5 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} -3 & \frac{5}{3} \\ 2 & -1 \end{pmatrix}
$$

15
Let's find the inverse matrix to A

16.
In matrix form this system of equations can be written as:

$$
\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}
$$

By Cramer rule we have:

$$u = \frac{\begin{vmatrix} 7 & 2 \\ 11 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}} = \frac{-1}{1} = -1$$

$$v = \frac{\begin{vmatrix} 3 & 7 \\ 4 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}} = \frac{5}{1} = 5$$

17

(a) We have to refer to the future questions. We cannot say much about first matrix, its determinant is positive and 2 is even number. As for the second matrix its determinant is negative hence the free term in the characteristic equation (which is quadratic in this case) is negative, hence by 19 at least one root is real, but since complex roots come only in pairs we have that all root are real. Moreover since free term is negative (it is equal to the propduct of the roots) we have that roots have different signs. As for the second matrix we see that it leaves second coordinat the same, hence $\lambda = 1$ must be the eigenvalue. As for the remaining roots consider matrix

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

its determinant is negative hence as before we get that both roots are real and of different signs. Thus for this matrix we get that there are two positive eigenvalues and one negative.

(b)

1. Characteristic equation is:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ -1 & 2-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 1) + 3 = \lambda^2 - 3\lambda + 5$$

$$D = 9 - 20 = -11$$

$$\lambda_1 = \frac{3 - i\sqrt{11}}{2}$$

$$\lambda_2 = \frac{3 + \iota\sqrt{11}}{2}$$

Eigenvectors can be computed as:

For λ_1

$$\begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{11}}{2} & 3 \\ -1 & \frac{1}{2} + \iota\frac{\sqrt{11}}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Letting e.g. $b = 1$ we get from the second equation that $a = \frac{1}{2} + \iota\frac{\sqrt{11}}{2}$. To verify our calculations substitute those values into the first equation:

$$-\left(\frac{1}{2} + \iota\frac{\sqrt{11}}{2}\right)\left(\frac{1}{2} - i\frac{\sqrt{11}}{2}\right) + 3 = -\left(\frac{1}{4} - i^2\frac{11}{4}\right) + 3 = 0$$

You can show that for λ_2 we would have as a eigenvector the following:

$$\begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{11}}{2} \\ 1 \end{pmatrix}$$

This matrix has distinct roots but cannot be diagonalized under real numbers.

2. Characteristic equation is:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 1) - 3 = \lambda^2 - 3\lambda - 1$$

$$D = 9 + 4 = 13$$

$$\begin{aligned} \lambda_1 &= \frac{3 - \sqrt{13}}{2} \\ \lambda_2 &= \frac{3 + \sqrt{13}}{2} \end{aligned}$$

Eigenvectors can be computed as:
For λ_1

$$\begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{13}}{2} & 3 \\ 1 & \frac{1}{2} + \frac{\sqrt{13}}{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Take $b = 1$ then $a = -\frac{1}{2} - \frac{\sqrt{13}}{2}$. Let's verify our calculations by substituting into the first equation:

$$-\left(-\frac{1}{2} + \frac{\sqrt{13}}{2}\right)\left(\frac{1}{2} + \frac{\sqrt{13}}{2}\right) + 3 = -\frac{13}{4} + \frac{1}{4} + 3 = 0$$

Thus the eiugenvector is

$$v_1 = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{13}}{2} \\ 1 \end{pmatrix}$$

We can show similarly that eigenvector corresponding to $\lambda_2 = \frac{3+\sqrt{13}}{2}$ can be chosen to be:

$$v_2 = \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{13}}{2} \\ 1 \end{pmatrix}$$

So, we can diagonalize our matrix. Consider matrix U

$$U = (v_1, v_2) = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{13}}{2} & -\frac{1}{2} + \frac{\sqrt{13}}{2} \\ 1 & 1 \end{pmatrix}$$

Clearly this matrix is non singular (i.e. there exists U^{-1}). We know that

$$\begin{aligned} AU &= (\lambda_1 v_1 \lambda_2 v_2) = (v_1 v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \\ &= U\Lambda \end{aligned}$$

Thus since U^{-1} exists we have that

$$A = U\Lambda U^{-1}$$

3. Characteristic equation is:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 9(1-\lambda) = \\ &= (1-\lambda)((1-\lambda)^2 - 9) \end{aligned}$$

Thus we have that

$$\begin{aligned} \lambda_1 &= -2 \\ \lambda_2 &= 1 \\ \lambda_3 &= 4 \end{aligned}$$

Eigenvector for λ_2 is easy to find take e.g.

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Eigenvector for λ_1 :

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We see that $b = 0$ and $3a + 3c = 0$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Eigenvector for λ_3 :

$$\begin{pmatrix} -3 & 0 & 3 \\ 0 & -3 & 0 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

It is clear that eigenvector is:

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

As before we can diagonalize our matrix by choosing martix U

$$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

And

$$\Lambda = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Also it we see that eigenvectors of matrix A_3 are orthogonal.

18

Consider characteristic polinomial

$$f(\lambda) = |A - \lambda I|$$

This is a polinomial of order n with unity as a coefficient on λ^n term and with real coefficients everywhere. By the main thorem of algebra this polinomial has exactly n roots. Since its coefficients ar real then complex roots come in conjugate pairs. Since n is odd then there is at least one real root.

19

If n is odd the this result is already established, hence it suffices to consider the cashe when n is even. in this case characteristic equation is the equation of n th order with unity coefficient on λ^n term. Since $|A| < 0$ then we have that $f(0) < 0$. We also know that $\lim_{\lambda \to \infty} f(\lambda) = +\infty$. Hence there exists a positive root (in case when n is odd this root may not be positive)

20

(a) Since λ_i is eigenvalue then there exists $x_i \neq 0$,

$$Ax_i = \lambda_i x_i$$

But then we would have:

$$A^k x_i = A^{k-1}(Ax_i) = A^{k-1}\lambda_i x_i = \lambda_i A^{k-1} x_i = ... = \lambda_i^k x_i$$

(b)

$$tr(A) = tr\left(U^{-1}JU\right)$$

where J is Jordan decomposition of U. We also know that $tr\,(ABC) = tr\,(CAB)$ thus we get that:

$$tr(A) = tr\left(JUU^{-1}\right) = tr\,(J) = \lambda_1 + \lambda_2 + ... + \lambda_n$$

Another solution:
Consider characteristic polinomial:

$$\tilde{f}(\lambda) = |\lambda I - A| = 0$$

This is a polinomial of degree n with unity coefficient on λ^n term.

$$\tilde{f}(\lambda) = \lambda^n - a_1\lambda^{n-1} + ... + a_n$$

Denote $\lambda_1, ..., \lambda_n$ roots of this polinomal. By the main theorem of algebra we also know that

$$\tilde{f}(\lambda) = (\lambda - \lambda_1)...(\lambda - \lambda_n)$$

Hence we see that

$$a_1 = \lambda_1 + ... + \lambda_n$$

Now from definition of $\tilde{f}(\lambda)$ we also have that:

$$\tilde{f}(\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & ... & ... & -a_{1n} \\ -a_{21} & \lambda - a_{22} & ... & ... & ... \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ -a_{n1} & -a_{n2} & ... & ... & \lambda - a_{nn} \end{vmatrix}$$

The only case when we get λ^{n-1} in this sum of products is when we take $n-1$ lambdas, i.e. select $n-1$ diagonal entries and multiply them by some number.

But this number can only be a_{ii} from the diagonal entry which we did not take among those $n-1$. Hence we get that

$$a_1 = a_{11} + a_{22} + ... + a_{nn}$$

21.

In this case A is diagonilizable:

$$A = U^{-1}\Lambda U$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$$

$$A^k = U^{-1}\Lambda^k U$$

Evidently the following series converge because all eigenvalues are less that unity in absolute value:

$$\sum_{k=0}^{T} A^k = \sum_{k=0}^{T} U^{-1}\Lambda^k U = U^{-1}\left(\sum_{k=0}^{T}\Lambda^k\right)U =$$

$$= \quad U^{-1}\begin{pmatrix} \sum_{k=0}^{T}\lambda_1^k & 0 & ... & 0 \\ 0 & \sum_{k=0}^{T}\lambda_2^k & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \sum_{k=0}^{T}\lambda_n^k \end{pmatrix} U \rightarrow$$

$$\rightarrow \quad U^{-1}\begin{pmatrix} \frac{1}{1-\lambda_1} & 0 & ... & 0 \\ 0 & \frac{1}{1-\lambda_2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \frac{1}{1-\lambda n} \end{pmatrix} U = \sum_{k=0}^{\infty} A^k$$

Consider now

$$\begin{aligned} (I-A)\sum_{k=0}^{\infty} A^k &= (I-A)\lim_{T\rightarrow\infty}\sum_{k=0}^{T} A^k = \\ \lim_{T\rightarrow\infty}(I-A)\sum_{k=0}^{T} A^k &= I - \lim_{T\rightarrow\infty} A^{T+1} = I \end{aligned}$$

$\lim_{T\rightarrow\infty} A^{T+1} = 0$ because series converge. Thus we see that $C = \sum_{k=0}^{\infty} A^k$ is the invere of $(I-A)$

22

Using result above we know that $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$ hence since all entries of all A^k are nonegative so will be the entries of $(I-A)^{-1}$.

23. Identity matrix will be such.

24

Consider characteristic equation:

$$|A - \lambda I| = 0$$

$$A = U^{-1}\Lambda U$$

Thus

$$\begin{aligned} |A - \lambda I| &= \left|U^{-1}\Lambda U - \lambda U^{-1}U\right| = \\ \left|U^{-1}(\Lambda - \lambda I)\,U\right| &= \left|U^{-1}\right| |\Lambda - \lambda I| \, |U| \end{aligned}$$

Thus all the root solving the original equation will solve characteristic equation for Λ.

25

Identity matrix will do, it is already diagonal. For the more general example consider some non-singular matrix A, and some diagonal matrix Λ with e.g. two same elements on the diagonal. And consider matrix $B = A\Lambda A^{-1}$. Evidently B would be diagonalizable and by the reasoning above it would have the same eigenvalues as Λ

26

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

Evidently $\lambda_1 = i, \lambda_2 = -i$ are complex eigenvalues. And eigenvectors corresponding to those would be:

For λ_1

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It is clear that all eigenvectors corresponding to this eigenvalue would be of the form: $b = ia, a = -ib$ i.e. at least one of the coordinates would be complex.

Another eigenvalue is considered similarly.

27.

Consider the upper triangular matrix and consider the characteristic equation for it:

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & ... & ... & a_{1n} \\ 0 & a_{22} - \lambda & a_{23} & ... & a_{2n} \\ 0 & 0 & a_{33} - \lambda & ... & a_{3n} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & a_{nn} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda)...(a_{nn} - \lambda)$$

28

Let the total size of population be 1. Then fractions would be actual quantities. Denote x_n number of people in the population who are well, y_n who are sick, and z_n those who are dead. The law of motion for those quantities are given by the following equations.

$$\begin{aligned} x_{n+1} &= \frac{1}{2}x_n \\ y_{n+1} &= \frac{1}{2}x_n + \frac{3}{4}y_n \\ z_{n+1} &= z_n + \frac{1}{4}y_n \end{aligned}$$

or in matrix notation:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

Assume that $\pi = (x^*, y^*, z^*)$
is the limiting distribution. Then it would solve the following system of equations:

$$\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix}$$

and also it would satisfy $x^* + y^* + z^* = 1$. Solving the equation above gives:

$$\begin{aligned} x^* &= \frac{1}{2}x^* => x^* = 0 \\ y^* &= \frac{3}{4}y^* => y^* = 0 \\ z^* &= z^* \end{aligned}$$

We see that $\pi = (0, 0, 1)$ i.e. eventually everyone dies.

29

Let there exists A^{-1} consider characteristic equation:

$$\begin{aligned} |AB - \lambda I| &= \left|A^{-1}\right| |AB - \lambda I| \, |A| = \\ \left|A^{-1}(AB - \lambda I)A\right| &= |BA - \lambda I| \end{aligned}$$

Another case is similar.

30

Since A has distinct eigenvalues it is diagonalizable and diagonal elements are eigenvalues themselves.

$$A = U^{-1}\Lambda U$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ 0 & 0 & \lambda_3 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda_n \end{pmatrix}$$

Consider matrix

$$\sqrt{\Lambda} = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & ... & 0 \\ 0 & \sqrt{\lambda_2} & 0 & ... & 0 \\ 0 & 0 & \sqrt{\lambda_3} & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \sqrt{\lambda_n} \end{pmatrix}$$

It is evident that $\left(\sqrt{\Lambda}\right)^2 = \Lambda$ then define matrix $B = U^{-1}\sqrt{\Lambda}U$, then we would have that:

$$\begin{aligned} B^2 &= BB = U^{-1}\sqrt{\Lambda}UU^{-1}\sqrt{\Lambda}U = \\ &= U^{-1}\left(\sqrt{\Lambda}\right)^2 U = U^{-1}\Lambda U = A \end{aligned}$$

31

1. Positive-semidefinite, but not positive definite take vector $(1, 1, 1)$
Consider an arbitrary vector in R^3 $x = (a, b, c)$

$$x'Ax = 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac$$

$$= (a-b)^2 + (b-c)^2 + (a-c)^2 \geq 0$$

2. Positive semi-definite

$$x'Ax = a^2 + b^2 + c^2 + 2ac = (a+c)^2 + b^2 \geq 0$$

3. Positive semi-definite
I tried to compute that directly and with the help of eigenvalues it doesn't help. So, let's use indirect approach: we need to show that all diagnoal minors are positive:

$$\Delta_1 = |1| = 1 > 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 > 0$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 6 + 4 - 2 - 3 - 4 = 1 > 0$$

Thus this matrix is positive definite.

32.

Assume that matrix A is positive semi-definite, i.e. A is symmetric and for each $x \in R^n$ we have that $x'Ax \geq 0$.

Consider matrix A^2 we then would have for each $x \in R^n$:

$$\begin{aligned} x'A^2x &= x'AAx = x'A'Ax = (Ax)'(Ax) \\ &= Ax \cdot Ax \geq 0 \end{aligned}$$

If there exists A^{-1} then since A^{-1} is also symmetric we would have that:

$$\begin{aligned} x'A^{-1}x &= x'A^{-1}AA^{-1}x = x'(A^{-1})'A(A^{-1}x) = \\ &= (A^{-1}x)'A(A^{-1}x) \geq 0 \end{aligned}$$

33.

$$\begin{aligned} x'Ax &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = \\ &= (a + b + c)^2 \geq 0 \end{aligned}$$

1.

(a) level sets are circuferences with center in the origin

(b) level sets are hyperbolas of the form $xy = C$

(c) level set for a particula vale $\mu : |\mu| \leq 1.$ are parallel lines with slope -1 of the form

$$\begin{aligned} x + y &= \arccos\mu + 2\pi n \\ x + y &= -\arccos\mu + 2\pi n \end{aligned}$$

where $n \in Z$.

2 Take any $\varepsilon > 0$ choose δ for this ε such that for all $(x, y) : ||(x, y) - (a, b)|| < \delta, (x, y) \neq (a, b)$ we have that $|f(x, y) - L| < \varepsilon$

Denote $g(x) = \lim_{y\to b} f(x, y)$, we know that this function is defined in the neighbourhood of point a.

Inequality $|f(x, y) - L| < \varepsilon$ is satisfied for all $y : |y - b| < \delta$ for each $x : |x - a| < \delta$ (I am using $||x|| = \max |x_i|$) hence taking limit $y \to b$

we get that for all such x :

$$\left| \lim_{y\to b} f(x, y) - L \right| \leq \varepsilon$$

But this is the definition that

$$\lim_{x\to a} \left[\lim_{y\to b} f(x, y) \right] = L$$

3.

For each $x \neq 0$ we have that

$$\lim_{y\to 0} f(x, y) = 0$$

Hence

$$\lim_{x\to 0} \left[\lim_{y\to 0} f(x, y) \right] = 0$$

But there is no $\lim_{(x,y)\to(0,0)} f(x, y)$

Indeed let $x_n = y_n = \frac{1}{n}$ then the

$$\lim_{n\to\infty} f(x_n, y_n) = 1$$

If on the other hand we have that $x_n = 0, y_n = \frac{1}{n}$ then

$$\lim_{n\to\infty} f(x_n, y_n) = 0$$

4.

(a)

$$\frac{\partial f}{\partial x} = 2x + y^3 \cos(xy)$$

$$\frac{\partial f}{\partial y} = 2y \sin(xy) + xy^2 \cos(xy)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 3y^2 \cos(xy) - xy^3 \sin(xy)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2y^2 \cos(xy) + y^2 \cos(xy) - xy^3 \sin(xy)$$

(b)

$$\frac{\partial f}{\partial x} = 2xy^3 z^4$$

$$\frac{\partial f}{\partial y} = 3x^2 y^2 z^4$$

$$\frac{\partial f}{\partial z} = 4x^2 y^3 z^3$$

(c)

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{xy}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\frac{xy}{\sqrt{x^2 + y^2}}$$

(d)

$$\frac{\partial f}{\partial x_i} = a_i$$

$$\nabla f = a$$

(e)

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{n} (a_{ij} + a_{ji}) x_j$$

$$\nabla f = (A + A') x$$

when A is symmetric we have

$$\nabla f = 2Ax$$

(f)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -\frac{2x\sin(x^2)}{y} \\
\frac{\partial f}{\partial y} &= -\frac{\cos(x^2)}{y^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{2x\sin(x^2)}{y^2} \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{2x\sin(x^2)}{y^2}
\end{aligned}$$

5.

$$\frac{\partial f}{\partial x} = \frac{e^{xy}}{2}\sqrt{\frac{y}{x}}$$

6.

$$\begin{aligned}
xdx &= vdv - udu \\
dy &= udv + vdu
\end{aligned}$$

Hence

$$\begin{aligned}
dv &= \frac{xdx + dy}{u+v} \\
du &= \frac{dy - xdx}{v-u}
\end{aligned}$$

Thus we get:

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{x}{u+v} \\
\frac{\partial v}{\partial y} &= \frac{1}{u+v} \\
\frac{\partial u}{\partial x} &= \frac{-x}{v-u} \\
\frac{\partial u}{\partial y} &= \frac{1}{v-u}
\end{aligned}$$

7.

We get:

$$\begin{aligned} F_x dx + F_y dy + F_u du + F_v dv &= 0 \\ G_x dx + G_y dy + G_u du + G_v dv &= 0 \end{aligned}$$

$$\begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{pmatrix} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}^{-1} \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}$$

8.

Let's introduce change of variables:

$$\begin{aligned} x' &= \frac{x-\mu_x}{\sigma_x}, y' = \frac{y-\mu_y}{\sigma_y} \\ dx &= \sigma_x dx', dy = \sigma_y dy' \end{aligned}$$

Thus we would have:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = K\sigma_x\sigma_y \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x'^2+y'^2)}dx'dy'$$

Now lets turn to polar coordinates:

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}$$

Jacobian of this transformation is:

$$\begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix} = r(\cos^2\phi + \sin^2\phi) = r$$

Thus

$$\begin{aligned} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy &= K\sigma_x\sigma_y \int_0^{+\infty}\left(\int_0^{2\pi} re^{-r^2}d\phi\right)dr = \\ K2\pi\sigma_x\sigma_y\int_0^{+\infty} re^{-r^2}dr &= K2\pi\sigma_x\sigma_y\int_0^{+\infty} e^{-u}dr = \\ K2\pi\sigma_x\sigma_y &= 1 \end{aligned}$$

Thus

$$K = \frac{1}{2\pi\sigma_x\sigma_y}$$

9.
(a)

$$\begin{aligned} f(x) &= 3e^x + 20x^3 - \frac{1}{x} \\ f''(x) &= 3e^x + 60x^2 + \frac{1}{x^2} > 0 \end{aligned}$$

Thus this function is convex (it is defined only for $x > 0$) Hence it is also quasiconvex, but no concave. It is easy to see that it is not quasiconcave, since it is not monotone

(it is dcreasing for small x and increasing for large x) we can find y such that set $\{x : f(x) \geq y\}$ is not convex.

(b)

Hessian matrix is:

$$H(x, y, z) = \begin{pmatrix} 3e^x & 0 & 0 \\ 0 & 60x^2 & 0 \\ 0 & 0 & \frac{1}{z^2} \end{pmatrix}$$

Evidently we have that $H(x, y, z) \geq 0$ for all $\{(x, y, z,) : z > 0\}$ .Hence this function is convex and hence quasiconvex. It is not concave since otherwise we should have $H(x, y, z) \leq 0$ and this condition is definitely violated here.

It is also not quasiconcave. Since othewise we should have that function $f(y) = y^4$ is quasiconcave and that is not true. Indeed, assuming that this funciton is quasiconvex we get that for every a the set $\{(x, y, z) | f(x, y, z) \geq a\}$ is convex. Hence the intersection of this set with the set $\{(x, y, z) : x = const, z = const\}$ should also be convex, but this is the set where $\left\{f(y) = y^4 \geq b\right\}$ and we can find b such that this set is not convex. i.e. consider $a = 10$. Then two points $(0, 1000, 0), (0, -1000, 0)$ belong to this set but their average $(0, 0, 0)$ does not.

(c)

Hessian matrix is:

$$H(x, y, z) = Af \begin{pmatrix} a(a-1)\frac{1}{x^2} & ab\frac{1}{xy} & ac\frac{1}{xz} \\ ab\frac{1}{xy} & b(b-1)\frac{1}{y^2} & bc\frac{1}{yz} \\ ac\frac{1}{xz} & bc\frac{1}{yz} & c(c-1)\frac{1}{z^2} \end{pmatrix}$$

If this function is convex we should have: $H(x, y, z) \geq 0$, if concanve we should have $H(x, y, z) \leq 0$

$$\begin{aligned}
\Delta_1 &= a(a-1)\frac{1}{x^2} \\
\Delta_2 &= ab[1-a-b]\frac{1}{x^2y^2} \\
\Delta_3 &= abc[(a-1)(b-1)(c-1) - ac(b-1) + 2abc - (a-1)bc - (c-1)ab]\frac{1}{x^2y^2z^2} \\
&= abc\left[abc - 1 - ab - bc - ac + a + b + c + 2abc - abc + bc - abc + ac - abc + ab\right]\frac{1}{x^2y^2z^2} \\
&= abc\left[a + b + c - 1\right]\frac{1}{x^2y^2z^2}
\end{aligned}$$

Several cases are possible. If one of the coefficients is greater than one. Let it be a then this matrix is neither positive nor negative smidefinite, since we would have $\Delta_1, \Delta_3 \geq 0$ and $\Delta_2 < 0$.

Let all coefficients be less than one. But their sum is greater than one then again this matrix ia neither positive nor negative definite.

Let now th sum of all coefficients be less than one. Then this matrix would be negative semidefinite and hence this function would be concave, and hence quasiconcave, but not convex.

What about quasiconvexity and quasiconvexity. Note that this question does not depend on the value of coefficients, since we can always renormalize them to be less than one. Or in fact it is easier to take log of this function (it is a monotonic transformation) and hence the function would be quasiconcave, quasiconvex irrespective of the fact that some coefficients are greater or less than one. Thus it is equivalent to consider the following funciton:

$$f(x, y, z) = a \log x + b \log y + c \log z$$

Evidently this funciton is concave as a sum of concave functions. Hence the original function is quasiconcave. But this function is not quasiconvex, just consider the projection of the level set $\{(x, y, z) : f(x, y, z) \leq 0\}$ with the plane $z = 1$ evidenly this set is a complement in (x, y) space to the set $\left\{x^a y^b \geq 1\right\}$ which is strictly convex, hence this set is not convex (just draw a picture, this is in case a Cobb-Douglas preferences case).

(d)

Hessian is

$$H(x, y) = \begin{pmatrix} ye^{-x} & -e^{-x} \\ -e^{-x} & 0 \end{pmatrix}$$

Since

$$\begin{aligned}
\Delta_1 &= ye^{-x} > 0 \\
\Delta_2 &= -e^{-2x} < 0
\end{aligned}$$

This matrix is neither positive nor negative semidefinite and hence the funciton is neither convex nor concave.

But it is easy to see the this function is quasiconcave, since itis a monotonic transformation of the concave funciton

$$\tilde{f}(x, y) = \ln y - x$$

The latter funciton is concave as a sum of two concave functions.

Also it is easy to show that this function is not quasiconvex. Indeed the sets $A(a) = \{(x, y) : \ln y \leq x + a\} = \{(x, y) : y \leq Ce^x\}$ is not convex.

(e)

Consider function

$$\tilde{f}(x) = x'Ax$$

It is easy to see that Hessian of this function is matrix A, and hence this function is convex, and the original function is quasiconvex. But this function is not quasiconvex (and hence the original function is not): indeed consider some $x \neq 0$ we know that $x'Ax > 0$ (if this matrix is positive semidefinite then such x exists, if matrix is positive definite then any non zero x is such.) we also know that $(-x)'A(-x) = x'Ax > 0$,but average of x and $-x$ is 0 and this function is zero at zero. $f(\frac{x-x}{2}) = 0 < \min\{f(x), f(-x)\}$.

What about convexity concavity of the original function.

$$f(x) = g(\tilde{f}(x)), \text{ where } g(y) = e^y$$

$g(y)$ is convex function.

$$\begin{aligned} H(x) &= \frac{\partial^2 f}{\partial x' \partial x} = \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x'}\left(2Axe^{x'Ax}\right) = \\ &= 2\left(Axx'A'e^{x'Ax} + 2A\right)e^{x'Ax} \end{aligned}$$

Evidently thius matrix is positive definite, hence the original function is convex, but not concave since otherwise this matrix should be negative semidefinite.

10.

Consider the second derivative of the composite function:

$$\begin{aligned} (f \circ g)' &= f'(g(x))g'(x) \\ (f(g(x)))'' &= f''(g(x))g'^2(x) + f'(g(x))g''(x) \end{aligned}$$

Thus is is sufficient to assume that both functions are concave and f is increasing.

11.

1=>2 If f has convex upper sets then the set $A(f(y)) = \{x : f(x) \geq f(y)\}$ is convex and hence since x, y are in this set then so is their convex combination, but that means that

$f(x_t) \geq f(y)$

2=>3 Evident

3=>1 Consider some upper level set $A(a) = \{x : f(x) \geq a\}$. If this set is empty or one point then it is convex. Let now x, y be some points in that set. Then $f(x) \geq a, f(y) \geq a$

but then for every convex combination we would have:

$$f(x_t) \geq \min \{f(x), f(y)\} \geq a$$

12

Take any two points x, y . Since every function f_i is convex we would have that:

$$f_i(tx + (1-t)y) \leq tf_i(x) + (1-t)f_i(y)$$

Hence for each function we would have:

$$f_i(tx + (1-t)y) \leq t \sup f_i(x) + (1-t) \sup f_i(y) = tf(x) + (1-t)f(y)$$

But since the last inequality is tru for each i then it is true for sup over i :

$$f(tx + (1-t)y) = \sup f_i(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Infimum of convex functions wiil be in general neither concave nor convex: Take e.g. fuctions $f_1(x) = x, f_2(x) = x^2$

13

Consider two points from this set: x, y since they are both global maximizors then we should have $f(x) = f(y) = f^*$. Since function f is concave we would have that

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y) = f^*$$

But strict inequality is not possible since f^* is the maximal value of the function, hence at $tx + (1-t)y$ function achieves its maximal value of f^*

14

(a)

$$\nabla f = \begin{pmatrix} 4x^3 + 2x - 6y \\ 6y - 6x \end{pmatrix}$$

$$H = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}$$

Equating gradient to zero one gets:

$$
\begin{aligned}
x &= y \\
4x^3 &= 4x
\end{aligned}
$$

Hence there are tow critical points: $(0,0),(1,1)$

When $(x,y) = (0,0)$ Hessian is neither negative nor positive semidefinite, sot his is a saddle point.

$$(\lambda-2)(\lambda-6)-36 = \lambda^2 - 8\lambda - 24$$

We see that eigenvalues are of different signs hence at point $(0,0)$ function is convex in the direction of one eigenvector and concave in the direction of other eigenvector.

When $(x,y) = (1,1)$ Hessian is positive semidefinite, thus this point is local maximum (but not global since function can attain arbitrarily high values.)

(b)

$$\nabla f = \begin{pmatrix} y^2 + 3x^2y - y \\ 2xy + x^3 - x \end{pmatrix}$$

$$H = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}$$

Equating gradient to zero we get:

$$
\begin{aligned}
y^2 &= y\left(3x^2 - 1\right) \\
\left(x^2 - 1\right)x &= 2xy
\end{aligned}
$$

Solutions are $(1,0),(0,0)(-1,0),(0,-1)$, or if both x,y are non zero we have:

$$
\begin{aligned}
3x^2 - 1 &= y \\
x^2 - 1 &= 2y
\end{aligned}
$$

$$y = 6y + 2$$

or $(x,y) = \left(\frac{1}{\sqrt{5}}, -\frac{2}{5}\right)$.

Consider now Hessian matrix at each of these points:

$$(x,y) = (1,0)$$

$$H = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$$

This matrix is neither positive nor negative semidefinite.($\Delta_1 = 0, \Delta_2 < 0$) So this is a saddle point.

$$(x,y) = (0,0)$$

$$H = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

The similar result $(\Delta_1 = 0, \Delta_2 < 0)$, again saddle point.

$$(x,y) = (-1,0)$$

$$H = \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}$$

This matrix is again neither positive nor negative semidefinite $(\Delta_2 < 0)$,again saddle point.

$$(x,y) = (0,-1)$$

$$H = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

Again saddle point

$$(x,y) = \left(\frac{1}{\sqrt{5}}, -\frac{2}{5}\right)$$

$$H = \begin{pmatrix} -\frac{12}{\sqrt{5}} & -\frac{6}{5} \\ -\frac{6}{5} & -\frac{2}{\sqrt{5}} \end{pmatrix}$$

We have that $\Delta_1 < 0, \Delta_2 = \frac{24}{5} - \frac{36}{25} > 0$, thus this is a local maximum, but not glabal since as x goes to $+\infty$ whole function goes there (at least for some y)

(c)

$$\nabla f = \begin{pmatrix} 2x + 6y - 10 \\ 2y + 6x - 3z - 5 \\ 8z - 3y - 21 \end{pmatrix}$$

$$H = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{pmatrix}$$

$\Delta_2 = 32 - 18 - 36 * 8 < 0$ thus this is a saddle point. Unique solution can easily be found.

(d)

$$\nabla f = \begin{pmatrix} 3x^2 \\ 2 - 2y \end{pmatrix}$$

$$H = \begin{pmatrix} 6x & 0 \\ 0 & -2 \end{pmatrix}$$

Critical point $(0,1)$ is a saddle point $|H| < 0$
(e)

$$\nabla f = \begin{pmatrix} 3x^2 - 6y \\ 3y^2 - 6x \end{pmatrix}$$

$$H = \begin{pmatrix} 6x & -6 \\ -6 & 6y \end{pmatrix}$$

Critical points solve:

$$\begin{aligned} x^2 &= 2y \\ y^2 &= 2x \end{aligned}$$

$$y^2 = \frac{x^4}{4} = 2x$$

Hence critical points are $(0,0)\,, (2,2)$
Consider Hessian at those points.
When $(x,y) = (0,0)$

$$H = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$$

$|H| < 0$, saddle point
When $(x,y) = (2,2)$

$$H = \begin{pmatrix} 12 & -6 \\ -6 & 12 \end{pmatrix}$$

We see that this matrix is positive definite, hence this is a local minimum, but not global — let x goes to - infinity.

15
(a)
When $a+b > 1$, or one of them is negative then this problem has no solution. Indeed if either α or β is negative then we get unpounded profis by setting one of the factors to zero. When sum of coefficients is greater than unity, then profits are unbounded (for each combination, which "maximimzes" our profits function consider λ times this combination and we will get even higher profits).

When $\alpha+\beta = 1$ then solution is not unique — only the ratio of factor prices can be determined. And solution is unique if $\alpha + \beta < 1$

Consider first order conditions, if the solution exists it must satisfy F.O.C.

$$\begin{aligned} \alpha p A k^{\alpha-1} l^{\beta} &= r \\ \beta p A k^{\alpha} l^{\beta-1} &= w \end{aligned}$$

Dividing first order conditions we get that:

$$\frac{\alpha}{\beta}\frac{l}{k} = \frac{r}{w}$$

substituting back into the first equation we get:

$$\alpha p A k^{\beta+\alpha-1}\left(\frac{r\beta}{w\alpha}\right)^{\beta} = r$$

We see that when $\alpha+\beta=1$ then solution is the whole line with the same capital labor ratio and value of the function is zero by Euler theorem.

When $\alpha+\beta<1$ we get that:

$$k = (Ap)^{\frac{1}{1-\alpha-\beta}}\left(\frac{\beta}{w}\right)^{\frac{\beta}{1-\alpha-\beta}}\left(\frac{\alpha}{r}\right)^{\frac{1-\beta}{1-\alpha-\beta}}$$

Similarly we get:

$$l = (Ap)^{\frac{1}{1-\alpha-\beta}}\left(\frac{\beta}{w}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha-\beta}}$$

16

Let a be a critical point of f

Let $H_{ii}>0, H_{jj}<0$ let's prove that a is saddle point.

The Hessian would be neither positive nore negative definite at this point, since if we consider vectors with all zeros except $i-th$ or j-th place

then we would have positive and negative values for bilinear form, given by H.

17

(a) For all $\alpha,\beta\geq 0$, (in fact we should have restrictions $x,y\geq 0$ otherwise) thus we would be maximizing over compact set a continuous function. Here condition $\alpha+\beta<1$ does not matter, since in effect the problem is the same whatever the power we raise our target function. (i.e. we can always guarantee $\alpha+\beta=1$)

(b) .First order conditions are:

$$\begin{aligned} \alpha\frac{x^{\alpha}z^{\beta}}{x} &= \lambda p_x \\ \beta\frac{x^{\alpha}z^{\beta}}{z} &= \lambda p_z \end{aligned}$$

Dividing both conditions we get:

$$p_x x = \frac{\alpha}{\beta} p_z z$$

Substituting back into the budget constraint we have:

$$\begin{aligned} z &= \frac{\beta}{\alpha+\beta}\frac{y}{p_z} \\ x &= \frac{\alpha}{\alpha+\beta}\frac{y}{p_x} \end{aligned}$$

(c) Since $u'_x = +\infty$ we know that solution would never be a corner $x = 0$ thus in the formulation of the Lagrangian we can ignore this constraint.

(d) If this inequality holds as strict then we can increase utility by adding a bit more to each of the goods.

18

(a)

Graphical solution, we are moving lines with slope -45 degrees as far to the left as possible. $x^2 + y^2 \leq 1$ is part or circle of radius 1 with center in $(0, 0)$.

Solution is $(0.5, 0.5)$

(b)

We are maximizing x hence we must select among all the points in the budget set the one with the highest x

Since $y^4 \geq 0$ then the maximal x is $x = 5$

(c)

Consider the Lagrangian:

$$L = 3xy - x - y + \lambda(5 - 2x - y) + \mu(38 - x - y) + \phi x + \kappa y$$

First order conditions:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 3y - 1 - 2\lambda - \mu + \phi = 0 \\ \frac{\partial L}{\partial y} &= 3x - 1 - \lambda - \mu + \kappa = 0 \end{aligned}$$

$$\begin{aligned} \mu(38 - x - y) &= 0 \\ \phi x &= 0 \\ \kappa y &= 0 \\ \mu, \phi, \kappa &\geq 0 \end{aligned}$$

Let at first constraint $x \geq 0$ binds, then we get that $y = 5$ and objective is equal to -5

If constraint $y \geq 0$ binds then $x = \frac{5}{2}$ and objective is equal to $-\frac{5}{2}$

Let now those constraints does not bind at all then we get that $\phi = \kappa = 0$ we then would have:

$$\mu = 3y - 1 - 2\lambda = 3x - 1 - \lambda$$

$$y - x = \lambda$$

We know that $2x + y = 5$ thus we get that:

$$2x + x + \lambda = 5$$

$$\begin{aligned} x &= \frac{5-\lambda}{3} \\ y &= \frac{5+2\lambda}{3} \end{aligned}$$

Assume that constraint $x + y \leq 38$ binds then we will get:

$$\begin{aligned} x + y &= \frac{10+\lambda}{3} = 38 \\ \lambda &= 38 * 3 - 10 > 5 \end{aligned}$$

Hence $x < 0$
Hence this constraint should not bind and hence $\mu = 0$ But then we have

$$\begin{aligned} 3x - 1 - \lambda &= 4 - 2\lambda = 0 \\ \lambda &= 2 \end{aligned}$$

$$\begin{aligned} x &= 1 \\ y &= 3 \end{aligned}$$

Value of the objective is:

$$9 - 3 - 1 = 5$$

Hence this is the solution.
19.
(a) this problem is equivalent to the sequence of one-period problems. For every t consumer would be solving:

$$\begin{aligned} &\max_{c_t, l_t} u(c_t) + v(l_t) \\ c_t &= w_t(1 - l_t) + d_t \\ c_t &\geq 0, 0 \leq l_t \leq 1 \end{aligned}$$

Consumer would be allocating his full diposable income $I_t = w_t + d_t$ between consumption and leisure, where he relative price of the leisure is w_t, solution would depend on the relative slopes of functions u and v

First order conditions to this problem are:

$$\begin{aligned} u'(c_t) &= \lambda_t + \mu_t \\ v'(l_t) &= w_t\lambda_t + \phi_t - \kappa_t \\ c_t\mu_t &= 0, \mu_t \geq 0 \\ l_t\phi_t &= 0, \phi_t \geq 0 \\ (1-l_t)\kappa_t &= 0, \kappa_t \geq 0 \end{aligned}$$

In the interior solution we would have:

$$u'(c_t)w_t = v'(l_t)$$

(b) In this case we know that constraints $c_t \geq 0, l_t \geq 0$ will not bind. If we assume the contrary then some of the lagrange multipliers would have to be infinite. So, we can solve the problem ignoring those constraints:

$$\begin{aligned} u'(c_t) &= \lambda_t \\ v'(l_t) &= w_t\lambda_t - \kappa_t \\ (1-l_t)\kappa_t &= 0, \kappa_t \geq 0 \end{aligned}$$

or more compactly:

$$u'(c_t)w_t \leq v'(l_t), \text{ with } = \text{ if } l_t < 1$$

(c)

Assuming that u is strictly increasing we would have that $l_t = 1$ is never a solution (provided $w_t > 0$) since if it is we can slightly decrease it causing decrease in the objective of $v'(1)dl = 0$ and gaining in consumption $u'(c_t)dc > 0$

More formally assume that $l_t = 1$ is the solution then $c_t = d_t > 0$ thus only one of inequality constraints bind: $l_t = 1$ and hence $\phi_t = \mu_t = 0$

but then we would have:

$$u'(c_t)w_t \leq v'(l_t) = 0, \text{ with } = \text{ if } \kappa_t = 0$$

which is not possible.

(d)

We know that $d_t > 0$ hence constraint $c_t \geq 0$ never binds, since the minimal level of consumption is d_t.

What about the effect of the wage, intuitively if wage is sufficiently high then individual may choose to never rest, if vice versa it is low then individual might choose to never work. Formally, these conditions can be written as:

$$w_t > \frac{v'(0)}{u'(w_t + d_t)}$$

If this conditions is satisfied then individual never has a rest. Indeed from the FOC we get

$$\begin{aligned} u'(c_t) &= \lambda_t \\ v'(l_t) &= w_t \lambda_t + \phi_t - \kappa_t = \\ &= u'(c_t) w_t + \phi_t - \kappa_t \end{aligned}$$

We know that $\phi_t \kappa_t = 0$, i.e. they never bind together, hence we see that in the solution we have

$$v'(l_t) < v'(0) < w_t u'(w_t + d_t) \leq w_t u'(c_t)$$

Hence we must have that $\kappa_t = 0$ and $\phi_t > 0$ and hence we get that $l_t = 0$
Similarly we can show that if

$$w_t < \frac{v'(1)}{u'(d_t)}$$

then the consumer will never work.

20
(a)
Kuhn-Tucker conditions would be:

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) - \lambda \frac{\partial g}{\partial x}(x, y) + \mu &= 0 \\ \frac{\partial f}{\partial y}(x, y) - \lambda \frac{\partial g}{\partial y}(x, y) + \kappa &= 0 \end{aligned}$$

$$\begin{aligned} \lambda g(x, y) &= 0, \lambda \geq 0 \\ \mu x &= 0, \mu \geq 0 \\ \kappa y &= 0, \kappa \geq 0 \end{aligned}$$

(b)
Follows directly from the slack conditions
(c)
Take $f(x, y) = -(x - 1)^2 - (y - 1)^2$ evidently point $(1, 1)$ maximizes this function. Assume now that constraint is $g(x, y) = x + y \leq 2$
In the optimum this constraint evidently binds but multiplier on it is zero.
(d)
we have:

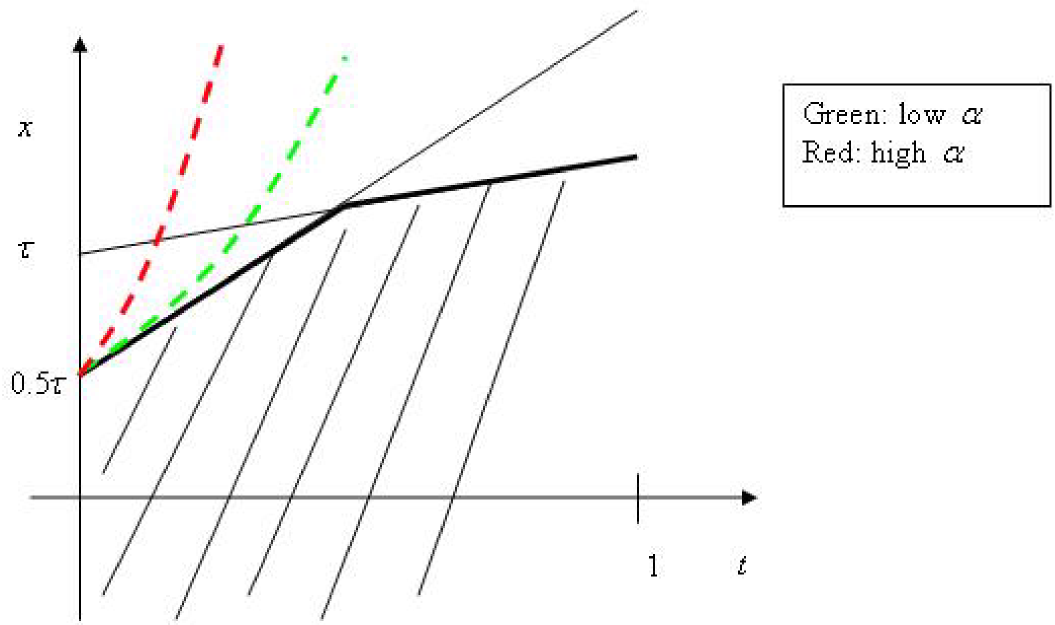

Figure 1:

$$\frac{\partial f}{\partial x}(x,y) - \lambda \frac{\partial g}{\partial x}(x,y) = 0$$
$$\frac{\partial f}{\partial y}(x,y) - \lambda \frac{\partial g}{\partial y}(x,y) = 0$$

Effectively the problem is to

$$\max f(x,y)$$

$$s.t. g(x,y) = 0$$

21.

(a)

(b) We see that two people can sit on one of the corners, e.g. on the corner where $t = 0$, i.e. for sufficiently high α it is

optimal to set t as low as possible.

(c) we can ignore the constraint $t \leq 1$ since this constraint is not going to bind, marginal utility of decreasing t a little bit at this point is infinite.

(d)
Do not use the Hint.:)
22.

When U is strictly quasiconcave then solution would be unique. Also budget constraint will be satisfied with equality. (increasing means increasing with respect to each variable.) Assume also that $U'(0, x_2) = U'(x_1, 0) = +\infty$. In order to apply implicit function theorem we need interior solution.

Solution can be found from the following equations.

$$\begin{aligned} \frac{\partial U}{\partial x_1} p_2 - \frac{\partial U}{\partial x_2} p_1 &= 0 \\ p_1 x_1 + p_2 x_2 &= y \end{aligned}$$

Taking full differential we get:

$$\begin{pmatrix} \frac{\partial^2 U}{\partial x_1^2} - \frac{\partial U}{\partial x_1 \partial x_2} & \frac{\partial U}{\partial x_1 \partial x_2} - \frac{\partial^2 U}{\partial x_2^2} \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial U}{\partial x_2} & -\frac{\partial U}{\partial x_1} & 0 \\ -x_1 & -x_2 & 1 \end{pmatrix} \begin{pmatrix} dp_1 \\ dp_2 \\ dy \end{pmatrix}$$

Second order conditions would guarantee that the determinant of this matrix is negative (just consider the problem as the problem of unconstrained optimization of one variable and write down the S.O.C.)

Hence when writing the inverse matrix we should take account of that:

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} p_2 & \frac{\partial^2 U}{\partial x_2^2} - \frac{\partial U}{\partial x_1 \partial x_2} \\ -p_1 & \frac{\partial^2 U}{\partial x_1^2} - \frac{\partial U}{\partial x_1 \partial x_2} \end{pmatrix} \begin{pmatrix} \frac{\partial U}{\partial x_2} & -\frac{\partial U}{\partial x_1} & 0 \\ -x_1 & -x_2 & 1 \end{pmatrix} \begin{pmatrix} dp_1 \\ dp_2 \\ dy \end{pmatrix}$$

Due to symmetry it is enough to consider the impact on x_1 only.

$$dx_1 = \frac{1}{\Delta} \begin{pmatrix} p_2 & \frac{\partial^2 U}{\partial x_2^2} - \frac{\partial U}{\partial x_1 \partial x_2} \end{pmatrix} \begin{pmatrix} \frac{\partial U}{\partial x_2} & -\frac{\partial U}{\partial x_1} & 0 \\ -x_1 & -x_2 & 1 \end{pmatrix} \begin{pmatrix} dp_1 \\ dp_2 \\ dy \end{pmatrix}$$

$$\frac{\partial x_1}{\partial y} = \frac{1}{\Delta} \left(\frac{\partial^2 U}{\partial x_2^2} - \frac{\partial U}{\partial x_1 \partial x_2} \right)$$

When cross derivative is small we immediately get that x_1 is increasing with income. i.e. it is a normal good, if on the contrary it negative and large in absolute value, then it might be that demand for this good is decreasing with income. i.e. marginal utility of the consumption of the other good decline rapidly with the consumption of the first good, then it might be optimal as a response to an increase in wealth to reduce the consumption of the first good, thus increasing marginal utility of consumption of the other good. Also note that it is impossible for the two goods to be inferior at the same time, this follows from budget constraint.

Consider now a change in own price of the good.

$$\frac{\partial x_1}{\partial p_1} = \frac{1}{\Delta}\left(p_2\frac{\partial U}{\partial x_2} - x_1\left(\frac{\partial^2 U}{\partial x_2^2} - \frac{\partial U}{\partial x_1 \partial x_2}\right)\right)$$

Again we see that if the second derivative is of moderate magnitude the own price effect is negative. For a normal good $\frac{\partial^2 U}{\partial x_2^2} - \frac{\partial U}{\partial x_1 \partial x_2} < 0$ we definitely have that. If the good is inferior but not too much then as price is increasing the consumption of this good falls. But there is also theoretical possiblity that income effect for the inferior good is too strong (this happens when $\frac{\partial U}{\partial x_1 \partial x_2} < 0$ and large in absolute value) then we might have that as the price of the good increases then its consumption goes up.

It might be useful to decompose a change in price into two effects, so called substitution and income effect. As our earlier calculations suggest the second term in the sum above is the income effect and the first one is substitution.

We see that own price substitution effect is always negative, whereas sign of income effect depends on whether the good is normal or not. A change in own price of the good have two effects. Assume e.g. decrease in price. Then substitution effect would make us consume more of the good, since it has become relatively cheaper than the other good. On the other hand income effect since our real income is increased is ambiguous, it is positive for the normal good and negative for the inferior. Also note that negative income effect is likely to dominate when the consumption of good under consideration is really high (x_1 multiplies it), when when we consume small amount of the good and its price decreases we are likely to consume more of this good iven if it is very inferior.

Consider now change in the price of the other good:

$$\frac{\partial x_1}{\partial p_2} = \frac{1}{\Delta}\left(-p_2\frac{\partial U}{\partial x_1} - x_2\left(\frac{\partial^2 U}{\partial x_2^2} - \frac{\partial U}{\partial x_1 \partial x_2}\right)\right)$$

Now we again see the work of two effects. Now the substitution effect is acting in the opposite direction. If the price of the other good decreases we want to consume less of the first good, since it is now relatively more expensive. Income effect acts as before in the same direction. when the price of the other good decreases then real income of the consumer increases and hence we might want to consume more of the first good if it is normal and less if it is inferior.

1.
(a) showed before
(b)

$$\nabla Az = A$$

(c) First order conditions are:

$$\begin{aligned} 2X'(Y - X\beta) &= 0 \\ X'X\beta &= X'Y \\ \beta &= \left(X'X\right)^{-1} X'Y \end{aligned}$$

(d) Matrix X must be invertible.

(e) Yes they might.

2.
consider the Lagrangian:

$$L(a) = f(x) + \lambda^T (a - g(x))$$

First order conditions:

$$\frac{df}{dx_i} - \sum \lambda_j \frac{dg_j}{dx_i} = 0$$

$$\lambda_i (g_i(x) - a_i) = 0$$

We know that for fixed λ optimal value of x maximizes the lagrangian.

$$\frac{dL}{da_i} = \sum \left(\frac{df}{dx_j} - \sum \lambda_k \frac{dg_k}{dx_j} \right) \frac{dx_i}{da_i} + \sum \frac{d\lambda_k}{da_i} \left(a_k - g_k(x)\right) + \lambda_i = \lambda_i$$

Since

$$f(x(a)) = L(x(a), \lambda(a))$$

we get the result.
3.
when $f'(0) = +\infty$ then non-negativity constraints on inputs will not bind.
Consider the Lagrangian:

$$L = \sum w_i x_i + \lambda(y - f(x_1, ..., x_n))$$

As before

$$\frac{dL}{dw_i} = x_i + \sum w_j \frac{dx_j}{dw_i} + \frac{d\lambda}{dw_i}(y - f(x)) - \sum \lambda \frac{df}{dx_j} \frac{dx_j}{dw_i} = x_i$$

4

(a)

Evidently this correspondence is not UHC, because its graph is not closed. But it is in fact LHC: for each sequence $x_n \to x$ and every $y : y'y < x$

Since $x_n \to x$ then from some n we have that

$$y'y < x_n$$

Take $y_n = y$ evidently this will do the job

(b) Again not UHC since graph is not closed.Will it be LHC, if y in question has the property that $y'y < x$ then everything is OK. Assume that $y'y = x$ take $y_n = y\sqrt{\frac{x_n}{x}}$

we have that $y_n \to y$ and since y has negative coordinate then from some n y$_n$ has negative coordinate, but then we also have that

$$y_n' y_n = x_n$$

and this means that $y_n \in f(x_n)$ since y_n has some negative coordinate.

(c) Not UHC since graph not closed (take $x > 0$) Also it is not LHC. Take sequence $x_n = -\frac{1}{n} \to 0$ and take $y = 0$ evidently for each $y_n \in f(x_n)$ we have that $|y_n| > 2$ so0 it cannot

converge to 0

5.

(a)

If we assume that prices can attain zero then it is not UHC since when price equals zero (compact set) the image is unbounded.

If we make additional assumption that prices are separated from zero. (All prices are greater or equal than some ε) Then it is UHC since graph is closed and images of bounded sets are bounded.

What about LHC? Assume that $p_n \to p, y_n \to y$

And tgake some $x : px \leq y$

If first $px < y$ then we can take $x_n = x$ and then by continuity we'll get that:

$$\begin{aligned} p_n x_n &= p_n x \to px < y \\ y_n &\to y \end{aligned}$$

We'll get that for sufficiently high n

$$p_n x < y_n$$

What if $px = y$

$$\begin{aligned} x_n &= \left(x_1 \frac{p_1}{p_{1n}}, ..., x_k \frac{p_k}{p_{kn}} \right) \frac{y_n}{y} \\ p_n x_n &= px = y_n \end{aligned}$$

And this is properly defined since all prices are positive.

(b) we already proved that
(c) it is not UHC since graph is not closed, but LHC
6.
(a) Let x_n br Cauchy sequence.

$$\forall(\varepsilon > 0)\exists(n_o \in N)\forall(n \geq n_0, p \in N)\left[||x_n - x_{n+p}|| < \varepsilon\right]$$

In particula in this norm it means that for each fixed t sequenceof values $x_n(t)$ is Cauchy and hence converges some value: $x(t)$. function $x(t)$ is continuous as the limit of unifromly converging continuous functions.
Also we have that

$$\begin{aligned}
\forall(\varepsilon &> 0)\exists(n_o \in N)\forall(n \geq n_0, p \in N)\forall t\left[|x_n(t) - x_{n+p}(t)| < \varepsilon\right] => \\
\forall(\varepsilon &> 0)\exists(n_o \in N)\forall(n \geq n_0)\forall t\left[|x_n(t) - x(t)| \leq \varepsilon\right] \\
\forall(\varepsilon &> 0)\exists(n_o \in N)\forall(n \geq n_0)\left[\max_t |x_n(t) - x(t)| = ||x_n - x|| \leq \varepsilon\right]
\end{aligned}$$

(b) Let $Lx = \alpha x$, where $0 < \alpha < 1$

$$||L(x) - L(y)|| = \alpha\,||x - y|| < ||x - y||$$

This is a contraction.

(c)(d) Unique fixed point is $x = 0$

$$\alpha x = x <=> x = 0$$

7.
(a) $S = [0, 1]$ and $f(x) = x$ for $x < 0.5$ and 1 otherwise
(b) Add to the previous example $[2, 3]$ and consider hte same funciton
(c) $S = [0, \infty)$ $f(x) = x$
(d) same as (b but instead $[2, \infty)$